DISTRIBUTIONAL EQUATION FOR LAGUERRE-HAHN FUNCTIONALS ON THE UNIT CIRCLE

A. BRANQUINHO AND M.N. REBOCHO

ABSTRACT: Let $u$ be a hermitian linear functional defined in the linear space of Laurent polynomials and $F$ its corresponding Carathéodory function. We establish the equivalence between a Riccati differential equation with polynomial coefficients for $F$, $zAF' = BF^2 + CF + D$ and a distributional equation for $u$, $\mathcal{D}(Au) = B_1u^2 + C_1u + H_1L$, where $L$ is the Lebesgue functional, and the polynomials $B_1, C_1, D_1$ are defined in terms of the polynomials $A, B, C, D$.

KEYWORDS: Hermitian functionals, measures on the unit circle, Carathéodory function, Laguerre-Hahn affine class on the unit circle, semi-classical functionals.

AMS SUBJECT CLASSIFICATION (2000): Primary 33C47, 42C05.

1. Introduction

In this paper we introduce the concept of Laguerre-Hahn class of hermitian functionals on the unit circle. Let $u$ be a hermitian regular linear functional defined in the linear space of Laurent polynomials and $F$ the corresponding Carathéodory function. The functional $u$ (respectively, the corresponding Carathéodory function, $F$) is said to be Laguerre-Hahn if $F$ satisfies a Riccati differential equation with polynomial coefficients,

$$zAF' = BF^2 + CF + D, \quad A \neq 0. \tag{1}$$

We shall call the set of all such functionals (respectively, Carathéodory function, $F$) the Laguerre-Hahn class on the unit circle. We remark that the Laguerre-Hahn class on the unit circle can be regarded as an extension of the Laguerre-Hahn class on the real line, studied in [5, 7, 8].

If $B \equiv 0$ in (1) we obtain the Laguerre-Hahn affine class on the unit circle (see [2, 3]). If $B \equiv 0$ and $C, D$ specific polynomials, we obtain the semi-classical class on the unit circle (see [2, 6, 11]). Moreover, the Laguerre-Hahn class on the unit circle includes the class of second degree functionals on

---

Received November 15, 2007.

This work was supported by CMUC/FCT. The second author was supported by FCT, with grant ref. SFRH/BD/25426/2005.
the unit circle (see [4]) and includes, as we will see on section 3, the linear-
fractional transformations of Carathéodory functions which are Laguerre-
Hahn.

Analogously to what has been done on the real case (cf. [7, 8]), we es-
tablish the equivalence between (1) and a distributional equation for the

corresponding $u$,

$$D(Au) = B_1 u^2 + C_1 u + H_1 \mathcal{L},$$

(2)

where $\mathcal{L}$ is the Lebesgue functional, and $B_1, C_1, D_1$ polynomials defined in
terms of $A, B, C, D$.

This paper is organized as follows. In section 2 we give the de-
\textbf{initions and notations which will be used in the forthcoming sections. In
section 3 we study the stability in the Laguerre-Hahn class, and give some
examples of sequences of polynomial orthogonal with respect to a functional
of Laguerre-Hahn type. In section 4 we state some auxiliary results which
enable us to establish, in section 5, the equivalence between (1) and (2).

\section{2. Preliminaries and notations}

Let $\Lambda = \text{span}\{z^k : k \in \mathbb{Z}\}$ be the space of Laurent polynomials with
complex coefficients, $\Lambda'$ its algebraic dual space, $\mathbb{P} = \text{span}\{z^k : k \in \mathbb{N}\}$ the
space of complex polynomials, and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (or, using the
parametrization $z = e^{i\theta}$, $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi]\}$) the unit circle.

Let $u \in \Lambda'$ be a linear functional. We denote by $\langle u, f \rangle$ the action of $u$ over
\textbf{f} \in \Lambda.

Given the sequence of moments $(c_n)$ of $u$, $c_n = \langle u, \xi^{-n} \rangle$, $n \in \mathbb{Z}$, $c_0 = 1$, the
\textbf{minors of the Toeplitz matrix are defined by}

$$\Delta_{-1} = 1, \Delta_0 = c_0, \Delta_k = \begin{vmatrix} c_0 & \cdots & c_k \\ \vdots & \ddots & \vdots \\ c_{-k} & \cdots & c_0 \end{vmatrix}, k \in \mathbb{N}.$$  

\textbf{Definition 1 (cf. [10])}. The linear functional $u$ is:

\begin{itemize}
  \item[a)] \textbf{hermitian} if $c_{-n} = \overline{c_n}, \forall n \geq 0$;
  \item[b)] \textbf{regular or quasi-definite} if $\Delta_n \neq 0, \forall n \geq 0$;
  \item[c)] \textbf{positive definite} if $\Delta_n > 0, \forall n \geq 0$.
\end{itemize}

As an example of hermitian functional we have the \textbf{Lebesgue functional}, $\mathcal{L}$,
def\textbf{ined in terms of its moments by} $\langle \mathcal{L}, \xi^{-n} \rangle = \delta_{0,n}, n \in \mathbb{Z}$. 

If \( u \) is a positive definite hermitian functional, then there exists a non-trivial probability measure \( \mu \) supported on the unit circle such that

\[
\langle u, \xi^{-n} \rangle = \int_{0}^{2\pi} \xi^{-n} d\mu(\theta), \quad \xi = e^{i\theta}, \quad n \in \mathbb{Z}.
\]

**Definition 2.** Let \( \{\phi_n\} \) be a sequence of complex polynomials with \( \text{deg}(\phi_n) = n \) and \( u \) a hermitian linear functional. We say that \( \{\phi_n\} \) is a sequence of orthogonal polynomials with respect to \( u \) (or \( \{\phi_n\} \) is a sequence of orthogonal polynomials on the unit circle) if

\[
\langle u, \phi_n(\xi)\overline{\phi_m(1/\xi)} \rangle = K_n \delta_{n,m}, \quad \xi = e^{i\theta}, \quad K_n \neq 0, \quad n, m \in \mathbb{N}.
\]

If for each \( n \in \mathbb{N} \) the leading coefficient \( \phi_n \) is 1, then \( \{\phi_n\} \) is said to be a sequence of monic orthogonal polynomials and will be denoted by MOPS.

**Remark.** In the positive definite case, as the hermitian functional \( u \) has an integral representation in terms of a measure \( \mu \), we will also say that \( \{\phi_n\} \) is orthogonal with respect to \( \mu \).

For a polynomial \( P \) with degree \( n \), the reciprocal polynomial \( P^* \) is defined by

\[
P^*(z) = z^n P(1/z).
\]

It is well known (see [10]) that a given sequence of complex polynomials \( \{\phi_n\} \) is orthogonal on the unit circle if, and only if, \( \{\phi_n\} \) satisfy a recurrence relation of the following type,

\[
\phi_n(z) = z\phi_{n-1}(z) + a_n\phi^*_{n-1}(z), \quad n \geq 1, \tag{3}
\]

with \( |a_n| \neq 1 \), and initial conditions \( \phi_0(z) = 1, \phi_{-1}(z) = 0 \).

Given a sequence of monic orthogonal polynomials \( \{\phi_n\} \) with respect to \( u \), the sequence of associated polynomials of the second kind \( \{\Omega_n\} \) is defined by

\[
\Omega_0(z) = 1, \quad \Omega_n(z) = \langle u_{\theta}, \frac{e^{i\theta} + z}{e^{i\theta} - z} (\phi_n(e^{i\theta}) - \phi_n(z)) \rangle, \quad n = 1, 2, \ldots,
\]

and verify \( \phi_n^*(z)\Omega_n(z) + \phi_n(z)\Omega^*_n(z) = 2K_n z^n \), for all \( n = 1, 2, \ldots \) (see for instance [10]).

We consider the formal series associated with the hermitian linear functional \( u \),

\[
F_u(z) = c_0 + 2 \sum_{k=1}^{+\infty} c_k z^k, \quad |z| < 1, \tag{4}
\]

\[
F_u(z) = -c_0 - 2 \sum_{k=1}^{+\infty} c_{-k} z^{-k}, \quad |z| > 1. \tag{5}
\]
Since, for each $\theta \in [0, 2\pi[$, the following expansions take place,
\[
\frac{e^{i\theta} + z}{e^{i\theta} - z} = 1 + 2 \sum_{k=1}^{+\infty} (e^{i\theta})^{-k} z^k, \ |z| < 1,
\]
\[
\frac{e^{i\theta} + z}{e^{i\theta} - z} = -1 - 2 \sum_{k=1}^{+\infty} (e^{i\theta})^k z^{-k}, \ |z| > 1
\]
then, taking into account the definition of the moments of $u$, formally we can write
\[
\langle u_\theta, \frac{e^{i\theta} + z}{e^{i\theta} - z} \rangle = F_u(z), \quad z \in \mathbb{C} \setminus \mathbb{T}
\] (6)
where $\langle u_\theta, . \rangle$ denotes the action of $u$ over the variable $\theta$, $\theta \in [0, 2\pi[$. Thus, we will also say that the series in (4), (5) formally correspond to the function $F_u$ defined by (6).

In the positive definite case, $F_u$ is the Carathéodory function corresponding to $u$, and is represented by
\[
F_u(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta), \quad z \in \mathbb{C} \setminus \mathbb{T},
\]
where $\mu$ is the probability measure associated with $u$.

Next we see some operations in the linear space $\Lambda'$. Given $u \in \Lambda'$, $f, g \in \Lambda$, the functionals $fu$ and $Du$, both elements of $\Lambda'$, are defined by
\[
\langle fu, p \rangle = \langle u, fp \rangle, \quad \langle Du, p \rangle = -i \langle u, \xi p' \rangle, \quad p \in \Lambda
\]
thus, $\langle Du, p \rangle = -i \langle u, \xi gp' \rangle$, $p \in \Lambda$. We remark that if $u$ is hermitian, then $Du$ is also hermitian.

We consider the generating function of the moments for a hermitian functional, $u$, defined in $\mathbb{C} \setminus \mathbb{T}$ by
\[
\mathcal{F}_u(z) = \sum_{n=0}^{+\infty} c_n z^n, \quad |z| < 1, \quad \mathcal{F}_u(z) = -\sum_{n=1}^{+\infty} c_{-n} z^{-n}, \quad |z| > 1.
\]
Then the following relation between the Carathéodory function and the generating function of the moments holds,
\[
\mathcal{F}_u(z) = \frac{F_u(z) + 1}{2}, \quad z \in \mathbb{C} \setminus \mathbb{T}. \tag{7}
\]
The next definition is the analogue of the definition given in [8] for the real case.

**Definition 3.** Let $u, v \in \mathcal{N}$ be hermitian linear functionals and $\mathcal{F}_u, \mathcal{F}_v$ the corresponding generating functions of the moments. The product of $u$ and $v$, $uv$, is the functional defined in terms of its moments by

$$ (uv)_n = \sum_{\nu+k=n, \text{sgn}(\nu) = \text{sgn}(k) = \text{sgn}(n)} u_\nu v_k , \quad n \in \mathbb{Z} , $$

that satisfies

$$ \mathcal{F}_u(z) \mathcal{F}_v(z) = \mathcal{F}_{uv}(z) $$

**Remark**. The functional $uv$ defined by (8) is hermitian.

As a consequence, we get the following result.

**Lemma 1.** Let $u$ be a hermitian linear functional and $F_u$ the corresponding formal Carathéodory function. Then,

$$ (F_u)^2 = 2F_u^2 - 2F_u + 1 . $$

**Proof**: Put $u = v$ in (9) and get, after using (7) that $F_u^2 + 1 = ((F_u)^2 + 2F_u + 1)/2$, and the required follows. \hfill \blacksquare

### 3. Stability in the Laguerre-Hahn class on the unit circle. Examples.

We begin this section studying the stability of the Laguerre-Hahn Carathéodory functions under a special kind of linear-fractional transformations.

**Theorem 1.** Let $F_1$ be a Carathéodory function satisfying a Riccati differential equations with polynomials coefficients,

$$ A_1 F_1' = B_1 F_1^2 + C_1 F_1 + D_1 , \quad A_1 \neq 0 , $$

and $F_2$ a linear-fractional transformation of $F_1$ of the following type

$$ F_2 = \frac{\alpha_1 \beta_1 F_1}{-\alpha_2 + \beta_2 F_1} , \quad \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{P} , \quad \alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0 . $$

Then, $F_2$ satisfies

$$ A_2 F_2' = B_2 F_2^2 + C_2 F_2 + D_2 , $$

(12)
with
\[
A_2 = -(\alpha_1\beta_2 - \alpha_2\beta_1)A_1 \neq 0, \quad (13)
\]
\[
B_2 = (\alpha_2'\beta_2 - \alpha_2'\beta_1)A_1 + \alpha_2^2B_1 + \alpha_2\beta_2C_1 + \beta_2^2D_1, \quad (14)
\]
\[
C_2 = (\alpha_2'\beta_1 + \alpha_1\beta_2 - \alpha_2'\beta_1 - \alpha_1'\beta_2)A_1
\]
\[+ 2\alpha_1\alpha_2B_1 + (\alpha_1\beta_2 + \alpha_2\beta_1)C_1 + 2\beta_1\beta_2D_1, \quad (15)
\]
\[
D_2 = (\alpha_1\beta_1' - \alpha_1'\beta_1)A_1 + \alpha_1^2B + \alpha_1\beta_1C_1 + \beta_1^2D_1. \quad (16)
\]

Thus, if \(F_2\) is a Carathéodory function, then \(F_2\) is Laguerre-Hahn.

**Proof:** We have \(F_2 = \frac{\alpha_1 - \beta_1F_1}{\alpha_2 + \beta_2F_1}\) or equivalently \(F_1 = \frac{\alpha_1 + \alpha_2F_2}{\beta_1 + \beta_2F_2}\). By substituting \(F_1 = \frac{\alpha_1 + \alpha_2F_2}{\beta_1 + \beta_2F_2}\) in (11) we obtain (12) with coefficients given by (13)-(16). Thus, the assertion follows.

Next we see some examples of Laguerre-Hahn sequences.

**Corollary 1.** Let \(\{\phi_n\}\) be a MOPS on \(\mathbb{T}\), \(\{\Omega_n\}\) the sequence of associated polynomials of the second kind and \(F, F_1\) the corresponding Carathéodory functions. If \(F\) is Laguerre-Hahn and satisfies \(zAF' = BF^2 + CF + D\), with \(A \neq 0\), then \(F_1\) is Laguerre-Hahn and satisfies \(zAF_1' = -DF_1^2 - CF_1 - B\).

**Proof:** It is well known that \(\{\Omega_n\}\) is orthogonal with respect to \(F_1 = 1/F\), with \(F\) the Carathéodory function associated to \(\{\phi_n\}\) (cf. for example [9]). Thus, the assertion follows from Theorem 1.

**Definition 4** (Peherstorfer [9]). Let \(\{\phi_n\}\) be a MOPS on the unit circle, \((a_n)\) the corresponding sequence of reflection coefficients and \(N \in \mathbb{N}\). The sequence \(\{\phi^N_n\}\) defined by (3) with \(\phi^N_n(0) = a_{n+N}, \ n = 0, 1, \ldots\) is said to be the sequence of associated polynomials of \(\{\phi_n\}\) of order \(N\).

Let \(\{\phi_n\}\) be a MOPS with respect to a Carathéodory function \(F\) and \(\{\Omega_n\}\) be the sequence of orthogonal polynomials of the first kind. In [9] it is established that \(\{\phi^N_n\}\) is orthogonal with respect to the Carathéodory function \(F^N\) given by
\[
F^N = \frac{(\Omega_N - \Omega^*_N) + (\phi_N + \phi^*_N)F}{(\Omega_N + \Omega^*_N) + (\phi_N - \phi^*_N)F}, \quad (17)
\]
and \((\Omega_N - \Omega^*_N)(\phi_N - \phi^*_N) - (\phi_N + \phi^*_N)(\Omega_N + \Omega^*_N) = -2K_N z^N \neq 0\).
Corollary 2. Let \( F \) be a Carathéodory function, \( \{\phi_n\} \) the corresponding MOPS, \( \{\Omega_n\} \) the sequence of associated polynomials of the second kind and \( F_N \) the Carathéodory function corresponding to the sequence of associated polynomials of \( \{\phi_n\} \) of order \( N \), \( \{\phi^*_n\} \). If \( F \) is Laguerre-Hahn and satisfies
\[
zAF' = BF^2 + CF + D , A \neq 0 ,
\]
then \( F^N \) is Laguerre-Hahn and satisfies
\[
A_N \left( F^N \right)' = B_N \left( F^N \right)^2 + C_N F^N + D_N ,
\]
with \( A_N = 4K_N z^{N+1} A \neq 0 \) and
\[
B_N = zA \left\{ \left( \Omega_N + \Omega^*_N \right)'(\phi_N - \phi^*_N) - (\phi_N - \phi^*_N)'(\Omega_N + \Omega_N)^* \right\}
+ (\Omega_N + \Omega^*_N)^2 B - (\Omega_N + \Omega_N^*)(\phi_N - \phi^*_N)C + (\phi_N - \phi^*_N)^2 D ,
\]
\[
C_N = zA \left\{ (\Omega_N - \Omega_N^*)(\phi_N - \phi^*_N)' - (\Omega_N + \Omega_N^*)'(\phi_N + \phi^*_N) \right\}
- 2(\Omega_N^2 - (\Omega_N^*)^2) B + 2(\phi_N \Omega_N + \phi_N^* \Omega_N^*) C - 2(\phi_N^2 - (\phi_N^*)^2) D ,
\]
\[
D_N = (\Omega_N - \Omega_N^*)^2 B - (\Omega_N - \Omega_N^*)(\phi_N + \phi^*_N) C + (\phi_N + \phi^*_N)^2 D .
\]

Proof: From (17) it follows that \( F = \frac{(\Omega_N + \Omega^*_N)F^N - (\Omega_N - \Omega^*_N)}{(\phi_N + \phi^*_N) - (\phi_N - \phi^*_N)F^N} \), and using Theorem 1 the assertion follows.

4. Auxiliary results

Next we establish results that will be used in next section. We consider hermitian linear functionals \( u \in \Lambda \) and \( F_u \) the corresponding formal Carathéodory function defined by (6).

Lemma 2 (cf. [2]). Let \( u \) be a hermitian linear functional and \( A, B \) polynomials. Then, for all \( z \in \mathbb{C} \setminus \mathbb{T} \),
\[
\langle B(\xi)u, \frac{\xi + z}{\xi - z} \rangle = P_{u,B}(z) + B(z) F_u(z) ,
\]
\[
A(z)F'_u(z) = -A'(z)F_u(z) + Q_{u,A}(z) + \frac{1}{iz} \langle \mathcal{D}(Au), \frac{\xi + z}{\xi - z} \rangle \quad (19)
\]
where $P_{u,B}$, $Q_{u,A}$ are polynomials with $\deg(P_{u,B}) = \deg(B)$, $\deg(Q_{u,A}) = \deg(A) - 1$, defined by

$$P_{u,B}(z) = \langle u, \frac{\xi + z}{\xi - z} (B(\xi) - B(z)) \rangle,$$  
(20)

$$Q_{u,A}(z) = -A'(z) - \langle u, 2\xi \sum_{k=2}^{\deg(A)} \frac{A^{(k)}(z)}{k!} (\xi - z)^{k-2} \rangle$$  
(21)

Remark. From (20) and (21) follows $P_{u,B + \tilde{B}} = P_{u,B} + P_{u,\tilde{B}}$, $P_{u,\alpha B} = \alpha P_{u,B}$, with $\alpha \in \mathbb{C}$ and $B, \tilde{B} \in \mathcal{P}$.

Lemma 3 (cf. [1]). Let $u \in \Lambda'$ be a hermitian linear functional and $A$ a polynomial. Then, $(A(z) + \overline{A}(1/z))u$ is hermitian.

Next lemma is a generalization of Theorem 3.4 of [1].

Lemma 4. Let $u$ be a hermitian linear functional. If there exist polynomials $A, B, C, H$ such that $D(Au) = Bu^2 + Cu + HL$, with $L$ the Lebesgue functional, then

$$D(A + \overline{A})u = (B + \overline{B})u^2 + (C + \overline{C})u + (H + \overline{H})L$$  
(22)

Conversely, if (22) holds, then $u$ satisfies the distributional equation with polynomial coefficients

$$D(A_1u) = B_1u^2 + (C_1 + sA_1)u + H_1L$$  
(23)

where $s = \max\{\deg(A), \deg(B), \deg(C), \deg(H)\}$, and the polynomials $A_1, B_1, C_1, H_1$ are given by $A_1(z) = z^s(A(z) + \overline{A}(1/z))$, $B_1(z) = z^s(B(z) + \overline{B}(1/z))$, $C_1(z) = z^s(C(z) + \overline{C}(1/z))$, $H_1(z) = z^s(H(z) + \overline{H}(1/z))$.

Proof: If $D(Au) = Bu^2 + Cu + HL$, then

$$\langle D(Au), \xi^k \rangle = \langle Bu^2 + Cu + HL, \xi^k \rangle, \forall k \in \mathbb{Z}. $$

Applying conjugates follows

$$\langle D(\overline{Au}), \xi^{-k} \rangle = \langle \overline{B}u^2 + \overline{C}u + \overline{H}L, \xi^{-k} \rangle, \forall k \in \mathbb{Z}. $$

Therefore, we obtain

$$\langle D((A + \overline{A})u), \xi^n \rangle = \langle (B + \overline{B})u^2 + (C + \overline{C})u + (H + \overline{H})L, \xi^n \rangle, \forall n \in \mathbb{Z}, $$

and (22) follows.
Conversely, if \( u \) satisfies (22), then we successively get for all \( k \in \mathbb{Z} \)
\[
\langle D((A + \overline{A})u), \xi^k \rangle = \langle (B + \overline{B})u^2, \xi^k \rangle + \langle (C + \overline{C})u, \xi^k \rangle + \langle (H + \overline{H})L, \xi^k \rangle,
\]
\[
- ik \langle u, (A(\xi) + \overline{A}(1/\xi))\xi^k \rangle = \langle u^2, (B(\xi) + \overline{B}(1/\xi))\xi^k \rangle
+ \langle (C(\xi) + \overline{C}(1/\xi))\xi^k \rangle + \langle L, (H(\xi) + \overline{H}(1/\xi))\xi^k \rangle.
\]
Let \( s = \max\{\deg(A), \deg(B), \deg(C), \deg(H)\} \), then the last equation is given by
\[
- ik \langle u, \xi^s(A(\xi) + \overline{A}(1/\xi))\xi^{k-s} \rangle = \langle u^2, \xi^s(B(\xi) + \overline{B}(1/\xi))\xi^{k-s} \rangle
+ \langle u, \xi^s(C(\xi) + \overline{C}(1/\xi))\xi^{k-s} \rangle + \langle L, \xi^s(H(\xi) + \overline{H}(1/\xi))\xi^{k-s} \rangle, \quad \forall k \in \mathbb{Z} \quad (24)
\]
If we write \( m = k - s \) and define \( A_1, B_1, C_1, H_1 \) as in the statement of the lemma, the equation (24) is given by
\[
- i(s + m) \langle u, A_1(\xi)\xi^m \rangle
= \langle u^2, B_1(\xi)\xi^m \rangle + \langle u, C_1(\xi)\xi^m \rangle + \langle L, H_1(\xi)\xi^m \rangle, \quad \forall m \in \mathbb{Z},
\]
and so,
\[
- im \langle u, A_1(\xi)\xi^m \rangle
= \langle u^2, B_1(\xi)\xi^m \rangle + \langle u, (C_1(\xi) + isA_1(\xi))\xi^m \rangle + \langle L, H_1(\xi)\xi^m \rangle, \quad \forall m \in \mathbb{Z}.
\]
From the definition of \( D \), the previous equation is equivalent to
\[
\langle D(A_1u), \xi^m \rangle = \langle B_1u^2, \xi^m \rangle + \langle (C_1 + isA_1)u, \xi^m \rangle + \langle H_1L, \xi^m \rangle, \quad \forall m \in \mathbb{Z},
\]
and we get (23).

**Lemma 5.** Let \( u \) be a hermitian linear functional and \( F_u \) the corresponding formal Carathéodory function. If \( F_u \) satisfies \( zAF_u' = BF_u^2 + CF_u + D \) in \( \mathbb{C} \setminus \mathbb{T} \), then \( u \) satisfies the distributional equation
\[
\langle D(Au) + L(\xi)u - 2iB(\xi)u^2, \xi + z \rangle = iH(z) \quad (25)
\]
with
\[
L = i(-zA' + 2B - C), \quad H = B - 2P_{u^2,B} + P_{u,-zA' + 2B - C} - zQ_{u,A} + D. \quad (26)
\]
Proof: If we use (19) and (10) in \(zAF'_u = BF^2_u + CF_u + D\) we get

\[
(-zA' + 2B - C)F_u - i\langle D(Au), \frac{\xi + z}{\xi - z} \rangle = 2BF_{u^2} + B + D - zQ_{u,A}
\] (27)

On the other hand, from (18) and (19) we know that,

\[
(-zA' + 2B - C)F_u = \langle (\xi A' - 2B\xi - C\xi, u, \frac{\xi + z}{\xi - z} \rangle - P_{u, -zA' + 2B - C}.
\]

and substituting these two equations in (27) we get

\[
\langle D(Au) + i(-\xi A' + 2B - C)u - 2iBu^2, \frac{\xi + z}{\xi - z} \rangle = i(B - 2P_{u^2, B} + P_{u, -zA' + 2B - C} - zQ_{u, A} + D).
\]

Thus, we get (25) with \(L, H\) given in (26).

**Theorem 2.** Let \(u\) be a hermitian linear functional and \(F_u\) the corresponding formal Carathéodory function. If \(F_u\) satisfies

\[
zAF'_u = BF^2_u + CF_u + D, \quad |z| < 1
\]

then \(u\) satisfies the distributional equation with polynomial coefficients

\[
\mathcal{D}(A_1u) = B_1u^2 + (isA_1 - L_1)u + H_1L,
\]

where \(s = \max\{\deg(A), \deg(B), \deg(L), \deg(H)\}\), and the polynomials \(A_1, B_1, L_1, H_1\) are given by \(A_1(z) = z^s(A(z) + \mathcal{A}(1/z)), B_1(z) = z^s(2iB(z) + 2i\mathcal{B}(1/z)), L_1(z) = z^s(L(z) + \mathcal{L}(1/z)), H_1(z) = z^s(iH(z) + i\mathcal{H}(1/z))\), and \(L, H\) are defined in (26).

**Proof:** If \(F_u\) satisfies \(zAF'_u = BF^2_u + CF_u + D, \quad |z| < 1\), then, from Lemma 5, we obtain (25), \(\langle \mathcal{D}(Au) + L(\xi)u - 2iB(\xi)u^2, \frac{\xi + z}{\xi - z} \rangle = iH(z)\). Applying conjugates and the transformation \(Z = 1/z\) to previous equation we get

\[
\langle \mathcal{D}(Au) + \overline{L}u - 2i\overline{B}u^2, \frac{\xi + 1/z}{\xi - 1/z} \rangle = -i\overline{H}(1/z).
\]

Since \(1/\xi + 1/z = \frac{\xi + z}{\xi - z}\), we obtain

\[
\langle \mathcal{D}(Au) + \overline{L}u - 2i\overline{B}u^2, \frac{\xi + z}{\xi - z} \rangle = i\overline{H}(1/z).
\] (28)
Summing (25) with (28) we get
\[
\langle \mathcal{D} ((A + A^c)u + (L + L^c)u - (2iB + 2iB)u^2, \xi + z \xi - z \rangle = i(H(z) + \overline{H}(1/z)).
\]
Therefore, if we compute the moments of the hermitian functional
\[
\mathcal{D} ((A + A^c)u + (L + L^c)u - (2iB + 2iB)u^2
\]
(using the asymptotic expansion of \(\xi + z \xi - z\) in \(|z| < 1\) and in \(|z| > 1\)) we get
\[
\mathcal{D} ((A + A^c)u + (L + L^c)u - (2iB + 2iB)u^2 = (iH + i\overline{H})\mathcal{L}.
\]
From Lemma 4 we obtain the required functional equation.

If we take \(B = 0\) in previous Theorem we obtain the result for the Laguerre-Hahn affine class.

**Corollary 3.** Let \(u\) be a hermitian linear functional. If \(u\) satisfies
\[
zAF'_u = CF_u + D\quad \text{for } |z| < 1,
\]
then \(u\) satisfies
\[
\mathcal{D}(A_1u) = (isA_1 - L_1)u + H_1\mathcal{L},
\]
where \(s = \max\{\deg(A), \deg(L), \deg(H)\}\), and \(A_1, B_1, L_1, H_1, L, H\) are given by
\[
A_1(z) = z^s(A(z) + A(1/z)),
L_1(z) = z^s(L(z) + L(1/z)),
H_1(z) = z^s(iH + i\overline{H}(1/z)),
\]
\(L(z) = i(-zA'(z) - C(z)), \quad H(z) = P_{u,-zA'-C}(z) - zQ_{u,A}(z) + D(z).\)

**5. The characterization theorem**

**Theorem 3.** Let \(u\) be a hermitian linear functional and \(F_u\) the corresponding formal Carathéodory function. If \(u\) satisfies \(\mathcal{D}(Au) = Bu^2 + Cu + \xi H\mathcal{L}\), where \(\mathcal{L}\) is the Lebesgue functional, then \(F_u\) satisfies the differential equations, in \(\mathbb{C} \setminus \mathbb{T}\),
\[
zAF'_u = -\frac{iB}{2}F_u^2 + (-zA' - iB - iC)F_u + \frac{ib}{2} - iP_{u,\cdot}', B + zQ_{u,A} - iP_{u,C} + E, \quad (29)
\]
with \(E(z) = -2izH(z)\mathbb{I}(z), \quad \mathbb{I}(z) = \begin{cases} 1, & |z| < 1 \\ 0, & |z| > 1 \end{cases}.
Conversely, if \(F_u\) satisfies the differential equations (29), then \(u\) satisfies the distributional equation
\[
\mathcal{D}(Au) = Bu^2 + Cu + \xi H\mathcal{L}, \quad (30)
\]
where $\mathcal{L}$ is the Lebesgue functional.

**Proof:** If we substitute $\mathcal{D}(Au) = Bu^2 + Cu + \xi H \mathcal{L}$ in (19) we get

$$zA(z)F_u'(z) = -zA'(z)F_u + zQ_{u,A}(z) - i \langle Bu^2, \frac{\xi + z}{\xi - z} \rangle - i \langle Cu, \frac{\xi + z}{\xi - z} \rangle - i \langle \xi H \mathcal{L}, \frac{\xi + z}{\xi - z} \rangle.$$

From (18) follows

$$zA(z)F_u' = -zA'(z)F_u + zQ_{u,A}(z) - i \left( P_{u,2,B} + B(z) F_u^2 \right)$$

$$- i \left( P_{u,C} + C(z) F_u \right) - i \langle \xi H \mathcal{L}, \frac{\xi + z}{\xi - z} \rangle. \quad (31)$$

Since $\langle \xi H \mathcal{L}, \frac{\xi + z}{\xi - z} \rangle = 2zH(z) \mathbb{I}(z)$ then, in $|z| < 1$, (31) is equivalent to

$$zA(z)F_u' = -iB(z)F_u^2 + (-zA'(z) - iC(z))F_u$$

$$+ zQ_{u,A}(z) - i P_{u,2,B}(z) - i P_{u,C}(z) - 2izH(z)$$

and, in $|z| > 1$, (31) is equivalent to

$$zA(z)F_u' = -iB(z)F_u^2 + (-zA'(z) - iC(z))F_u$$

$$+ zQ_{u,A}(z) - i P_{u,2,B}(z) - i P_{u,C}(z).$$

Using (10) we get (29).

We now prove the converse. If $F$ satisfies

$$zAF_u' = -\frac{iB}{2} F_u^2 + \left( -A' - iB - iC \right) F_u + \frac{iB}{2} - i P_{u,2,B}$$

$$+ zQ_{u,A}(z) - i P_{u,C}(z) + E(z), \ |z| < 1,$$

then, from (25), we get

$$\langle V, \frac{\xi + z}{\xi - z} \rangle = 2zH \mathbb{I}(z) \quad \text{with} \quad V = \mathcal{D}(Au) - Bu^2 - Cu.$$
Using the asymptotic expansion of \( \frac{\xi + z}{\xi - z} \), in \( |z| < 1 \) and in \( |z| > 1 \), the two previous equations are, respectively, equivalent to

\[
\langle V, 1 + 2 \sum_{k=1}^{+\infty} \xi^{-k} z^k \rangle = 2zH(z), \quad |z| < 1 \tag{32}
\]

\[
\langle V, -1 - 2 \sum_{k=1}^{+\infty} \xi^{k} z^{-k} \rangle = 0, \quad |z| > 1 \tag{33}
\]

Writing \( zH(z) = h_1 z + \cdots + h_l z^l \) then, from (32) we get

\[
\langle V, 1 \rangle = 0, \quad \langle V, \xi^{-k} \rangle = h_k, \quad k = 1, \ldots, l, \quad \langle V, \xi^{-k} \rangle = 0, \quad k \geq l + 1 \tag{34}
\]

and, from (33) we get

\[
\langle V, 1 \rangle = 0, \quad \langle V, \xi^k \rangle = 0, \quad k \geq 1 \tag{35}
\]

Finally, from (34) and (35), we conclude that \( V = zH \mathcal{L} \) and (30) holds.

If we do \( B = 0 \) in previous Theorem we get the following result to the Laguerre-Hahn affine class (see [2, 3]).

**Corollary 4.** Let \( u \) be a hermitian linear functional and \( F_u \) the corresponding formal Carathéodory function. If \( u \) satisfies \( \mathcal{D}(Au) = Cu + \xi H(\xi) \mathcal{L} \), then \( F_u \) is such that

\[
zA(z)F_u'(z) = (-zA'(z) - iC(z))F_u(z) + zQ_{u,A}(z) - iP_{u,C}(z) + E(z), \tag{36}
\]

for \( z \in \mathbb{C} \setminus \mathbb{T} \), with \( E(z) = -2izH(z)\mathbb{I}(z) \).

Conversely, if \( F_u \) satisfies (36), then \( u \) satisfies \( \mathcal{D}(Au) = Cu + \xi H(\xi) \mathcal{L} \).

As a consequence of previous Theorem, we get a characterization for semi-classical functionals (see [2]).

**Corollary 5.** Let \( u \) be a hermitian linear functional and \( F \) the corresponding formal Carathéodory function, satisfying a differential equation with polynomial coefficients

\[
zA(z)F'(z) = (-zA'(z) - iC(z))F(z) + P(z), \quad z \in \mathbb{C} \setminus \mathbb{T}. \tag{37}
\]

A necessary and sufficient condition for \( u \) to be semi-classical and satisfy \( \mathcal{D}(Au) = Cu \) is that the polynomial \( P \) of (37) satisfy \( P = zQ_{u,A} - iP_{u,C} \), where \( Q_{u,A} \) and \( P_{u,C} \) are given by (20) and (21).
References


A. BRANQUINHO
CMUC, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COIMBRA, LARGO D. DINIS, 3001-454 COIMBRA, PORTUGAL.

_E-mail address:_ ajplb@mat.uc.pt

URL: http://www.mat.uc.pt/~ajplb

M.N. REBOCHO
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BEIRA INTERIOR, 6201-001 COVILHÃ, PORTUGAL, CMUC, UNIVERSITY OF COIMBRA, 3001-454 COIMBRA, PORTUGAL.

_E-mail address:_ mneves@mat.ubi.pt