Quantum deformation of quantum cosmology: 
A framework to discuss the cosmological constant problem

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Abstract

We endorse the context that the cosmological constant problem is a quantum cosmology issue. Therefore, in this paper we investigate the q-deformed Wheeler-DeWitt equation of a spatially closed homogeneous and isotropic Universe in the presence of a conformally coupled scalar field. Specifically, the quantum deformed Universe is a quantized minisuperspace model constructed from quantum Heisenberg-Weyl \( U_q(\mathfrak{su}(1, 1)) \) groups. These intrinsic mathematical features allow to establish that (i) the scale factor, the scalar field and corresponding momenta are quantized and (ii) the phase space has a non-equidistance lattice structure. On the other hand, such quantum group structure provides us a new framework to discuss the cosmological constant problem. Subsequently, we show that a ultraviolet cutoff can be obtained at \( 10^{-3}eV \), i.e., at a scale much larger than the expected Planck scale. In addition, an infrared cutoff, at the size of the observed Universe, emerges from within such quantum deformation of Universe. In other words, the spectrum of the scale factor is upper bounded. Moreover, we show that the emergent cosmological horizon is a quantum sphere \( S^2_q \) or, alternatively, a fuzzy sphere \( S^2_F \) which explicitly exhibits features of the holographic principle. The corresponding number of fundamental cells equals the dimension of the Hilbert space and hence, the cosmological constant can be presented as a consequence of the quantum deformation of the FLRW minisuperspace.

Keywords: Cosmological constant problem, Quantum cosmology, Quantum groups, Holographic principle

1. Introduction

Since the mid-1980s, astrophysicists have been compiling evidence – such as cosmic microwave background observations, the supernova type Ia data and large scale structure – that the late time Universe is accelerating. The simplest candidate to explain this acceleration, within the framework of general relativity (GR), is a positive cosmological constant (CC). Many theoretical physicists were reluctant to consider the CC as a bona fide explanation regarding the mentioned acceleration, because the natural predicted value for the CC from particle physics is \( \rho_{\Lambda} \approx M_P^4 \approx (10^{18} \text{GeV})^4 \), which has a enormous discrepancy with the astronomical bound for CC, \( \rho_{\Lambda} \approx (10^{-3} eV)^4 \) – some \( 10^{22} \) times too small. In other words, from an effective field theory (EFT) point of view, the CC is an IR scale problem and affecting the large scale structure of the Universe, when we investigate the whole Universe. Hence, the CC problem seems to violate our prejudice about decoupling UV and IR scales, which underlies the use of EFT. The CC, being interpretable both as the zero point energy and as the scale of the observed Universe, goes against the notion of local quantum fields and suggests a mixing between local UV and global IR physics. In this direction, some physicists believe that CC problem is essentially a quantum gravity and quantum cosmology problem [1]. A candidate theory for quantum gravity must provide a classical continuum spacetime geometry at macroscopic scales with a global IR cutoff (CC) but also involving quantum corrections at the local UV scale.

The quantum spacetime hypothesis asserts that the classical continuum should break down at the Planck scale. As we know, the Planck length provides a natural length unit involving gravitational and quantum features. It defines a distance scale at which quantum corrections to GR are expected to be significant. It is also commonly thought to provide a momentum cutoff, rendering finite otherwise divergent particle self-energies and enabling gravity to play
the role of a universal regulator to other fundamental inter-
actions.

It is well known that noncommutative (NC) geometry
provides an approach to deal with possible properties of
Planck scale physics. In fact, the idea of the quantization
of a spacetime manifold, as well as phase space sym-
plectic manifolds, using noncommuting coordinates \(\hat{x}^\mu\), is
an old one. In this context, the quantum spacetime of
Hartland Snyder type introduces the
\[
[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu},
\]
where the \(\theta^{\mu\nu}\) generates the Lorentz group. The cor-
responding NC gravity has been considered in various mod-
els (see for example [7, 8]). In particular in Ref. [8] a
NC Einstein gravity is constructed by using the Seiberg-
Witten map and gauging the NC ISO(3,1) group. Also, the
Snyder type of NC has been studied in the context of homogen-
ous cosmologies for various min-
issorspace models.

Moreover, fuzzy NC models proposed by ’t Hooft have
\[
[\hat{x}^\mu, \hat{x}^\nu] = 2i\hbar\epsilon_{\mu\nu\rho}\hat{x}^\rho.
\]

It is, in fact, a toy model of quantum gravity in a 3D
manifold with a Euclidean signature. A subsequent de-
velopment, including quantum differential calculus and an
action of a certain “quantum double” quantum group as
NC Euclidean group of motions, was proposed by Majid
and E. Batista. Furthermore, the Majid-Ruegg bicro-
product model spacetimes bear
\[
[\hat{x}^i, \hat{\xi}^j] = 0, \quad [\hat{x}^i, \hat{i}] = i\lambda\hat{x}^i, \quad i, j = 1, 2, 3,
\]
with corresponding deformed Poincaré group and convey
a physically testable prediction of a variable speed of light.

In Ref. [13] the authors showed that in Snyder type NC
spacetimes defined in [11] there is a minimum distance but
no minimum area. This suggests that we need to be care-
ful about the possible emergence of minimum area at the
Planck scale in this kind of NC spacetimes.

Let us also mention a specific category of NC space-
times featuring anisotropic or q-deformed manifolds, moti-
vated from quantum groups theory, given by
\[
\hat{x}^\mu \hat{x}^\nu = q^{\nu\mu} \hat{x}^\mu.
\]
This model was developed independently in [15] and also
by Majid and coworkers, in a series of papers on braided
matrices.

In general, quantum groups give us symmetries which
are richer than the classical Lie algebras, which are con-
tained in the former as a special case. It is therefore
possible that quantum group can turn out to be suit-
able for describing symmetries of physical systems which
are outside the realm of Lie algebras [23]. Therefore,
the q-deformed models spacetimes are applicable for any
physical manifold. In addition, the great advantage of q-
deformed models is that the corresponding Hilbert space is
finite dimensional when \(q\) is root of unity [24]. This sug-
gests that the use of quantum groups when deformation
parameter is root of unity is a powerful tool to build mod-
els with a finite number of states, aiming at applications
in quantum gravity and quantum cosmology that are con-
sistent with the assumption of the holographic principle
and a UV/IR mixing to solve CC problem.

Quantum groups may appear in q-versions of gravity in
various situations. In Ref. [29] Finkelstein constructed a q-
deformation of GR by replacing the Lorentz group by the
quantum Lorentz group. Furthermore, in [30] a q-gravity is
constructed by “gauging” the quantum analogue of a
Poincaré algebra. They also occur in Hamiltonian quanti-
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2Quantum groups and algebras emerged from studies on quantum
integrable models using the quantum inverse scattering method and led to certain deformations of classical matrix groups and the
corresponding Lie algebras. The original main reason for the great
significance of quantum groups was that they are related to the so-
called quantum Yang-Baxter equation which plays a major role in
quantum integrable systems, conformal field theory, solvable
lattice models, knot theory and topological quantum compu-
tation. Phenomenological applications of quantum groups in
nuclear and molecular spectroscopy lead to significant results showing that the vibrational-rotational spectra of nuclei and
molecules can be fit into schemes in which the number of phenomeno-
logical deformation parameters required are very much fewer than
the number of traditional phenomenological parameters.

3Any consistent theory of quantum gravity requires dramatically
new ideas like the holographic principle which have recently at-
ttracted increasing attention. Perhaps the most remarkable aspect
of this principle is that the entropy is finite, suggesting a finite dimen-
sional Hilbert space of excitations describing the interior of a region
bounded by a surface of area. This is also a feature found in loop
quantum gravity, where quantum groups at roots of unit and finite
number of states naturally enter the formalism.

1One of the most interesting applications of this type of noncom
mutativity to quantum field theory, is that concerning a descrip-
tion of Yang-Mills instantons in NC spacetime; In these NC spaces,
instantons acquire an effective size proportional to the noncommut-
ativity parameter \(\theta\). As a consequence, the moduli space of NC
instantons no longer has the singularities corresponding to small in-
stantons [4].
route. More specifically, we consider the quantum deformation of the phase space variables from a spatially closed homogeneous and isotropic universe minisuperspace, with $q$ being a root of unit.

This paper is organized as follows. In Sec. II, we present the model that assists in our investigation. In Sec. III, the corresponding WDW equation and boundary conditions of model are extracted. In Sec. IV, we describe the hidden symmetries and corresponding quantum groups that are deformations of the enveloping algebras of Heisenberg-Weyl and $su(1,1)$ Lie algebras. We obtain the eigenvectors and discrete eigenvalues of scale factor, scalar field and the corresponding momenta. We show that the scale factor and the corresponding momenta are non-singular. In Sec. V, we explain that the finite dimensional Hilbert torus and the corresponding momenta are non-singular. We show that the scale factor vanishes at the classical singularity (Dirichlet boundary condition) \[50\]. Two other proposals have been used as explicit procedures to deal with the presence of classical singularities. More precisely, the wave function should vanish at the classical singularity \cite{13}. Two other proposals have been used as explicit procedures to deal with the presence of classical singularities. More precisely, the wave function should vanish at the classical singularity (DeWitt or Neumann boundary condition) \cite{49}, or its derivative with respect to the scale factor vanishes at the classical singularity (Dirichlet boundary condition) \cite{50}.

Our physical interpretation of quantum mechanical operators depends on their Hermicity and self-adjointness.

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However, in some cases boundary conditions must be specified in order for Hamiltonians to be Hermitian and self-adjoint. In particular, Hamiltonians with singular potentials or restricted domain of definition require us to specify how wave functions behave at the boundaries or at the singularities of potential. Thus, a satisfactory treatment of the WDW equation of our cosmological model requires to solve the equation in a Hilbert space and the solutions have to be associated with a self-adjoint operator.

In coordinate representation, the canonical quantization of our model is accomplished by setting $x = x_1$ and $\Pi_1 = -i \frac{\partial}{\partial x_1}$. Then, the Hamiltonian constraint \( 11 \) becomes the WDW equation

$$
-\frac{1}{2M_\rho} \left( \partial_1^2 - \partial_2^2 \right) \Psi(x_1) + \frac{1}{2} M_\rho \omega^2 (x_1^2 - x_2^2) \Psi(x_1) = 0. \tag{11}
$$

Due to the hyperbolic character of \( 11 \), we can separate the scalar field part from the gravitational sector, i.e., $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ represent the gravitational and scalar field parts of super-Hamiltonian \( 10 \) respectively. By assuming $\Psi(x_1, x_2) = \Theta(x_1) \Phi(x_2)$, for the scalar field part $\Phi(x_2)$, with a separation constant $E$, we find

$$
\left( -\frac{1}{2M_\rho} \frac{d^2}{dx_2^2} + \frac{1}{2} M_\rho \omega^2 x_2^2 \right) \Phi(x_2) = E \Phi(x_2). \tag{12}
$$

The solution to the above equation is

$$
\Phi_n(x_2) = C_n H_n \left( \sqrt{\frac{M_\rho}{\omega}} x_2 \right) e^{-\frac{M_\rho}{2} x_2^2}, \quad E_n = \omega (n + \frac{1}{2}),
$$

where $H_n$ are the Hermite polynomials.

The domain of definition of scale factor, $x_1 = a$, is $\mathbb{R}^+$. Therefore, the gravitational part of WDW equation

$$
\left( -\frac{1}{2M_\rho} \frac{d^2}{dx_1^2} + \frac{1}{2} M_\rho \omega^2 x_1^2 \right) \Theta(x_1) = E \Theta(x_1), \tag{14}
$$

is defined on a dense domain $\mathcal{D}(\mathcal{H}_1) = C^\infty_0 (\mathbb{R}^+)$. The operator $\mathcal{H}_1 := -\frac{1}{2M_\rho} \frac{d^2}{dx_1^2} + \frac{1}{2} M_\rho \omega^2 x_1^2$ is in the limit point case at $+\infty$ and in the limit circle case at zero, hence it is not essentially self-adjoint \( 51 \). $\mathcal{H}_1$ is Hermitian if

$$
(\Theta_1 | \mathcal{H}_1 | \Theta_2) = (\mathcal{H}_1 | \Theta_1 \Theta_2), \quad \Theta_1, \Theta_2 \in \mathcal{D}(\mathcal{H}_1). \tag{15}
$$

Since $\mathcal{H}_1$ is not singular, such that all subtleties related to Hermiticity and self-adjointness are associated entirely with the behavior at $x_2 = 0$, this is the case if

$$
\lim_{x_2 \to 0^+} \left( \frac{d\Theta_1}{dx_1} \Theta_2 - \Theta_1 \frac{d\Theta_2}{dx_1} \right) = 0. \tag{16}
$$

It can be shown \( 52 \) that to ensure the validity of this condition it is necessary and sufficient for the domain of $\mathcal{H}_1$ to be restricted to those wave functions that satisfy the Robin boundary condition

$$
\frac{d\Theta_1}{dx_1}|_{x_1 = 0^+} + \gamma \Theta_1(0^+) = 0, \tag{17}
$$

where $\gamma$ is an arbitrary real constant which has the dimension of inverse of length. The parameter $\gamma$ thus characterizes a 1-parameter family of self-adjoint extensions of the $\mathcal{H}_1$ on the half-line.

There is a peculiar a difficulty here with these extensions, which is that each extension leads to a different physics and the problem is not just of technical nature. Nevertheless, due to the existence of the conformal scalar field we will show that the scalar field part solves the problem. The general square-integrable solution of Eq. \( 14 \) with boundary condition \( 17 \) is given by

$$
\Theta(x_1) = \frac{\sqrt{\pi} e^{-\frac{1}{2} M_\rho x_1^2}}{2^{\frac{M_\rho}{2}} \Gamma(\frac{M_\rho}{2} + \frac{1}{2})} F_1(\frac{1}{2} - \frac{E}{2\omega}; \frac{1}{2}; \frac{M_\rho}{2} x_1^2) \tag{18}
$$

$$
-\frac{\sqrt{\pi} e^{-\frac{1}{2} M_\rho x_1^2}}{2^{\frac{M_\rho}{2}} \Gamma(\frac{M_\rho}{2} + \frac{1}{2})} F_1(\frac{3}{4} - \frac{E}{2\omega}; \frac{3}{4}; \frac{M_\rho}{2} x_1^2),
$$

where $F_1(\alpha; \beta; x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} I_1(x)$ denotes confluent hypergeometric function. Making use of the properties, $F_1(\alpha; \beta; 0) = 1$ and $F_1(\alpha + \gamma; \beta; x) = F_1(\alpha; \beta; x)$, we can rewrite the boundary condition \( 17 \) as

$$
\gamma = 2 \sqrt{\frac{M_\rho}{\omega}} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{M_\rho}{2})}, \quad n = 0, 1, 2, \ldots \tag{19}
$$

For odd values of $n$, this equation fixes the length parameter as $\gamma = +\infty$. In addition, for even values of $n$, we find $\gamma = 0$. Therefore, the scalar field part of the WDW equation restricts the self-adjoint extension of gravitational part to the self-adjoint extension operator with Dirichlet boundary condition \( 50 \)

$$
\frac{d\Theta(x_1)}{dx_1}|_{x_1 \to 0^+} = 0, \tag{20}
$$

where the spectrum coincide with the spectrum of the odd parity eigenfunctions of the harmonic oscillator, or by DeWitt boundary (or Neumann boundary) condition \( 19 \)

$$
\Theta(x_1)|_{x_1 \to 0^+} = 0, \tag{21}
$$

with eigenvalues $E_n = \omega (2n + \frac{1}{2})$, which coincides with the even parity sector of the harmonic oscillator spectrum.

### 4. Quantum deformation of quantum cosmology

Two basic concepts both in classical and quantum systems are states and observables. In classical mechanics states are points of a phase space manifold, $\Gamma$, and observables (physical quantities) are smooth functions $f \in C^\infty(\Gamma)$. Every state determines the value of the observables on it \( 53 \). In
quantum mechanics states are one-dimensional subspaces of a Hilbert space and observables are operators in Hilbert space. By the above duality relation between states and observables in both classical and quantum cases, the relation between classical and quantum mechanics is easier to understand in terms of an algebra of observables: observables form an associative algebra which is commutative (abelian) in the classical mechanics and NC (non-abelian) in the corresponding quantum system. In this regard, quantization (in quantum mechanics where non-commutativity controlled by \( \hbar \)) amounts to replace the commutative algebras by NC ones [5] and all related fundamental mathematical concepts are expressed in such a way that it does not require commutativity of the algebra. In this manner we may arrive at the concept of NC geometry: the usual (algebraic) geometry is the study of commutative algebras and NC (algebraic) geometry is the study of NC algebras. In this regards, NC Hopf algebras are like non-abelian groups.

Hopf structures in ordinary Lie groups and Lie algebras. If \( F(G) \) denotes the set of differentiable functions from a Lie group \( G \) into the complex numbers \( \mathbb{C} \), then the algebraic structure is given by the usual pointwise sum and product of functions and the unit of algebra \( 1 \in F(G) \) is the constant function \( f(y) = 1, \forall y \in G \). Using the group structure of \( G \) we can introduce on \( F(G) \) three other linear operations, the coproduct \( \Delta \), the counit \( \varepsilon \) and the antipode (or coinverse) \( S \):

\[
\begin{align*}
\Delta(f)(g_1,g_2) &= f(g_1g_2), \\
\Delta : F(G) &\rightarrow F(G) \otimes F(G), \\
\varepsilon(f) &= f(e), \quad \varepsilon : F(G) \rightarrow \mathbb{C}, \\
(Sf)(g) &= f(g^{-1}), \quad S : F(G) \rightarrow F(G),
\end{align*}
\]

where \( e \) is the unit of \( G \) and \( g_1,g_2 \in G \). Algebra \( F(G) \) equipped with the linear maps \( \Delta, \varepsilon \) and \( S \) is called the Hopf algebra [55]. Coproducts are commonly used in the familiar addition of momentum, angular momentum and of other so-called primitive operators in quantum mechanics. Since additivity of observables is an essential requirement, the coproduct, and therefore the Hopf algebra structure, appears to provide an essential algebraic tool in quantum mechanics. Therefore, Hopf algebra is a bialgebra with an antipode.

Formally, quantum groups are defined to be Hopf algebras which are in general, NC. From a physical point of view, quantum group includes two basic ideas, namely the \( q \)-deformation of an algebraic structure and the notion of a NC comultiplication. Physicists are familiar with the idea of deformation. For example, the Poincaré group is a deformation of the Galilei group with deformation parameter \( c \), which is recovered in the limit \( c \rightarrow \infty \) or quantum mechanics can be considered as a deformation of classical mechanics with deformation parameter Planck’s constant which is regained in the limit \( \hbar \rightarrow 0 \). In the \( q \)-deformation of an algebraic structure, usually a deformation parameter \( q \) (a dimensionless parameter) is introduced in which a commutative algebra becomes noncommuting and in the \( q \rightarrow 1 \) limit, the original algebraic structure is recovered.

### 4.1. Quantum deformation of the scalar field

To construct the quantum deformation of the scalar field part of the super-Hamiltonian given in Eq.(12), let us employ the Heisenberg-Weyl algebra of \( \mathcal{H}_2, h_4 \). This is a non-semisimple Lie algebra with four generators \( \{A_+, A_-, N, e\} \) that satisfy following commutation relations

\[
[A_-, A_+] = e, \quad [N, A_\pm] = \pm A_\pm,
\]

where \( e \) is the central charge and

\[
A_+ := \frac{M_p \omega}{2} \left( x_2 + \frac{\hbar}{M_p \omega} \frac{d}{dx_2} \right), \\
A_- := \frac{M_p \omega}{2} \left( x_2 - \frac{1}{M_p \omega} \frac{d}{dx_2} \right).
\]

Since \( h_4 \) is a Lie algebra, its universal enveloping algebra \( \mathcal{U}(h_4) \) is a Hopf algebra with the usual maps given by

\[
\Delta(y) = y \otimes 1 + 1 \otimes y, \\
\varepsilon(y) = 0, \quad S(y) = -y,
\]

where \( y \in \{A_+, A_-, N, e\} \), \( \Delta \) is the comultiplication of algebra, \( \varepsilon \) denotes its counit and \( S \) is antipode. For the unit element of the algebra the Hopf maps are \( \Delta(1) = 1, \quad \varepsilon(1) = 1 \) and \( S(1) = 1 \). It is usual to work with the quotient algebra \( \mathcal{U}'(h_4) = \mathcal{U}(h_4)/(e-1) \), where the Heisenberg relation is \([A_-, A_+] = 1\).

In the Fock space, \( \mathcal{F}_2 \), with the basis \( \{|n\}, N|n\rangle = n|n\rangle \) the pairs of operators \( A_\pm \) act in the following form

\[
\begin{align*}
A_+|n\rangle &= \sqrt{n+1}|n+1\rangle, \\
A_-|n\rangle &= \sqrt{n}|n-1\rangle.
\end{align*}
\]

The quantum Heisenberg-Weyl algebra, \( \mathcal{U}'_q(h_4) \), is the associative unital \( \mathbb{C}(q)\)-algebra with generators \( \{A_+, A_-, q^{\frac{N}{2}}, q^{-\frac{N}{2}}\} \) with the following q-deformed commutation relations

\[
\begin{align*}
A_-A_+ - q^{\frac{1}{2}}A_+A_- &= q^{-\frac{1}{2}}N, \\
A_-A_+ - q^{-\frac{1}{2}}A_+A_- &= q^{\frac{1}{2}}N,
\end{align*}
\]

\([N, A_\pm] = \pm A_\pm\).

Note that in this definition we do not postulate any relation among the generators of the algebra. We can show that for this representation, the first two relations act actually equivalent to the following relations

\[
A_+A_- = [N], \quad A_-A_+ = [N + 1],
\]

where

\[
[y] := q^\frac{y}{2} - q^{-\frac{1}{2}}q^\frac{y}{2}.
\]

The relation \( A_+A_- = [N] \) can be compared to the central element (it commutes with all generators of algebra) of \( \mathcal{U}'(h_4), A_+A_- = -N \), which acts as zero on the standard
Fock space representation of vacuum. If we define the vacuum state of the quantum deformed Fock space, $F_2(q)$, by

$$\mathcal{A}_-|0\rangle = 0, \quad q^{\pm \frac{1}{2N}}|0\rangle = |0\rangle,$$  

then we can construct the representation of the $U'_q(h_4)$ in the Fock space spanned by normalized eigenvectors $|n\rangle$

$$|n\rangle = \frac{1}{\sqrt{|n|!}} A^n_+ |0\rangle,$$

where the $q^\frac{1}{2}$-factorial defined by $[n]! := \prod_{m=1}^{n}[m]$. The basis $|n\rangle$ defined above is orthonormal due to the identities

$$\mathcal{A}_- A^n_+ = q^{\frac{n}{2}} A^n_+ A_-, \quad [n] A^n_+ - q^{\frac{n}{2}} A^n_+ = q^{\frac{n-1}{2}} A^n_+ - q^{\frac{n-1}{2}} A^n_+,$$

Hence, in the Fock space $F_2(q)$ the set up operators act due

$$\mathcal{A}_-|n\rangle = \sqrt{|n+1|} |n\rangle, \quad \mathcal{A}_+|n\rangle = \sqrt{|n|} (n-1) |n\rangle, \quad \mathcal{N}|n\rangle = n |n\rangle.$$

Note that in the Fock space there exists a deforming map to the classical $U'\langle h_4\rangle$ given by

$$\mathcal{A}_- = \sqrt{\frac{|N+1|}{N+1}} A_-, \quad \mathcal{A}_+ = A_+ \sqrt{\frac{|N+1|}{N+1}}, \quad \mathcal{N} = N.$$  

The operator

$$\mathcal{H}_2 = \frac{\omega}{2} (A_+ A_+ + A_- A_-) = \frac{\omega}{2} ([N+1] + [N]),$$

can be considered to be the $q$-analog of the scalar field part of super-Hamiltonian defined in (12). Furthermore, the phase space realization of the $q$-deformed oscillator is given by

$$\hat{x}_2 = \frac{1}{\sqrt{2\omega}} (A_+ + A_-), \quad \hat{p}_2 = i \sqrt{2\omega} (A_+ - A_-).$$

Hence, the commutation relation

$$[x_2, \hat{p}_2] = i ([N+1] - [N]),$$

shows that the effective Planck's constant is no longer a constant, but depends on the state of the scalar field.

Let us now consider the case in which $q$ is a primitive root of unity, i.e.,

$$q = \exp \left( \frac{2\pi i}{\mathfrak{N}} \right),$$

where $\mathfrak{N}$ is a natural number, $\mathfrak{N} \in \mathbb{N}^+$, and $\mathfrak{N} \geq 2$. In the remainder of this section we will show that it is equivalent to the existence of the finite number of possible quantum states in the Universe.

It is clear that for $\mathfrak{N} \rightarrow \infty$, the deformation parameter $q \rightarrow 1$ and all of the deformed quantities will reduce to the ordinary undeformed ones. The quantum number defined in (30) will be

$$|y| = \frac{q^\frac{1}{2} y - q^{-\frac{1}{2}} y}{q^\frac{1}{2} - q^{-\frac{1}{2}}} = \sin \left( \frac{\pi}{\mathfrak{N}} \right),$$

Evidently, $q^\mathfrak{N} = 1$, $|\mathfrak{N}| = |\mathfrak{N}| = 0$ and $|\mathfrak{N} + k| = |k|$ where $k$ is an integer. By using identities (33) and the defining relations of $U'_q(h_4)$ on can show that at the root of unity the elements $\{A^n_+, A^n_-, q^{\frac{N}{2}}, q^{-\frac{N}{2}}\}$ lie in the center of $U'_q(h_4)$. The action of pairs of operators $\{A_+, A_-\}$ on the basis eigenvectors are

$$A_+|n\rangle = \sqrt{|n+1|} |n\rangle, \quad A_-|n\rangle = \sqrt{|n|} (n-1) |n\rangle, \quad A_+|0\rangle = 0, \quad A_+|0\rangle = 0.$$  

Therefore, $A_+$ annihilate the state $|\mathfrak{N} - 1\rangle$ and the Fock space $F_2(q)$ is finite $\mathfrak{N}$-dimensional vector space with basis $\{|0\rangle, |1\rangle, ..., |\mathfrak{N}-1\rangle\}$. The Fock space matrix representation of the generators then become finite dimensional 57

$$A_+ = \sum_{n=0}^{\mathfrak{N}-2} \sqrt{|n+1|} |n+1\rangle \langle n|, \quad A_- = \sum_{n=1}^{\mathfrak{N}-1} \sqrt{|n|} |n-1\rangle \langle n|, \quad N = \sum_{n=0}^{\mathfrak{N}-1} n |n\rangle \langle n|.$$  

This representation follows the nilpotency of the generators of $U'_q(h_4)$

$$A^\mathfrak{N}_\pm = 0.$$  

Now, Eqs. (30) and (40) admit the following eigenvalues for $\mathcal{H}_2$

$$E_n = \frac{\omega \sin \left( \frac{\pi}{\mathfrak{N}} (n + \frac{1}{2}) \right)}{2 \sin \left( \frac{\pi}{\mathfrak{N}} \right)}, \quad n = 0, ..., \mathfrak{N} - 1.$$  

Note that for $\mathfrak{N} \rightarrow \infty$ the earlier eigenvalues will reduce to (13). Since $\sin \left( \frac{\pi}{\mathfrak{N}} (n + \frac{1}{2}) \right) = \sin \left( \frac{\pi}{\mathfrak{N}} (\mathfrak{N} - n - 1 + \frac{1}{2}) \right)$, there is a two-fold degeneracy at the eigenvalues.

An interesting question is the determination of the eigenvalues of the pairs operators $\{\hat{x}_2, \hat{p}_2\}$ and the corresponding eigenvalues. We shall perform this for the scalar field operator $\hat{x}_2$ (for corresponding momenta the analysis is similar). Let $|x_2\rangle$ and $x_2$ be the eigenvector and the corresponding eigenvalue of the operator $\hat{x}_2$, satisfying

$$\hat{x}_2 |x_2\rangle = x_2 |x_2\rangle.$$  

As we mentioned before, the Fock space is a $\mathfrak{N}$-dimensional $\mathbb{C}(q)$-vector space. Therefore, the expression of $|x_2\rangle$ in the $q^\mathfrak{N}$ representation will be

$$|x_2\rangle = \sum_{n=0}^{\mathfrak{N}-1} c_n |x_2\rangle |n\rangle.$$  

On the other hand, Eqs. (30) and (41) give

$$|x_2\rangle = \frac{1}{\sqrt{2\mathfrak{N} \omega}} \left( \sqrt{|n|} |n-1\rangle + \sqrt{|n+1|} |n+1\rangle \right).$$
By Eqs. (45)-(47) one obtains the following recurrence relations for the coefficients $c_n$

\[
\begin{align*}
c_{0}(x_2) &= H_0(\sqrt{M_w}x_2) = 2\sqrt{M_w}x_2c_0(x_2), \\
c_{n+1}(x_2) &= 2\sqrt{M_w}x_2c_n(x_2) - 2[n]c_{n-1}(x_2), \\
c_{0}(x_2) &= 0.
\end{align*}
\]

These relations have the solution $c_n(x_2) = H_n(\sqrt{M_w}x_2; q^{\frac{2}{2}})$, \[51\]

where $H_n(y; q^{\frac{2}{2}})$ are the $q$-Hermite polynomials satisfy the equation

\[
H_{n+1}(y; q^{\frac{2}{2}}) + 2[n]H_{n-1}(y; q^{\frac{2}{2}}) - 2yH_n(y; q^{\frac{2}{2}}) = 0,
\]

(50)

The last condition in (48) is equivalent to

\[
H_{\pi}(\sqrt{M_w}x_2; q^{\frac{2}{2}}) = 0,
\]

(51)

where $x_{2,\mu}$ are the roots of $q$-Hermite polynomial. Hence, the eigenvalues of operator $\hat{x}_2$ in a $\pi$-dimensional Fock space are the roots of the corresponding $q$-Hermite polynomial. Also, the $q$-Hermite polynomials defined by the recurrence relation of Eq. (50) are odd (or even) functions of $x_2$ for $n$ odd (or even), therefore the half of eigenvalues of $x$ are negative. The number of these eigenvalues is equal to the order $\pi$ of the corresponding polynomial and these roots are real. \[52\] The discrete values of $q$-Hermite polynomial are indexed by the following convention

\[
\mu = -l, -l + 1, ..., l - 1, l,
\]

(52)

where $\pi = 2l + 1$ for odd values of $\pi$ and $\pi = 2l$ for even values of $\pi$.

The $q$-deformed Hermite polynomials for the first few values of $n$ are listed below:

\[
\begin{align*}
H_1(y; q^{\frac{2}{2}}) &= 2y, \\
H_2(y; q^{\frac{2}{2}}) &= 4y^2 - 2, \\
H_3(y; q^{\frac{2}{2}}) &= 8y^3 - 4[2][4]y, \\
H_4(y; q^{\frac{2}{2}}) &= 16y^4 - 8[2][4][6]y^2 + 4[3], \\
H_5(y; q^{\frac{2}{2}}) &= 32y^5 - 16[2][4][6]y^3 + 8([3] + [2][4][4])y.
\end{align*}
\]

(53)

In simplifying the above relations we used $[1] + [2] + ... + [n] = [2][\frac{n+1}{2}][\frac{n}{2}]$. With similar analysis, one can see that $q$ being a root of unity induces a discretization of the spectrum of the momenta

\[
H_{\pi}(\frac{1}{\sqrt{M_w}}\Pi_{2,\mu}; q^{\frac{2}{2}}) = 0,
\]

(54)

which shows that the spectra of the operators $\sqrt{M_w}\hat{x}_2$ and $\frac{1}{\sqrt{M_w}}\Pi_{2,\mu}$ are identical and the same $q$-Hermite polynomials appearing in both cases.

As a result of the discretization of the “position” and “momenta” eigenvalues found above, the phase space of scalar field, $(\hat{x}_2, \Pi_{2,\mu})$, is not the whole real plane, but it is a two-dimensional lattice with non-uniformly distributed points. For very large values of $\pi$ the $q$-numbers are reduced to the ordinary reals, so $|\pi| \simeq \pi$ and consequently the $q$-Hermite and defined in Eqs. [59] will be reduced to the ordinary Hermite polynomials. Also, the largest root of Hermite polynomial $H_{\pi}(y)$ is $y_{\pi, \pi} \simeq \sqrt{2\pi}$. Therefore, according to Eq. (51) for the large values of $\pi$ the largest value of scalar field is $x_{2,\mu} \simeq LP\sqrt{\pi}$.

### 4.2. Quantum deformation of the gravitational sector

Let us now investigate the quantum deformation of the gravitational part of super-Hamiltonian, $H_1$ defined in Eq. (13). We should write the Heisenberg-Weyl Lie algebra for gravitational part with generators \{ $A_-, A_+, \pi$ \}

\[
[A_-, A_+] = 1, \ [\pi, A_\pm] = \pm A_\pm
\]

(55)

In the Fock space, $\mathcal{F}_1$, with the basis \{|$n$\}, $\pi$|n\} the pairs of operators $A_\pm$ act due

\[
A_+(n) = \sqrt{n + 1}|n + 1\rangle,
\]

\[
A_-(n) = \sqrt{n}|n - 1\rangle.
\]

(56)

But, as we showed is section II, the gravitational part of super-Hamiltonian $H_1$ has the self-adjoint extension if the wave function obeys the standard Dirichlet or Neumann boundary conditions.

The states of the Fock space are thus classified into two disjoint odd and even subspaces. Therefore, it seems that the $h_4$ does not represent the suitable symmetry of the gravitational part. To split the Hilbert space into odd and even subspaces and obtain the true symmetry of gravitational part, let we introduce the generators

\[
K_0 = \frac{1}{2}(\pi + \frac{1}{2}), \quad K_\pm = \frac{1}{2}(A_\pm)^2.
\]

(57)

It is not difficult to verify that these generators satisfy the commutation relations

\[
[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0,
\]

(58)

of the algebra $su(1, 1)$. The $*$-involution on the elements of algebra is defined by $K^*_0 = K_0$ and $K^*_\pm = -K_-$. The positive discrete series representations of this Lie algebra, $D^+_\lambda$, are labeled by a positive real number $\lambda > 0$. Suppose that \{|$k, m\rangle, m = 0, 1, 2, ... \} is the basis in the Hilbert space $\mathcal{H}$ of representation $D^+_\lambda$. Then, the actions of the above generators on a set of basis eigenvectors $|k, m\rangle$ are given by

\[
K_0|k, m\rangle = (k + m)|k, m\rangle,
\]

\[
K_+|k, m\rangle = \sqrt{(2k + m)(m + 1)}|k, m + 1\rangle,
\]

\[
K_-|k, m\rangle = \sqrt{(2k + m - 1)(m - 1)}|k, m - 1\rangle.
\]

(59)

The Casimir operator, $C_2$, is a central self-adjoint element of the universal enveloping algebra $U(su(1, 1))$ and

\[
C_2 := K_0(K_0 + 1) - K_+K_-.
\]

(60)
In addition, the gravitational part of super-Hamiltonian \([H]_{G}\) can be presented as

\[ H_{1} = 2\omega K_{0}, \quad (61) \]

which leads us to point out that the Casimir operator commutes with \(H_{1}\). A direct calculation of the Casimir operator \([69]\) with the aid of \([67]\) shows that the eigenvalue \(k(k - 1)\) in this particular case is equal to \(-\frac{1}{4}\).

This means that \(k\) is equal to either \(k = \frac{1}{2}\) or \(k = \frac{3}{2}\), where each of these two values of \(k\) defines a unitary irreducible representation of the algebra \(su(1,1): D_{1}^{L}\) consists of those eigenstates of \(H_{1}\), which correspond to the eigenvalues \(k_{1} + n = n + \frac{1}{2} = \frac{1}{2}(2n + 1 + \frac{1}{2})\), \(n = 0, 1, 2, \ldots\), of the generator \(K_{0}\); whereas \(D_{1}^{+}\) corresponds to the eigenvalues \(k_{2} + n = n + \frac{1}{2} = \frac{1}{2}(2n + 1 + \frac{1}{2})\).

As we make progress, let us summarize the \(q\)-deformation of the universal enveloping algebra of \(su(1,1)\), \(U_{q}(su(1,1))\), when the deformation parameter \(q\) is generic. The quantized enveloping algebra \(U_{q}(su(1,1))\) is an associative unital algebra generated by the four generators \(\{K_{+}, K_{-}, q^{K_{0}}, q^{-K_{0}}\}\) with the commutation relations

\[ [K_{+}, K_{-}] = \frac{2^{q}q^{-q} - 2^{-q}q^{q}}{q - q^{-1}}, \quad [K_{0}, K_{\pm}] = \pm K_{\pm}. \]  

These relations reduce to those of \(su(1,1)\) defined in \([68]\) in the classical limit \(q \to 1\). \(U_{q}(su(1,1))\) is equipped with a structure of Hopf algebra: \(\Delta(K_{0}) = K_{0} \otimes K_{0}, \quad \Delta(K_{\pm}) = K_{\pm} \otimes K_{0} + K_{0} \otimes K_{\pm}, \quad \epsilon(K_{\pm}) = 1, \epsilon(K_{0}) = 0, \quad S(K_{0}) = K_{0}^{-1} \quad \text{and} \quad S(K_{\pm}) = -q^{1/2}K_{\pm}\). Also the Casimir operator is

\[ C_{2} = [K_{0} + \frac{1}{2}]_{q}^{2} - 2q^{1/2}K_{0} - K_{0} = [K_{0} + \frac{1}{2}]_{q}^{2} + K_{+}L_{-}. \]

where

\[ [y]_{q} = \frac{q^{y} - q^{-y}}{q - q^{-1}}. \]

Suppose that \(\{k, m\}, m = 0, 1, 2, \ldots\} is the basis in the Hilbert space \(F_{k}(k)\) of representation \(D_{k}^{-}\). The action of the generators then has the form \([56]\)

\[ K_{0}[k, m] = (m + k)[k, m], \quad K_{+}[k, m] = \sqrt{m + 1}q^{m}[m + 2k][k, m + 1], \quad K_{-}[k, m] = \sqrt{m}[m + 2k - 1]q[k, m - 1]. \]  

Also

\[ C_{2}[k, m] = k - \frac{1}{2}[k]_{q}^{2}[k, m], \quad [K_{0}]_{q}[k, m] = [k + m]_{q}[k, m]. \]

The elements of this basis are obtained from the highest vector \([k, 0]\) by a second application of the operator \(K_{+}\),

\[ [k, m] = \frac{1}{\sqrt{[m]_{q}![2k]_{q}m}}K_{m}^{n}[k, 0], \]  

where

\[ ([y]_{q})_{m} = [y]_{q}[y + 1]_{q}[y + 2]_{q}\ldots[y + m - 1]_{q}, \]

is the Pochhammer \(q\)-symbol. The generators of \(U_{q}(su(1,1))\) can be realized \([56]\) with the aid of the generators of the algebra \(U_{q}(su(1,1))\)

\[ K_{0} = \frac{1}{2}(\hat{N} + \frac{1}{2}), \quad K_{\pm} = \frac{1}{[2]}(\hat{A}_{\pm})^{2}. \]

In this case the Fock space representation of the \(q\)-oscillator splits into the direct sum of two irreducible components \(F_{1} = F_{+} \oplus F_{-}\), whereas \(F_{n}(F)_{0}\) is formed by states with an even (odd) number of quanta. Using \([69]\) and \([63]\) we obtain

\[ C_{2}[n] = \frac{1}{2}n[n]_{q} = [-\frac{1}{4}]_{q}[n], \]  

where \([n]_{q} \in F_{1}\). It then follows from \([63]\) that \(k = \frac{1}{2}\) or \(k = \frac{3}{2}\). Taking \([65]\) into account, we see that the representation with \(k = \frac{1}{2}\) and \([2m]\) acts in \(F_{0}\) and in \(F_{+}\) we have \(k = \frac{1}{2}\) and \([2m + 1]\) like as the classical \(U_{q}(su(1,1))\).

Let us now return to the case in which \(q\) is a primitive root of unity as defined in \([69]\). The new quantum number defined in Eq. \((61)\) will be

\[ [y]_{q} = \frac{q^{y} - q^{-y}}{q - q^{-1}} = \frac{\sin (\frac{\pi q}{4})}{\sin (\frac{\pi}{4})}. \]

If we define, \(l\) as: \(\Re = 2l + 1\) for odd values of \(\Re\) and \(\Re = 2l\) for even values of \(\Re\), then for odd values of \(\Re\) and \(m = l - 1\) \((k = \frac{1}{2})\) we obtain \([m + 2k]_{q} = \frac{[m]_{q}}{2}\) and \([2m]_{q} = \frac{[m]_{q}}{2}\). Also for even values of \(\Re\) and \(m = l \quad (k = \frac{1}{2})\) we have \([m]_{q} = \frac{[m]_{q}}{4}\). Therefore, elements \(\{K_{+}^{l}, K_{-}^{l}, q^{\pm iK_{0}}\}\) lie in the center of \(U_{q}(su(1,1))\). These lead us to

\[ K_{+}[k, l - 1] = 0. \]

Hence, the Fock space split into two \(l\)-dimensional spaces with bases \(\{[\frac{1}{4}, m], m = 0, \ldots, l - 1\}\) and \(\{[\frac{3}{4}, m], m = 1, \ldots, l\}\).

The domain of definition of scale factor, \(x_{1} = a_{1}\), is \(\Re^{+}\), so the conjugate momenta \(\Pi_{1}\) is not Hermitian, but one can easily show that \(\Pi_{1}^{*}\) is Hermitian. Let us then obtain the eigenvalues and eigenvalues of \(\hat{x}_{1}^{2}\) and \(\Pi_{1}^{2}\). Let \(|x_{1}\rangle = (\Pi_{1}\rangle)\) and \(x_{1}^{2}\) be the eigenvector and eigenvalue of the operator \(\hat{x}_{1}^{2}\) \((\Pi_{1}^{2})\), satisfying

\[ \hat{x}_{1}^{2}|x_{1}\rangle = x_{1}^{2}|x_{1}\rangle, \quad \Pi_{1}^{2}|\Pi_{1}\rangle = \Pi_{1}^{2}|\Pi_{1}\rangle. \]

Likewise to the scalar field part, the expression of \(|x_{1}\rangle\) \((\Pi_{1}\rangle)\) in the \(K_{0}\) representation is given by

\[ |x_{1}\rangle = \sum_{n} e_{n}(\frac{-1}{n})^{n} \sqrt{n!} \sqrt{\frac{2m+n}{2k+n}} \quad |k, n\rangle, \]

\[ |\Pi_{1}\rangle = \sum_{n} e_{n}(\frac{-1}{n})^{n} \sqrt{n!} \sqrt{\frac{2m+n}{2k+n}} \quad |k, n\rangle, \]

where the \(q\)-factorial defined by \([n]_{q}! = \prod_{m=1}^{n}[m]_{q}q\). Using the phase space realizations

\[ \hat{x}_{1} = \sqrt{\frac{1}{\sqrt{2m+n}}}(\hat{A}_{+} + \hat{A}_{-}), \quad \hat{\Pi}_{1} = i\sqrt{\frac{1}{\sqrt{2m+n}}}(\hat{A}_{+} - \hat{A}_{-}), \]

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and \[\tilde{H}_2 = \frac{1}{x^2}(K_+ + K_-) - \frac{1}{x^2} |\tilde{K}_0\rangle \langle \tilde{K}_0|, \]
\[2M\p^2 \tilde{L}_1^2 = \frac{1}{x^2}(K_+ + K_-) + \frac{1}{x^2} |\tilde{K}_0\rangle \langle \tilde{K}_0| = \text{Eqs. (65), (70), (74) and (76) give recursion relations}
\]
\[\frac{1}{2i\tilde{x}_2} [n + 1]_q c_{n+1} = \left( \frac{\tilde{x}_2^2}{x^2} |\tilde{K}_0\rangle \langle \tilde{K}_0| - M\p^2 \right) c_n - \frac{1}{2i\tilde{x}_2} [n + 2k - 1]_q c_{n-1}, \]
\[c_0 = 1,
\]
\[c_1 = 0, \]

which is the recursion relation for the q-deformed generalized Laguerre polynomials
\[c_n = L_n^{(2k-1)}(M\p^2 x^2; q), \]
\[L_n^{(2k-1)}(M\p^2 x^2; q) = 0, \]
\[\text{where } x^2_2 \text{ and } \Pi^2_2 \text{ are the positive roots of } q\text{-Laguerre polynomials}. \]
\[\text{One can show that the } q\text{-deformed Hermite polynomials defined in recursion relation (50) satisfy following relations}
\]
\[H_{2n+2}(y; q^2) = 4 \left( y^2 - \frac{1}{x^2} [n + \frac{1}{2}]_q \right) H_{2n}(y; q^2) - \frac{4}{x^2} |\tilde{K}_0\rangle \langle \tilde{K}_0| H_{2n-2}(y; q^2), \]
\[H_{2n+3}(y; q^2) = 4 \left( y^2 - \frac{1}{x^2} [n + \frac{1}{2}]_q \right) H_{2n+1}(y; q^2) - \frac{4}{x^2} |\tilde{K}_0\rangle \langle \tilde{K}_0| H_{2n-1}(y; q^2). \]

Using these recursion relations, it is easy to show that the generalized q-Laguerre polynomials are related to the q-Hermite polynomials
\[H_{2n}(y; q^2) = \left( \frac{2}{x^2} \right)^n H_{2n}(y; q^2), \]
\[H_{2n+1}(y; q^2) = \left( \frac{2}{x^2} \right)^n H_{2n+1}(y; q^2), \]

where \( y \in \{\sqrt{M\p^2 x_1, \mu}, \frac{\Pi_1}{\sqrt{M\p^2}}\} \). The second equation shows that for odd values of \( \Omega \) the zero eigenvalue is removed from the set of eigenvalues of scale factor and conjugate momenta.

In order to get the discrete eigenvalue spectrum of the four operator \( \{x^2_1, x^2_2, \Pi^2_1, \Pi^2_2\} \), let us consider the special case \( \Omega = 5 \) as an example. Then the roots of \( q\)-Hermite polynomials defined in Eqs. (51) and (54) give us the following approximate eigenvalues for \( x_2 \) and corresponding conjugate momenta
\[\tilde{x}_2^2 \approx \{ -0.12, -0.07, 0, 0.07, 0.12 \}, \]
\[L_\p \Pi_2 \approx \{ -13.94, -8.33, 0, 8.33, 13.94 \}. \]

Also, the roots of \( q\)-Laguerre polynomials (79), or equivalently the second equation in (82), give us the eigenvalues of scale factor and the square of the conjugate momenta
\[\tilde{x}_2^2 \approx \{ 0.07, 0.12, \} \]
\[L_\p \Pi_2 \approx \{ 69.37, 194.41 \}. \]

where \( L_\p = \sqrt{8\pi G} \) is the reduced Planck length.

5. The IR/UV mixing and “new” cosmological constant problem

Let us briefly review the “old” and the “new” CC problems. Regarding locality and unitarity of quantum field theory, the vacuum has an energy. To obtain the corresponding energy we need to calculate the vacuum loop diagrams for each matter field species. For example the one loop diagram result for a scalar field of mass \( m \) is given by
\[V_{\text{vac}}^{1\text{-loop}} \approx - \frac{m^4}{(8\pi)^2} \left( \frac{2}{\kappa} + \ln \left( \frac{M_\p^2}{m^2} \right) + \text{finite} \right), \]

where we work in the UV cut-off at \( M_\p^2 \text{UV} \) is the UV regulator scale. If we add a counter term
\[V_c^{1\text{-loop}} \approx - \frac{m^4}{(8\pi)^2} \left( \frac{2}{\kappa} + \ln \left( \frac{M_\text{UV}}{M^2} \right) + \text{finite} \right), \]

where \( M \) is the renormalisation scale to eliminate the divergences, then the renormalised vacuum energy at 1-loop level will be
\[\Lambda_\text{ren}^{1\text{-loop}} \approx \frac{m^4}{(8\pi)^2} \left( \frac{m^2}{M^2} + \text{finite} \right). \]

It is clear that the finite contributions to vacuum energy are completely arbitrary since they can always be absorbed into a re-definition of the subtraction scale \( M \). This emphasises that QFT does not have a concrete prediction for the renormalised vacuum energy. In comparison to \( V_{\text{vac}}^{1\text{-loop}} \), the counter term \( V_c^{1\text{-loop}} \) has a divergent and finite part where this finite part can be consider as the “bare” classical CC that we were free to add to the Einstein-Hilbert action. Since QFT cannot theoretically predict the magnitude of the CC, we have to measure it and adjust the finite part of \( \Lambda_c \) appropriately such that the theory
matches with cosmological observations. For example, if we assume that our scalar field is the Higgs boson of the standard model then it has a mass $m = 126\text{GeV}$. Given that observations place an upper bound on the total CC of (meV)$^4 \simeq 10^{-60}(\text{TeV})^4$, the finite contributions to the vacuum energy at the 1-loop level must cancel to an accuracy of 1 part in $10^{60}$. Let us now consider the 2-loop correction to the vacuum energy from the massive scalar field. At two loops, we consider the so-called scalar “figure of eight” with external graviton legs. Its contribution to vacuum energy is given by

$$V_{\text{vac}}^{2\text{-loop}} \simeq \lambda n^4.$$  \hspace{1cm} (88)

For perturbative theories without finely tuned couplings, where $\lambda \sim O(0.1)$, (as for example the Standard Model Higgs) this is a huge contribution to the cosmological constant relative to the observed value. Now having already fixed the bare CC to match observations at the 1-loop level, we need to re-tune its value to a high order of accuracy to cancel the unwanted contributions at the 2-loop level. Similarly we have to re-tune its value as we go to 3-loops and so on. In other words, it is radiatively unstable and we need to re-tune the bare CC at every order in loop perturbation theory to deal with this instability.

Because this radiatively instability of vacuum energy was already a problem before the late time acceleration of Universe was discovered, this is sometimes called the “old” CC problem [63]. In fact, it is distinguished from a “new” problem that is more to do with understanding the precise value and the origin of the CC that has been observed. The “new” CC problem [64] has its origin in the discovery that the vacuum energy is not exactly zero and it causes observed late time acceleration of Universe. In this direction, it is usually presupposed that there is a solution to the “old” CC problem that makes the vacuum energy precisely zero and radiatively stable [63]. Then it would be remain to explain why the measured value of the CC is not precisely zero, and instead has a nonzero but very small value. The “new” version of the CC problem divides itself into two parts: i) why it is so small? ii) How come that the energy density of CC and baryonic matter have the same order of magnitude at the present epoch? This is called the coincidence problem. Let us now describe how the “new” CC problem can be discussed from quantum deformation of quantum cosmology perspective [65].

For very large values of $\Re$ the $g$-numbers are reduced to the ordinary reals, so $[\Re] \simeq [\Re]_g \simeq \Re$ and consequently the $q$-Hermite and $q$-Laguerre polynomials defined in Eqs. (50) and (72) will be reduced to the ordinary Hermite and Laguerre polynomials. Let $\chi_{n,k}, k = 1, 2, \ldots$, denote the zeros of the Laguerre polynomial $L_n^{(\alpha)}(y)$, in increasing order, for large values of $n$. It is well known that these zeros lie in the oscillatory region $0 < \chi_{n,k} < 4n + 2\alpha + 2$. Then the smallest root, $\chi_{n,1}$, and the largest, $\chi_{n,n}$, are given by [64]

$$\chi_{n,1} \simeq \frac{(\alpha + 1)(\alpha + 3)}{2n + \alpha + 1}, \quad \chi_{n,n} \simeq 4n.$$  \hspace{1cm} (89)

Using the above results in Eqs. (89) it is easy to show the minimum and maximum eigenvalues of the square of the scale factor and the corresponding momenta

$$a^2_{\text{min}} \simeq \frac{L^2_{\text{min}}}{\Re}, \quad a^2_{\text{max}} \simeq \Re L^2_{\text{max}},$$

The above relations thus suggest to introduce the smallest and largest distances respectively as

$$L_{\text{min}} = \frac{L_{\text{P}}}{\sqrt{\Re}}, \quad L_{\text{max}} = \sqrt{\Re}L_{\text{P}}.$$  \hspace{1cm} (91)

These further suggest that the deformation parameter [65] may be rewritten as

$$q = \exp i \left( \frac{L_{\text{P}}}{L_{\text{max}}} \right)^2.$$  \hspace{1cm} (92)

This yields a simple geometrical interpretation for the relation between the quantum deformation parameter and the maximal possible value of the scale factor. In GR, a maximal distance of the order $L_{\text{max}} \simeq \Lambda^{-\frac{1}{2}}$ is also essentially implied when a cosmological constant $\Lambda$ is present. Hence, it seems that the quantum deformation of cosmological model induce a cosmological constant and Eq. (92) should be

$$q = \exp(i\Lambda L^2_{\text{P}}).$$  \hspace{1cm} (93)

We can take the limit $q \rightarrow 1$ by taking $\Lambda \rightarrow 0$ which leads to the original quantum cosmology without cosmological constant [4]. Let us stress that in our original cosmological model the CC does not exist but the quantum deformation of the model introduced a cosmological constant, where it is related to the natural number $\Re \in \mathbb{N}^+$ by

$$\Lambda \simeq \frac{1}{L^2_{\text{max}}} \simeq \frac{1}{\Re L^2_{\text{P}}},$$  \hspace{1cm} (94)

In such a line of reasoning a CC should be understood as a direct consequence of the finite number of states in the Hilbert space which itself is a result of $q$-deformation. The minimum value of scalar curvature is $R_{\text{min}} \simeq \frac{1}{\Re L^2_{\text{P}}} \simeq \Lambda$. It immediately leads to a suggestion for the explanation for the late time acceleration of the Universe: the Universe reaches the minimally possible curvature and has to stay in this state.

Let us now consider an observer located at 3D sphere of curvature radius $a_{3\text{D}} = L_{\text{max}}$. If one measures the apparent size of a small sphere of diameter $r$ located at large distance $L$, ($r \ll L$), an observer will see it under an angular size $\delta \theta \simeq \frac{r}{L}$. Consider this sphere located at horizon (largest distance), she(he) will never see [65].

\textsuperscript{4}Loop quantum gravity (LQG) and spinfoams use the cosmological constant as a coupling constant just like the gravitational constant. A $q$-deformation has been derived in LQG as a way to implement the dynamics of the theory with cosmological constant [39, 61] and the deformation parameter, $q$, then is given by [60].
the sphere under an angle smaller than $\approx 1 = \frac{1}{L_{\text{max}}} \approx r \sqrt{\Lambda}$. Therefore, it is natural to assume that it is impossible to measure areas having angular size smaller than 

$$\delta \phi_{\text{min}} \approx \frac{r_{\text{min}}}{L_{\text{max}}} \approx \frac{L_{\text{min}}}{L_{\text{max}}} = \frac{1}{\mathcal{R}}.$$  

(95)

In such a situation, at the presence of minimal length $L_{\text{min}}$ everything that the observer sees is captured, on the local celestial 2-sphere formed by the directions around him, by spherical harmonics with $j = j_{\text{max}}$. In quantum groups language, a 2-sphere not resolved at small angles is a Po-

dleś “quantum sphere”, $S^2_q$. If we denote the generators of algebra $A_q$ for $S^2_q$ by $\{\hat{X}_+, \hat{X}_-, \hat{X}_3\}$ then they satisfy the following commutation relations 

$$\hat{X}_+ \hat{X}_- - \hat{X}_- \hat{X}_+ + \lambda \hat{X}_3^2 = \mu \hat{X}_3,$$

$$q^{\hat{X}_x \hat{X}_+ - q^{-1} \hat{X}_- \hat{X}_3} = \mu \hat{X}_x,$$

$$q^{(\hat{X}_- \hat{X}_{-3} - q^{-1} \hat{X}_x \hat{X}_3)} = \mu \hat{X}_x,$$

$$\hat{X}_3^2 + q^{(\hat{X}_- \hat{X}_+ + q^{-1} \hat{X}_x \hat{X}_3)} = L_{\text{max}}^2,$$

(96)

where

$$\lambda = q - q^{-1}, \quad \mu = L_{\text{max}} \frac{2(\mathcal{R} + 1)}{[\mathcal{R} + 1] Q \sqrt{[\mathcal{R} + 2]} q}. $$

(97)

The above relations define quantum sphere $S^2_q$ when $q$ is root of unity.

An interesting limit of the above quantum sphere appears when $\mathcal{R}$ is a very big natural number. In this case Eqs. (95) will be reduce to

$$[\hat{X}_i, \hat{X}_j] = i \lambda_{\mathcal{R}} \epsilon_{ijk} \hat{X}_k, \quad \hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = L_{\text{max}}^2,$$

(98)

where, $i, j, k = 1, 2, 3$,

$$\lambda_{\mathcal{R}} = \frac{2L_{\text{max}}}{\sqrt{\mathcal{R}(\mathcal{R} + 2)}} \simeq 2 \frac{\mathcal{R}}{\mathcal{R}_{\text{max}}},$$

(99)

and

$$\hat{X}_1 = - \frac{1}{\sqrt{2}} (\hat{X}_+ + \hat{X}_-), \quad \hat{X}_2 = - \frac{i}{\sqrt{2}} (\hat{X}_+ - \hat{X}_-).$$

(100)

This algebra represents the fuzzy sphere $S^2_F$ which can appear as vacuum solutions in Eulidean gravity. The parameter $\lambda_{\mathcal{R}}$ has dimension of length, and plays a role analogue to the Planck’s constant in quantum mechanics, as a quantization parameter. From (98) and definition of $\lambda_{\mathcal{R}}$ and $L_{\text{min}}$, we can see that in the limit $\lambda_{\mathcal{R}} \to 0$ ($\mathcal{R} \to \infty$) the matrices $\hat{X}_i$ become commutative, and we recover the commutative sphere $S^2$ with radius $L_{\text{max}} \to \infty$. In other words, at the limit $q \to 1$, the emerged cosmological constant and corresponding noncommutative horizon will be disappear. Eqs. (99) or alternatively Eqs. (98) bear an interesting statement of holography. In order to see the holographic screen, that one can probe on $S^2_F$ is given by [71]

$$\text{Area}(\text{min}) = L_{\text{min}}L_{\text{max}} = L_{\text{F}}^2.$$  

(101)

Furthermore, the surface area of $S^2_F$ is given by

$$\text{Area}(S^2_F) = 4\pi L_{\text{F}}^2 \frac{\mathcal{R} + 1}{\sqrt{\mathcal{R}(\mathcal{R} + 1)}} \simeq 4\pi L_{\text{max}}^2.$$  

(102)

Therefore, the number of smallest cells one can fit into the fuzzy sphere of surface $L_{\text{max}}^2 \simeq \frac{1}{\mathcal{R}}$ is given by

$$\frac{\text{Area}(S^2_F)}{\text{Area}(\text{min})} = \mathcal{R}. $$

(103)

This shows that the number of fundamental cell of the surface of fuzzy horizon is equals to the dimension of Hilbert space of the scalar field. The holographic principle asserts that the total number of degrees of freedom, or entropy $S_{\text{DS}}$, living on the holographic screen is bounded by one quarter of the area in Planck units

$$S_{\text{DS}} \simeq \frac{1}{2 \mathcal{R} \Lambda_{\text{F}}^2}.$$  

(104)

Using Eqs. (101), (102) and (104) one can readily check that

$$S_{\text{DS}} \simeq (\text{the number of cells on the } S^2_F) \simeq \mathcal{R},$$  

(105)

which is basically the statement of holography. In this sense the cosmological constant and holographic principle are emerged as the result of the quantum deformation.

Moreover, noting that the minimum area defined in (104) involves both the UV character $L_{\text{min}}$ and the IR character $L_{\text{max}}$ one expects the IR/UV mixing phenomena. It is generally assumed that particle physics can be accurately described by an EFT with an ultraviolet UV cutoff, $M_{\text{UV}}$, less than the Planck mass, provided that all momenta and field strengths are small compared with this cutoff to the appropriate power. Consequently the length $L$, which acts as an IR cutoff, cannot be chosen independently of the UV cutoff, and scales relations obtained in Eq. (101). If $\rho_{\text{F}} \simeq M_{\text{UV}}^4$ is the quantum zero point energy density caused by a UV cutoff, the total energy of vacuum in a region of size $L$ should not exceed the maximum energy scale $\frac{1}{L_{\text{min}}}$, thus

$$M_{\text{UV}}^4 L^3 \leq \frac{1}{L_{\text{min}}}, $$

(106)

The largest $L = L_{\text{max}}$ allowed is the one saturating this inequality. Thus

$$M_{\text{UV}}^4 = \frac{1}{L_{\text{min}}^3 L_{\text{max}}^3} = \mathcal{R}^{-1} \Lambda_{\text{F}}^3.$$  

(107)

To estimate the numerical value of the emerged CC obtained in (104) we need to know the value of the inverse of deformation parameter, $\mathcal{R}$, which is equal to the entropy of degrees of freedom living on the emerged holographic screen. It is known [73] that the total entropy of dust, $S_{(\text{dust})} \simeq 10^{80}$ and radiation $S_{(\text{radiation})} \simeq 10^{80}$ in observable Universe [74] are related to the entropy of holographic screen, $S_{\text{DS}}$ via $S_{\text{DS}} \simeq S_{(\text{dust})}^4 \simeq S_{(\text{radiation})}^4$. Hence, Eq. (105) leads us to

$$\mathcal{R} = S_{\text{DS}} \simeq S_{(\text{dust})}^4 \simeq S_{(\text{radiation})}^4 \simeq 10^{120}. $$

(108)
We can also obtain the value of the emerged CC if we find the value of $L_{\text{max}}$ which is related to the CC by Eq. (103). Since observations show that our Universe is presently entering dark energy domination, the growth of the event horizon has slowed, and it is almost as large now as it will ever become. Therefore, we can estimate the value of $L_{\text{max}}$ as the present value of cosmic event horizon. The present radius of the cosmic event horizon is $L_{\text{CEH}} \simeq 15.7 \pm 0.4$ Gyr $\simeq 10^{43} L_P$. [72.] These values together with the Eq. (104) lead us to
\[
\Lambda \simeq \frac{1}{L_{\text{max}}^2} = \frac{1}{9 \Pi L_P^2} \simeq 10^{-122} M_P^2,
\]
(109)
which is consistent with the observed value of the CC.

At the end of this section, let us concentrate on the coincidence problem. Cosmological observations suggest that we live in an remarkable period in the history of the Universe when $\rho_L \simeq \rho_m$, where $\rho_L$ and $\rho_m$ are the energy density of the CC and the matter respectively. Within the standard model of cosmology, this equality of energy densities just at the present epoch can be seen as coincidental since it requires very special initial conditions in the very early Universe. The corresponding “why now” question constitutes the cosmological “coincidence problem”. If $M_U$ denotes the total mass of the pressureless matter (dust) content of the Universe, then $M_U = m_b N_b$, where $m_b$ and $N_b$ are the mass and the total number of particles of matter content of Universe. As we know the total number of particles is approximately equal to the entropy of the matter, $N_b \simeq S_{\text{(dust)}}$. [74.] Also, if we use the well-known relation between the radius of the Universe (herein the present value of cosmic event horizon, $L_{\text{CEH}} \simeq L_{\text{max}}$) and mass of nucleons, $m_b$, as a result of the uncertainty principle, $m_b L_{\text{max}} \sim \sqrt{\hbar}$, we obtain
\[
m_b \simeq \frac{\sqrt{N_b}}{L_{\text{max}}} \simeq \frac{S_{\text{(dust)}}^{1/3}}{\Pi^{1/3} L_P} \simeq \frac{\hbar^{1/3}}{\Pi^{1/3} L_P} = \hbar^{1/3} M_P.
\]
(110)
As a consequence, the total mass of Universe can be rewritten as
\[
M_U = N_b m_b \simeq S_{\text{(dust)}}^{1/3} M_P \hbar^{1/3} = \hbar^{1/3} \Pi^{1/3} M_P.
\]
(111)

Now we can summarize Eqs. (110), (111), (110), and (111) as the following scaling relations
\[
M_U \simeq M_P \hbar^{1/3}, \quad m_b \simeq M_P \Pi^{1/6},
\]
\[
M_{UV} \simeq M_P \hbar^{-1/6},
\]
\[
L_{\text{max}} \simeq L_P \hbar^{1/2},
\]
\[
L_{\text{min}} \simeq L_P \hbar^{-1/2},
\]
\[
\Lambda \simeq L_P^{-2} \hbar^{-1},
\]
(112)
which are in fact the extension of the Dirac large numbers hypothesis (LNH) explained in [76]. Note that all of these scaling relations are established at the present time, because the observations show the entrance of Universe in the acceleration phase just at the present epoch. Therefore, the LNH of Dirac can actually be explained in terms of the quantum deformation of quantum Universe. Eliminating $\Pi$ from the second and the fourth scaling relations gives us
\[
m_b \simeq \left( \frac{1}{G L_{\text{max}}} \right)^{1/3},
\]
(113)
which is the empirical Weinberg formula for the mass of the nucleon [77]. Also, by eliminating $\Pi$ from the second (or third one) and the last scaling relations in (112) we obtain
\[
\rho_\Lambda = \frac{\hbar}{G L_{\text{max}}} \simeq M_P^3 \hbar^{-1/2} \approx G m_b^6,
\]
\[
M_{UV} \simeq M_B^3 \hbar^{-1} \approx G m_b^6.
\]
(114)
These equations are identical to the scaling law proposed by Zeldovich [78] for the value CC. Let us now obtain the energy density of dust at the present epoch. The linear size of Universe at the present time is approximately equal to the $L_{\text{CEH}} \simeq L_{\text{max}}$. Hence, by inserting the first, the second and the third scaling relations obtained in (112) into the definition of the energy density of dust, we find
\[
\rho_m \simeq \frac{M_U}{L_{\text{max}}^{3/2}} \simeq m_b M_P^3 \hbar^{-1/2} \approx G m_b^6.
\]
(115)
It is clear that the last equality is established just at the presence epoch of cosmic evolution. Therefore, Eqs. (114) and (115) show that the present values of the densities of dark energy and matter are of the same order of magnitude, $\rho_\Lambda / \rho_m \simeq O(1)$.

The last interesting equation which can be derived from the first scaling equation of (112) is
\[
\hbar \simeq 4 \pi G M_P^2.
\]
(116)
The right hand side of this relation is the entropy of a black hole with size of Universe. On the other hand, the left hand side, as we showed in Eq. (105), represents the entropy of the holographic screen, $S_{\text{AS}}$. In other words, the Universe can have no more states than that of a black hole of the same size.

6. Conclusion

In this paper, we have investigated the quantum deformation of a spatially closed Friedmann-Lemaître-Robertson-Walker universe in the presence of a conformally coupled scalar field. The gravitational part of super-Hamiltonian has a self-adjoint extension if the wave function satisfies the standard Dirichlet or Neumann boundary condition. As was shown in Ref. [76], there is a deep relation between the boundary conditions and the symmetries of cosmological models. In the model investigated here, the conformal invariance of the scalar field part of action functional leads us to the Heisenberg-Weyl symmetry. On the other hand, the spatial closeness of the spacetime along with the boundary conditions, mentioned
previously, demands that symmetry of gravitational part is $SU(1, 1)$ group. Consequently, the corresponding quantum groups of the model after deformation will be $U_q(h_4)$ and $U_q(su(1, 1))$ quantum groups.

The quantum deformation of our cosmological model, causes the quantization of the scalar field, scale factor and the corresponding momenta. In addition, the initial Big-Bang singularity is absent in the sense that the quantized scale factor does not have the zero eigenvalue. On the other hand, the scale factor operator is bounded from above. This means that the Universe reaches the minimum possible value of the curvature and has to stay in this state. Also, we show that the energy densities of dark matter and dark energy (CC) and the baryonic matter are of the same order of magnitude at the present epoch of cosmic evolution. Also, the quantum deformation causes a quantum sphere $S^2_q$ (or equivalently a fuzzy sphere $S^2_f$) as the causal horizon with elementary cells of Planck's area. The number of fundamental cells of the surface of a fuzzy horizon is equal to the dimension of Hilbert space of the scalar field part of super-Hamiltonian. This allows to suggest that the CC and holographic principle are emerged quantities as a result of the quantum deformation of quantum cosmology. Interestingly, as it was shown in [31], gravitational holography is argued to render the CC stable against divergent quantum corrections. Thus gravitational holography provides a technically natural solution to the radiatively instability of the CC [61].

Nevertheless, our setting here must be viewed as only one of the many attempts trying to include quantum gravity effects into cosmological models. It is necessarily partial and incomplete. In order to reach more robust conclusions regarding e.g., the status of singularities in realistic situations, we may need to quantize more degrees of freedom as compared to the only two treated by us.

References
