

# Generalized derivations of multiplicative $n$ -ary $Hom$ - $\Omega$ color algebras<sup>1</sup>

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**Abstract.** We generalize the results of Leger and Luks, Zhang R. and Zhang Y.; Chen, Ma, Ni, Niu, Zhou and Fan; Kaygorodov and Popov (see, [6, 28, 31, 41, 44–49]) about generalized derivations of color  $n$ -ary algebras to the case of  $n$ -ary  $Hom$ - $\Omega$  color algebras. Particularly, we prove some properties of generalized derivations of multiplicative  $n$ -ary  $Hom$ - $\Omega$  color algebras. Moreover, we prove that the quasiderivation algebra of any multiplicative  $n$ -ary  $Hom$ - $\Omega$  color algebra can be embedded into the derivation algebra of a larger multiplicative  $n$ -ary  $Hom$ - $\Omega$  color algebra.

**Keywords:** generalized derivation, color algebra,  $Hom$ -algebra,  $Hom$ -Lie superalgebra,  $n$ -ary algebra.

## INTRODUCTION

It is well known that the algebras of derivations and generalized derivations are very important in the study of Lie algebras and its generalizations. There are many generalizations of derivations (for example, Leibniz derivations [27] and Jordan derivations [15]). The notion of a  $\delta$ -derivation appeared in the paper of Filippov [11], in which he studied  $\delta$ -derivations of prime Lie and Malcev algebras [12, 13]. After that,  $\delta$ -derivations of Jordan and Lie superalgebras were studied in [18–21, 26, 43, 50] and many other works. The notion of a generalized derivation is a generalization of  $\delta$ -derivation. The most important and systematic research on the generalized derivations algebras of a Lie algebra and their subalgebras was due to Leger and Luks [31]. In their article, they studied properties of generalized derivation algebras and their subalgebras, for example, the quasiderivation algebras. They have determined the structure of algebras of quasiderivations and generalized derivations and proved that the quasiderivation algebra of a Lie algebra can be embedded into the derivation algebra of a larger Lie algebra. Their results were generalized by many authors. For example, Zhang and Zhang [41] generalized the above results to the case of Lie superalgebras; Chen, Ma, Ni, Zhou and Fan considered generalized derivations of color Lie algebras,  $Hom$ -Lie superalgebras, Lie triple systems,  $Hom$ -Lie triple systems and  $n$ - $Hom$  Lie superalgebras [6, 29, 44–49]. Generalized derivations of color  $n$ -ary  $\Omega$ -algebras were studied in [28].

Generalized derivations of simple algebras and superalgebras were investigated in [16, 30, 38, 39]. Pérez-Izquierdo and Jiménez-Gestal used generalized derivations to study non-associative algebras [17, 32]. Derivations and generalized derivations of  $n$ -ary algebras were considered in [3, 5, 22–25, 34, 40] and other works. For example, Williams proved that, unlike the case of binary algebras, for any  $n \geq 3$  there exist a non-nilpotent  $n$ -Lie algebra with invertible derivation [40], Kaygorodov described  $(n+1)$ -ary derivations of simple  $n$ -ary Malcev algebras [24] and generalized derivations algebras of semisimple Filippov algebras over an algebraically closed field of characteristic zero [25].

The study of  $Hom$ -structures was started in the classic paper of Hartwig, Larsson and Silvestrov [14]. After  $Hom$ -Lie algebras were studied some questions related to  $Hom$ -Lie bialgebras [37],  $Hom$ -Lie color algebras [1],  $Hom$ -alternative and  $Hom$ -Malcev algebras [9], ternary  $Hom$ -Nambu-Lie algebras [2],  $Hom$ -Sabinin algebras [7] and many others.

The main purpose of our work is to generalize the results of Leger and Luks [31]; Zhang R. and Zhang Y. [41]; Chen, Ma, Ni, Niu, Zhou and Fan [6, 44–49]; Kaygorodov and Popov [28] to the case of multiplicative  $n$ -ary  $Hom$ - $\Omega$  color algebras for an arbitrary variety of  $Hom$ -identities  $Hom$ - $\Omega$ . Particularly, we prove some properties of generalized derivations of multiplicative  $n$ -ary  $Hom$ - $\Omega$  color algebras; prove that the quasiderivation algebra of a multiplicative  $n$ -ary  $Hom$ - $\Omega$  color algebra can be embedded into the derivation algebra of a larger multiplicative  $n$ -ary  $Hom$ - $\Omega$  color algebra.

## 1. PRELIMINARIES

In this section we consider some well known  $Hom$ -algebraic structures that are twisted versions of the original algebraic structures. In fact, they are  $(n$ -ary) algebras where the identities defining the structure are twisted by homomorphisms, usually called twisting maps. All of the following  $Hom$ -algebras are particular cases of  $n$ -ary  $Hom$ - $\Omega$  color algebras that we consider in the present work.

**Definition 1.** Let  $\mathbb{F}$  be a field and  $\mathbb{G}$  be an abelian group. A map  $\epsilon : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{F}^*$  is called a bicharacter on  $\mathbb{G}$  if the following relations hold for all  $f, g, h \in \mathbb{G}$  :

- (1)  $\epsilon(f, g+h) = \epsilon(f, g)\epsilon(f, h)$ ;
- (2)  $\epsilon(g+h, f) = \epsilon(g, f)\epsilon(h, f)$ ;
- (3)  $\epsilon(g, h)\epsilon(h, g) = 1$ .

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**Definition 2.** A color  $n$ -ary algebra  $T$  is an  $n$ -ary  $\mathbb{G}$ -graded vector space  $T = \bigoplus_{g \in \mathbb{G}} T_g$  with a graded  $n$ -linear map  $[\cdot, \dots, \cdot] : T \times \dots \times T \rightarrow T$  satisfying

$$[T_{\theta_1}, \dots, T_{\theta_n}] \subseteq T_{\theta_1 + \dots + \theta_n}, \quad \theta_i \in \mathbb{G}.$$

The main examples of color  $n$ -ary algebras are color Lie algebras [6], color Leibniz algebras [8], Filippov ( $n$ -Lie) superalgebras [4, 5, 33, 35, 36] and 3-Lie color algebras [42].

Let  $T = \bigoplus_{g \in \mathbb{G}} T_g$  be a color algebra. An element  $x$  is called a homogeneous element of degree  $t \in \mathbb{G}$  if  $x \in T_t$ . We denote this by  $hg(x) = t$ . A linear map  $D$  is homogeneous of degree  $t$  if  $D(T_g) \subseteq T_{g+t}$  for all  $g \in \mathbb{G}$ . We denote this by  $hg(D) = t$ . If  $t = 0$  then  $D$  is said to be even. From now on, unless stated otherwise, we assume that all elements and maps are homogeneous. Let  $\epsilon$  be a bicharacter on  $\mathbb{G}$ . For two homogeneous elements  $a$  and  $b$  we set  $\epsilon(a, b) := \epsilon(hg(a), hg(b))$ .

Let  $D_1, D_2$  be linear maps on  $T$  and  $x_1, \dots, x_n \in T$ . We use the following notations:

$$(1) \quad [D_1, D_2] = D_1 D_2 - \epsilon(D_1, D_2) D_2 D_1,$$

$$X_i = hg(x_1) + \dots + hg(x_i).$$

In particular, we set  $X_0 = 0$ .

By  $\text{End}(T)$  we denote the set of all linear maps of  $T$ . Obviously,  $\text{End}(T) = \bigoplus_{g \in \mathbb{G}} \text{End}_g(T)$  endowed with the color bracket (1) is a color Lie algebra over  $\mathbb{F}$ .

**1.1. Binary Hom-algebras.** In this subsection we give some definitions of *Hom*-algebras with a binary multiplication. It is known that for any family of polynomial identities  $\Omega$ , every  $\Omega$ -algebra is a  $\Omega$ -superalgebra and every  $\Omega$ -superalgebra is a color  $\Omega$ -algebra. In what follows we only give the definitions of color *Hom*- $\Omega$  algebras. The definitions of *Hom*- $\Omega$  superalgebras and *Hom*- $\Omega$  algebras are particular cases of the former ones for  $\mathbb{G} = \mathbb{Z}_2$  and  $\mathbb{G} = \{0\}$ , respectively. On the other hand, the definitions of  $\Omega$ -algebras,  $\Omega$ -superalgebras and color  $\Omega$ -algebras can be obtained from suitable *Hom*-definitions for  $\alpha = id$ .

**Definition 3.** A color *Hom*-Lie algebra  $(T, [\cdot, \cdot], \epsilon, \alpha)$  is a  $\mathbb{G}$ -graded vector space  $T = \bigoplus_{g \in \mathbb{G}} T_g$  with a bicharacter  $\epsilon$ , an even bilinear map  $[\cdot, \cdot] : T \times T \rightarrow T$  and an even linear map  $\alpha : T \rightarrow T$  satisfying

$$[x, y] = -\epsilon(x, y)[y, x],$$

$$\epsilon(z, x)[\alpha(x), [y, z]] + \epsilon(x, y)[\alpha(y), [z, x]] + \epsilon(y, z)[\alpha(z), [x, y]] = 0.$$

**Definition 4.** A color *Hom*-Jordan algebra  $(T, [\cdot, \cdot], \epsilon, \alpha)$  is a  $\mathbb{G}$ -graded vector space  $T = \bigoplus_{g \in \mathbb{G}} T_g$  with a bicharacter  $\epsilon$ , an even bilinear map  $[\cdot, \cdot] : T \times T \rightarrow T$  and an even linear map  $\alpha : T \rightarrow T$  satisfying

$$[x, y] = \epsilon(x, y)[y, x],$$

$$\epsilon(z, x + w)as_T([x, y], \alpha(w), \alpha(z)) + \epsilon(x, y + w)as_T([y, z], \alpha(w), \alpha(x)) + \epsilon(y, z + w)as_T([z, x], \alpha(w), \alpha(y)) = 0,$$

where the trilinear map  $as_T : T \times T \times T \rightarrow T$  is the *Hom*-associator of  $T$  defined by

$$as_T(x, y, z) = [[x, y], \alpha(z)] - [\alpha(x), [y, z]].$$

**1.2.  $n$ -ary Hom-algebras.** In this subsection we give some definitions related to multiplicative *Hom*-algebras with  $n$ -ary multiplication. In general (see, [7]), for the definition of  $n$ -ary *Hom*-algebras we must use a family of homomorphisms  $\{\alpha_i\}_I$ , but  $\alpha_i = \alpha$  and  $\alpha([x_1, \dots, x_n]) = [\alpha(x_1), \dots, \alpha(x_n)]$  in the case of multiplicative  $n$ -ary *Hom*-algebras.

**Definition 5.** A multiplicative  $n$ -ary *Hom*-Lie color algebra  $(T, [\cdot, \dots, \cdot], \epsilon, \alpha)$  is a  $\mathbb{G}$ -graded vector space  $T = \bigoplus_{g \in \mathbb{G}} T_g$  with an  $n$ -linear map  $[\cdot, \dots, \cdot] : T \times \dots \times T \rightarrow T$  and an even linear map  $\alpha : T \rightarrow T$  satisfying

$$[x_1, \dots, x_i, x_{i+1}, \dots, x_n] = -\epsilon(x_i, x_{i+1})[x_1, \dots, x_{i+1}, x_i, \dots, x_n],$$

$$[\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n]] = \sum_{i=1}^n \epsilon(X_{n-1}, Y_{i-1})[\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha(y_{i+1}), \dots, \alpha(y_n)].$$

**Definition 6.** A multiplicative  $n$ -ary *Hom*-associative color algebra  $(T, [\cdot, \dots, \cdot], \epsilon, \alpha)$  is a  $\mathbb{G}$ -graded vector space  $T = \bigoplus_{g \in \mathbb{G}} T_g$  with an  $n$ -linear map  $[\cdot, \dots, \cdot] : T \times \dots \times T \rightarrow T$  and an even linear map  $\alpha : T \rightarrow T$  satisfying

$$[\alpha(x_1), \dots, \alpha(x_i), [x_{i+1}, \dots, x_{i+n}], \alpha(x_{i+n+1}), \dots, \alpha(x_{2n-1})] = [\alpha(x_1), \dots, \alpha(x_{n-1}), [x_n, \dots, x_{2n-1}]].$$

Now we define the notion of  $n$ -ary multiplicative *Hom*- $\Omega$  color algebra for arbitrary family of polynomial identities  $\Omega$ .

**Definition 7.** For a (maybe  $n$ -ary) multilinear polynomial  $f(x_1, \dots, x_n)$  we fix the order of indices  $\{i_1, \dots, i_n\}$  of one non-associative word  $[x_{i_1} \dots x_{i_n}]_\beta$  from the polynomial  $f$ . Here,  $f = \sum_{\beta, \sigma \in S_n} \alpha_{\beta, \sigma} [x_{\sigma(i_1)} \dots x_{\sigma(i_n)}]_\beta$ , where  $\beta$  is an arrangement of brackets in the non-associative word. For the shift  $\mu_i : \{j_1, \dots, j_n\} \mapsto \{j_1, \dots, j_{i+1}, j_i, \dots, j_n\}$  we define the element  $\epsilon(x_{j_i}, x_{j_{i+1}})$ . Now, for arbitrary non-associative word  $[x_{\sigma(i_1)} \dots x_{\sigma(i_n)}]_\beta$  its order of indexes is a composition of suitable shifts  $\mu_i$ , and for this word we set  $\epsilon_\sigma$  defined as the product of corresponding  $\epsilon(x_{j_i}, x_{j_{i+1}})$ . Now, for the multilinear polynomial  $f$ , we define the color multilinear polynomial

$$f_{co} = \sum_{\beta, \sigma \in S_n} \alpha_{\beta, \sigma} \epsilon_\sigma [x_{\sigma(i_1)} \dots x_{\sigma(i_n)}]_\beta.$$

To construct a color multiplicative multilinear Hom-polynomial  $f_{co}^{Hom}$  from a color multilinear polynomial  $f_{co}$  we use the algorithm from [7] and we are changing all free letters  $x_j$  to  $\alpha(x_j)$  in all words of the color multilinear polynomial  $f_{co}$ .

**Definition 8.** Let  $\Omega = \{f_i\}$  be a family of  $n$ -ary multilinear polynomials. Then

(1) A color  $n$ -ary  $\Omega$ -algebra  $L$  is a color  $n$ -ary algebra satisfying the family of color multilinear polynomials  $\Omega_{co} = \{(f_i)_{co}\}$ .

(2) A multiplicative  $n$ -ary Hom- $\Omega$  color algebra  $T$  is a color  $n$ -ary algebra satisfying the family of color multiplicative multilinear Hom-polynomials  $\Omega_{co}^{Hom} = \{(f_i)_{co}^{Hom}\}$  for a homomorphism  $\alpha$ .

**1.3. Linear maps on Hom-algebras.** From now on, we denote a multiplicative  $n$ -ary Hom- $\Omega$  color algebra  $(T, [\cdot, \dots, \cdot], \epsilon, \alpha)$  by  $T$ .

**Definition 9.** A  $\mathbb{G}$ -graded subspace  $M \subseteq T$  is a color Hom-subalgebra of  $T$  if  $\alpha(M) \subseteq M$  and  $[M, \dots, M] \subseteq M$ . A  $\mathbb{G}$ -graded subspace  $I \subseteq T$  is a color Hom-ideal of  $T$  if  $\alpha(I) \subseteq I$  and  $[T, \dots, I, \dots, T] \subseteq I$ .

**Definition 10.** The vector subspace  $S$  of  $\text{End}(T)$  is defined by  $S = \{u \in \text{End}(T) \mid u\alpha = \alpha u\}$  with  $\tilde{\alpha} : S \rightarrow S$  given by  $\tilde{\alpha}(u) = \alpha u$ .

Notice that  $(S, [\cdot, \cdot], \epsilon, \tilde{\alpha})$ , where  $[\cdot, \cdot]$  is the color bracket in (1) and  $\epsilon$  is a bicharacter, is a color Hom-Lie algebra over  $\mathbb{F}$ . In what follows, we only consider linear maps from  $S$ . Let  $\xi \in \mathbb{G}$ .

**Definition 11.**  $D \in \text{End}_\xi(T)$  is a homogeneous  $\alpha^k$ -derivation of  $T$ , with  $k \in \mathbb{N}$  if it satisfies

$$D([x_1, \dots, x_n]) = \sum_{i=1}^n \epsilon(D, X_{i-1})[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)].$$

We denote the set of all  $\alpha^k$ -derivations of  $T$  by  $\text{Der}_{\alpha^k}(T)$ . Notice that  $\text{Der}(T) := \bigoplus_{k \geq 0} \text{Der}_{\alpha^k}(T)$  equipped with the color bracket defined by (1) and the even map  $\tilde{\alpha} : \text{Der}(T) \rightarrow \text{Der}(T)$  defined by  $\tilde{\alpha}(D) = D\alpha$ , is a color Hom-subalgebra of  $S$ , which is called the derivation algebra of  $T$ .

**Definition 12.**  $D \in \text{End}_\xi(T)$  is a homogeneous generalized  $\alpha^k$ -derivation of degree  $\xi$  of  $T$  if there exist endomorphisms  $D^{(i)} \in \text{End}_\xi(T)$  satisfying

$$\begin{aligned} [D(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)] + \sum_{i=2}^n \epsilon(D, X_{i-1})[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D^{(i-1)}(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)] \\ = D^{(n)}([x_1, \dots, x_n]). \end{aligned}$$

**Definition 13.**  $D \in \text{End}_\xi(T)$  is a homogeneous  $\alpha^k$ -quasiderivation of degree  $\xi$  of  $T$  if there exists  $D' \in \text{End}_\xi(T)$  satisfying

$$\sum_{i=1}^n \epsilon(D, X_{i-1})[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)] = D'([x_1, \dots, x_n]).$$

Let  $\text{GDer}_{\alpha^k}(T)$  and  $\text{QDer}_{\alpha^k}(T)$  be the sets of homogeneous generalized  $\alpha^k$ -derivations and of homogeneous  $\alpha^k$ -quasiderivations, respectively. We denote

$$\text{GDer}(T) := \bigoplus_{k \geq 0} \text{GDer}_{\alpha^k}(T), \quad \text{QDer}(T) := \bigoplus_{k \geq 0} \text{QDer}_{\alpha^k}(T).$$

**Definition 14.**  $D \in \text{End}_\xi(T)$  is a homogeneous  $\alpha^k$ -centroid of  $T$ , where  $k \in \mathbb{N}$  if it satisfies

$$D([x_1, \dots, x_n]) = \epsilon(D, X_{i-1})[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)]$$

for all  $i$ .

**Definition 15.**  $D \in \text{End}_\xi(T)$  is a homogeneous  $\alpha^k$ -quasicentroid of degree  $\xi$  of  $T$ , where  $k \in \mathbb{N}$  if it satisfies

$$[D(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)] = \epsilon(D, X_{i-1})[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)]$$

for all  $i$ .

Denote the sets of homogeneous  $\alpha^k$ -centroids and homogeneous  $\alpha^k$ -quasicentroids by  $C_{\alpha^k}(T)$  and  $QC_{\alpha^k}(T)$ , respectively. Denote also

$$C(T) := \bigoplus_{k \geq 0} C_{\alpha^k}(T), \quad QC(T) := \bigoplus_{k \geq 0} QC_{\alpha^k}(T).$$

**Definition 16.**  $D \in \text{End}_\xi(T)$  is a homogeneous  $\alpha^k$ -center derivation of degree  $\xi$  of  $T$  if it satisfies

$$D([x_1, \dots, x_n]) = [\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)] = 0.$$

Denote the set of homogeneous  $\alpha^k$ -center derivations by  $\text{ZDer}_{\alpha^k}(T)$ . We also denote

$$\text{ZDer}(T) := \bigoplus_{k \geq 0} \text{ZDer}_{\alpha^k}(T).$$

Notice that we have the following chain of inclusions:

$$\text{ZDer}(T) \subseteq \text{Der}(T) \subseteq \text{QDer}(T) \subseteq \text{GDer}(T) \subseteq \text{End}(T).$$

## 2. GENERALIZED DERIVATION ALGEBRAS AND THEIR COLOR *Hom*-SUBALGEBRAS

In this section we present some basic properties of generalized derivations, quasiderivations and center derivations of a multiplicative  $n$ -ary *Hom*- $\Omega$  color algebra.

**Lemma 17.** *Let  $T$  be a multiplicative  $n$ -ary *Hom*- $\Omega$  color algebra. Then the following statements hold:*

- (1)  $\text{GDer}(T)$ ,  $\text{QDer}(T)$  and  $\text{C}(T)$  are color *Hom*-subalgebras of  $S$ ;
- (2)  $\text{ZDer}(T)$  is a color *Hom*-ideal of  $\text{Der}(T)$ .

*Proof.* (1) Let  $D_\xi \in \text{GDer}_{\alpha^k}(T)$ ,  $D_\eta \in \text{GDer}_{\alpha^s}(T)$ , where  $k, s \in \mathbb{N}$ . For arbitrary  $x_1, \dots, x_n \in T$  we have

$$\begin{aligned} & [\tilde{\alpha}(D_\xi)(x_1), \alpha^{k+1}(x_2), \dots, \alpha^{k+1}(x_n)] = [(D_\xi \alpha)(x_1), \alpha^{k+1}(x_2), \dots, \alpha^{k+1}(x_n)] = \alpha([D_\xi(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)]) \\ &= \alpha(D_\xi^{(n)}([x_1, x_2, \dots, x_n]) - \sum_{i=2}^n \epsilon(D_\xi, X_{i-1})[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D_\xi^{(i-1)}(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)]) \\ &= \tilde{\alpha}(D_\xi^{(n)}([x_1, x_2, \dots, x_n]) - \sum_{i=2}^n \epsilon(D_\xi, X_{i-1})[\alpha^{k+1}(x_1), \dots, \alpha^{k+1}(x_{i-1}), \tilde{\alpha}(D_\xi^{(i-1)})(x_i), \alpha^{k+1}(x_{i+1}), \dots, \alpha^{k+1}(x_n)]). \end{aligned}$$

For any  $i \in \mathbb{N}$ ,  $\tilde{\alpha}(D_\xi^{(i)})$  belongs to  $\text{End}_\xi(T)$ , thus  $\tilde{\alpha}(D_\xi) \in \text{GDer}_{\alpha^{k+1}}(T)$  and is obviously of degree  $\xi$ . For arbitrary  $x_1, \dots, x_n \in T$  we obtain

$$\begin{aligned} & [D_\xi D_\eta(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= D_\xi^{(n)}([D_\eta(x_1), \alpha^s(x_2), \dots, \alpha^s(x_n)]) \\ &= \sum_{i=2}^n \epsilon(D_\xi, D_\eta + X_{i-1})[\alpha^k(D_\eta(x_1)), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_{i-1}), D_\xi^{(i-1)}(\alpha^s(x_i)), \alpha^{k+s}(x_{i+1}), \dots, \alpha^{k+s}(x_n)] \\ &= D_\xi^{(n)} D_\eta^{(n)}([x_1, \dots, x_n]) \\ &= \sum_{j=2}^n \epsilon(D_\eta, X_{j-1}) D_\xi^{(n)}[\alpha^s(x_1), \dots, \alpha^s(x_{j-1}), D_\eta^{(j-1)}(x_j), \alpha^s(x_{j+1}), \dots, \alpha^s(x_n)] \\ &= \sum_{i=2}^n \epsilon(D_\xi, D_\eta + X_{i-1}) D_\eta^{(n)}[\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D_\xi^{(i-1)}(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)] \\ &+ \sum_{i=2, j=2, j < i}^n \epsilon(D_\xi, D_\eta + X_{i-1}) \epsilon(D_\eta, X_{j-1})[\alpha^{k+s}(x_1), \dots, D_\eta^{(j-1)}(\alpha^k(x_j)), \dots, D_\xi^{(i-1)}(\alpha^s(x_i)), \dots, \alpha^{k+s}(x_n)] \\ &+ \sum_{i=2, j=2, i < j}^n \epsilon(D_\xi, X_{i-1}) \epsilon(D_\eta, X_{j-1})[\alpha^{k+s}(x_1), \dots, D_\xi^{(i-1)}(\alpha^s(x_i)), \dots, D_\eta^{(j-1)}(\alpha^k(x_j)), \dots, \alpha^{k+s}(x_n)] \\ &+ \sum_{i=2}^n \epsilon(D_\xi, D_\eta + X_{i-1}) \epsilon(D_\eta, X_{i-1})[\alpha^{k+s}(x_1), \dots, \alpha^{k+s}(x_{i-1}), D_\eta^{(i-1)} D_\xi^{(i-1)}(x_i), \alpha^{k+s}(x_{i+1}), \dots, \alpha^{k+s}(x_n)]. \end{aligned}$$

Thus, for arbitrary  $x_1, \dots, x_n \in T$ , we have

$$\begin{aligned} & [[D_\xi, D_\eta](x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] = [(D_\xi D_\eta - \epsilon(D_\xi, D_\eta) D_\eta D_\xi)(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= [D_\xi^{(n)}, D_\eta^{(n)}]([x_1, \dots, x_n]) - \sum_{i=2}^n \epsilon(D_\xi + D_\eta, X_{i-1})[\alpha^{k+s}(x_1), \dots, [D_\xi^{(i-1)}, D_\eta^{(i-1)}](x_i), \dots, \alpha^{k+s}(x_n)]. \end{aligned}$$

For all  $2 \leq i \leq n$ ,  $[D_\xi^{(i-1)}, D_\eta^{(i-1)}] \in \text{End}_{\xi+\eta}(T)$ . Therefore  $[D_\xi^{(n)}, D_\eta^{(n)}] \in \text{GDer}_{\alpha^{k+s}}(T)$  and  $\text{GDer}(T)$  is a color *Hom*-subalgebra of  $S$ .

Similarly,  $\text{QDer}(T)$  is a color *Hom*-subalgebra of  $S$ .

Let  $D_\xi \in C_{\alpha^k}(T)$ ,  $D_\eta \in C_{\alpha^s}(T)$ . For arbitrary  $x_1, \dots, x_n \in T$  we have

$$\tilde{\alpha}(D_\xi)([x_1, \dots, x_n]) = \alpha D_\xi([x_1, \dots, x_n]) = \alpha([D_\xi(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)]) = [\tilde{\alpha}(D_\xi)(x_1), \alpha^{k+1}(x_2), \dots, \alpha^{k+1}(x_n)]$$

and

$$\begin{aligned} & [\tilde{\alpha}(D_\xi)(x_1), \alpha^{k+1}(x_2), \dots, \alpha^{k+1}(x_n)] = \alpha([D_\xi(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)]) \\ &= \epsilon(D_\xi, X_{i-1}) \alpha([\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D_\xi(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)]) \\ &= \epsilon(D_\xi, X_{i-1})[\alpha^{k+1}(x_1), \dots, \alpha^{k+1}(x_{i-1}), \tilde{\alpha}(D_\xi)(x_i), \alpha^{k+1}(x_{i+1}), \dots, \alpha^{k+1}(x_n)]. \end{aligned}$$

Thus,  $\tilde{\alpha}(D_\xi) \in C_{\alpha^{k+1}}(T)$ .

Notice that

$$\begin{aligned} & [[D_\xi, D_\eta](x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= [D_\xi D_\eta(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] - \epsilon(D_\xi, D_\eta)[D_\eta D_\xi(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= D_\xi D_\eta([x_1, x_2, \dots, x_n]) - \epsilon(D_\xi, D_\eta) D_\eta D_\xi([x_1, x_2, \dots, x_n]) = [D_\xi, D_\eta]([x_1, x_2, \dots, x_n]). \end{aligned}$$

Similarly, we have

$$\epsilon(D_\xi + D_\eta, X_{i-1})[\alpha^{k+s}(x_1), \dots, [D_\xi, D_\eta](x_i), \dots, \alpha^{k+s}(x_n)] = [D_\xi, D_\eta]([x_1, x_2, \dots, x_n]).$$

Thus,  $[D_\xi, D_\eta] \in C_{\alpha^{k+s}}(T)$  of degree  $\xi + \eta$  and  $\text{C}(T)$  is a color *Hom*-subalgebra of  $S$ .

(2) Let  $D_\xi \in \text{ZDer}_{\alpha^k}(T)$ ,  $D_\eta \in \text{Der}_{\alpha^s}(T)$ . For arbitrary  $x_1, \dots, x_n \in T$  we have

$$\begin{aligned} & [\tilde{\alpha}(D_\xi)(x_1), \alpha^{k+1}(x_2), \dots, \alpha^{k+1}(x_n)] = [(\alpha D_\xi)(x_1), \alpha^{k+1}(x_2), \dots, \alpha^{k+1}(x_n)] \\ &= \alpha([D_\xi(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)]) = \alpha D_\xi([x_1, x_2, \dots, x_n]) = \tilde{\alpha}(D_\xi)([x_1, x_2, \dots, x_n]) = 0. \end{aligned}$$

So,  $\tilde{\alpha}(D_\xi) \in \text{ZDer}_{\alpha^{k+1}}(T)$ .

Observe that

$$[D_\xi, D_\eta]([x_1, x_2, \dots, x_n]) = D_\xi D_\eta([x_1, x_2, \dots, x_n]) - \epsilon(D_\xi, D_\eta) D_\eta D_\xi([x_1, x_2, \dots, x_n])$$

$$= D_\xi([D_\eta(x_1), \alpha^s(x_2), \dots, \alpha^s(x_n)]) + \sum_{i=2}^n \epsilon(D_\eta, X_{i-1}) D_\xi([\alpha^s(x_1), \dots, \alpha^s(x_{i-1}), D_\eta^{(i-1)}(x_i), \alpha^s(x_{i+1}), \dots, \alpha^s(x_n)]) =$$

0

and

$$\begin{aligned} & [[D_\xi, D_\eta](x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= [D_\xi D_\eta(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] - \epsilon(D_\xi, D_\eta)[D_\eta D_\xi(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= -\epsilon(D_\xi, D_\eta)[D_\eta(D_\xi(x_1)), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] = -\epsilon(D_\xi, D_\eta) D_\eta([D_\xi(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)]) \\ &+ \sum_{i=2}^n \epsilon(D_\xi, D_\eta) \epsilon(D_\eta, D_\xi + X_{i-1}) [\alpha^s(D_\xi(x_1)), \alpha^{k+s}(x_2), \dots, D_\eta^{(i-1)}(\alpha^k(x_i)), \dots, \alpha^{k+s}(x_n)] = 0. \end{aligned}$$

Hence  $[D_\xi, D_\eta] \in ZDer_{\alpha^{k+s}}(T)$  and is of degree  $\xi + \eta$ . Therefore,  $ZDer(T)$  is a color  $Hom$ -ideal of  $Der(T)$ .

**Lemma 18.** *Let  $T$  be a multiplicative  $n$ -ary  $Hom$ - $\Omega$  color algebra. Then*

- (1)  $[Der(T), C(T)] \subseteq C(T)$ ;
- (2)  $[QDer(T), QC(T)] \subseteq QC(T)$ ;
- (3)  $C(T) \cdot Der(T) \subseteq Der(T)$ ;
- (4)  $C(T) \subseteq QDer(T)$ ;
- (5)  $[QC(T), QC(T)] \subseteq QDer(T)$ ;
- (6)  $QDer(T) + QC(T) \subseteq GDer(T)$ .

*Proof.* (1) Let  $D_\xi \in Der_{\alpha^k}(T)$ ,  $D_\eta \in C_{\alpha^s}(T)$ . For arbitrary  $x_1, \dots, x_n \in T$  we have

$$\begin{aligned} & [D_\xi D_\eta(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= D_\xi([D_\eta(x_1), \alpha^s(x_2), \dots, \alpha^s(x_n)]) - \sum_{i=2}^n \epsilon(D_\xi, D_\eta + X_{i-1}) [D_\eta(\alpha^k(x_1)), \alpha^{k+s}(x_2), \dots, D_\xi(\alpha^s(x_i)), \dots, \alpha^{k+s}(x_n)] \\ &= D_\xi D_\eta([x_1, x_2, \dots, x_n]) - \sum_{i=2}^n \epsilon(D_\xi, D_\eta) \epsilon(D_\xi + D_\eta, X_{i-1}) [\alpha^{k+s}(x_1), \dots, D_\eta D_\xi(x_i), \dots, \alpha^{k+s}(x_n)] \end{aligned}$$

and

$$\begin{aligned} & [D_\eta D_\xi(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= D_\eta([D_\xi(x_1), \alpha^k(x_2), \dots, \alpha^k(x_n)]) \\ &= D_\eta D_\xi([x_1, x_2, \dots, x_n]) - \sum_{i=2}^n \epsilon(D_\xi, X_{i-1}) D_\eta([\alpha^k(x_1), \dots, D_\xi(x_i), \dots, \alpha^k(x_n)]) \\ &= D_\eta D_\xi([x_1, x_2, \dots, x_n]) - \sum_{i=2}^n \epsilon(D_\xi + D_\eta, X_{i-1}) [\alpha^{k+s}(x_1), \dots, D_\eta D_\xi(x_i), \dots, \alpha^{k+s}(x_n)]. \end{aligned}$$

Hence

$$\begin{aligned} & [[D_\xi, D_\eta](x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= [D_\xi D_\eta(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] - \epsilon(D_\xi, D_\eta) [D_\eta D_\xi(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= [D_\xi, D_\eta]([x_1, \dots, x_n]). \end{aligned}$$

Similarly,

$$[[D_\xi, D_\eta](x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] = \epsilon(D_\xi + D_\eta, X_{i-1}) [\alpha^{k+s}(x_1), \dots, [D_\xi, D_\eta](x_i), \dots, \alpha^{k+s}(x_n)].$$

Thus,  $[D_\xi, D_\eta] \in C_{\alpha^{k+s}}(T)$ , is of degree  $\xi + \eta$  and  $[Der(T), C(T)] \subseteq C(T)$ .

(2) Is similar to the proof of (1).

(3) Let  $D_\xi \in C_{\alpha^k}(T)$ ,  $D_\eta \in Der_{\alpha^s}(T)$ . For arbitrary  $x_1, \dots, x_n \in T$  we have

$$\begin{aligned} & D_\xi D_\eta([x_1, \dots, x_n]) \\ &= D_\xi(\sum_{i=1}^n \epsilon(D_\eta, X_{i-1}) [\alpha^s(x_1), \dots, D_\eta(x_i), \dots, \alpha^s(x_n)]) \\ &= \sum_{i=1}^n \epsilon(D_\xi + D_\eta, X_{i-1}) [\alpha^{k+s}(x_1), \dots, D_\xi D_\eta(x_i), \dots, \alpha^{k+s}(x_n)]. \end{aligned}$$

Therefore  $D_\xi D_\eta \in Der_{\alpha^{k+s}}(T)$  and is of degree  $\xi + \eta$ . Hence  $C(T) \cdot Der(T) \subseteq Der(T)$ .

(4) Let  $D_\xi \in C_{\alpha^k}(T)$ . For arbitrary  $x_1, \dots, x_n \in T$  we have

$$D_\xi([x_1, \dots, x_n]) = \epsilon(D_\xi, X_{i-1}) [\alpha^k(x_1), \dots, D_\xi(x_i), \dots, \alpha^k(x_n)].$$

$$\text{Hence } \sum_{i=1}^n \epsilon(D_\xi, X_{i-1}) [\alpha^k(x_1), \dots, D_\xi(x_i), \dots, \alpha^k(x_n)] = n D_\xi[x_1, \dots, x_n].$$

Therefore  $D_\xi \in QDer_{\alpha^k}(T)$  since  $D' = n D_\xi \in C_{\alpha^k}(T)$ .

(5) Let  $D_\xi \in QC_{\alpha^k}(T)$ ,  $D_\eta \in QC_{\alpha^s}(T)$ . For arbitrary  $x_1, \dots, x_n \in T$  we have

$$\begin{aligned} & [\alpha^{k+s}(x_1), \dots, [D_\xi, D_\eta](x_i), \dots, \alpha^{k+s}(x_n)] \\ &= \epsilon(X_{i-1}, D_\xi) \epsilon(X_{i-1} - X_1, D_\eta) [D_\xi(\alpha^s(x_1)), D_\eta(\alpha^k(x_2)), \dots, \alpha^{k+s}(x_i), \dots, \alpha^{k+s}(x_n)] \\ &\quad - \epsilon(D_\xi, D_\eta) \epsilon(X_{i-1} - X_1, D_\eta) \epsilon(X_{i-1} + D_\eta, D_\xi) [D_\xi(\alpha^s(x_1)), D_\eta(\alpha^k(x_2)), \dots, \alpha^{k+s}(x_i), \dots, \alpha^{k+s}(x_n)] = 0. \end{aligned}$$

Hence

$$\sum_{i=1}^n \epsilon(D_\xi + D_\eta, X_{i-1}) [\alpha^{k+s}(x_1), \dots, [D_\xi, D_\eta](x_i), \dots, \alpha^{k+s}(x_n)] = 0.$$

therefore  $[D_\xi, D_\eta] \in QDer_{\alpha^{k+s}}(T)$  and is of degree  $\xi + \eta$ .

(6) Is obvious. □

**Lemma 19.** *If  $T$  is a multiplicative  $n$ -ary Hom- $\Omega$  color algebra, then  $\text{QC}(T) + [\text{QC}(T), \text{QC}(T)]$  is a color Hom-subalgebra of  $\text{GDer}(T)$ .*

*Proof.* By Lemma 18 (5) and (6), we have

$$\text{QC}(T) + [\text{QC}(T), \text{QC}(T)] \subseteq \text{GDer}(T) \text{ and}$$

$$[\text{QC}(T) + [\text{QC}(T), \text{QC}(T)], \text{QC}(T) + [\text{QC}(T), \text{QC}(T)]]$$

$$\subseteq [\text{QC}(T) + \text{QDer}(T), \text{QC}(T) + [\text{QC}(T), \text{QC}(T)]]$$

$$\subseteq [\text{QC}(T), \text{QC}(T)] + [\text{QC}(T), [\text{QC}(T), \text{QC}(T)]] + [\text{QDer}(T), \text{QC}(T)] + [\text{QDer}(T), [\text{QC}(T), \text{QC}(T)]]$$

Using the color Hom-Jacobi identity and (2) of the previous lemma, it is easy to verify that

$$[\text{QDer}(T), [\text{QC}(T), \text{QC}(T)]] \subseteq [\text{QC}(T), \text{QC}(T)].$$

Thus,

$$[\text{QC}(T) + [\text{QC}(T), \text{QC}(T)], \text{QC}(T) + [\text{QC}(T), \text{QC}(T)]] \subseteq \text{QC}(T) + [\text{QC}(T), \text{QC}(T)]. \quad \square$$

**Definition 20.** *Let  $T$  is a multiplicative  $n$ -ary algebra. The annihilator of  $T$  is defined by*

$$\text{Ann}(T) = \{x \in T \mid [x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n] = 0 \text{ for all } i\}.$$

**Lemma 21.** *Let  $\alpha$  be a surjective map. If  $T$  is a multiplicative  $n$ -ary Hom- $\Omega$  color algebra, then  $[\text{C}(T), \text{QC}(T)] \subseteq \text{End}(T, \text{Ann}(T))$ . Moreover, if  $\text{Ann}(T) = \{0\}$ , then  $[\text{C}(T), \text{QC}(T)] = \{0\}$ .*

*Proof.* Assume that  $D_\xi \in C_{\alpha^k}(T)$ ,  $D_\eta \in \text{QC}_{\alpha^s}(T)$  and  $x \in T$ . As  $\alpha$  is surjective, for any  $y'_i \in T$ ,  $1 \leq i \leq n$ , there exists  $y_i \in T$  such that  $y'_i = \alpha^{k+s}(y_i)$ . We have

$$\epsilon(Y_{i-1}, D_\xi + D_\eta)[\alpha^{k+s}(y_1), \dots, [D_\xi, D_\eta](x), \dots, \alpha^{k+s}(y_n)] =$$

$$\epsilon(Y_{i-1}, D_\xi + D_\eta)[\alpha^{k+s}(y_1), \dots, (D_\xi D_\eta - \epsilon(D_\xi, D_\eta)D_\eta D_\xi)(x), \dots, \alpha^{k+s}(y_n)] =$$

$$D_\xi([D_\eta(y_1), \alpha^s(y_2), \dots, \alpha^s(x), \dots, \alpha^s(y_n)] - [D_\eta(y_1), \alpha^s(y_2), \dots, \alpha^s(x), \dots, \alpha^s(y_n)]) = 0.$$

Hence  $[D_\xi, D_\eta](x) \in \text{Ann}(T)$  and  $[D_\xi, D_\eta] \in \text{End}(T, \text{Ann}(T))$  as desired. Furthermore, if  $\text{Ann}(T) = \{0\}$ , it is clear that  $[\text{C}(T), \text{QC}(T)] = \{0\}$ .  $\square$

**Lemma 22.** *Let  $\mathbb{F}$  be of characteristic  $\neq 2$  and  $(S, \bullet, \tilde{\alpha})$  be a color Hom-algebra with multiplication*

$$D_\xi \bullet D_\eta = \frac{1}{2}(D_\xi D_\eta + \epsilon(D_\xi, D_\eta)D_\eta D_\xi)$$

where  $D_\xi, D_\eta$  are  $\alpha$ -derivations of  $S$ . Then

- (1)  $(S, \bullet, \alpha)$  is a color Hom-Jordan algebra;
- (2)  $(\text{QC}(T), \bullet, \alpha)$  is a color Hom-Jordan algebra.

*Proof.* (1) Let  $D_\xi, D_\eta \in S$ . We have

$$D_\xi \bullet D_\eta$$

$$= \frac{1}{2}(D_\xi D_\eta + \epsilon(D_\xi, D_\eta)D_\eta D_\xi)$$

$$= \frac{1}{2}\epsilon(D_\xi, D_\eta)(D_\eta D_\xi + \epsilon(D_\eta, D_\xi)D_\xi D_\eta)$$

$$= \epsilon(D_\xi, D_\eta)D_\eta \bullet D_\xi.$$

Consider now the second Hom-Jordan identity:

$$((D_\xi \bullet D_\eta) \bullet \alpha(D_\theta)) \bullet \alpha^2(D_\gamma)$$

$$= \frac{1}{2}((D_\xi D_\eta + \epsilon(D_\xi, D_\eta)D_\eta D_\xi) \bullet \alpha(D_\theta)) \bullet \alpha^2(D_\gamma)$$

$$= \frac{1}{4}((D_\xi D_\eta + \epsilon(D_\xi, D_\eta)D_\eta D_\xi)\alpha(D_\theta) + \epsilon(D_\xi + D_\eta, D_\theta)\alpha(D_\theta)(D_\xi D_\eta + \epsilon(D_\xi, D_\eta)D_\eta D_\xi)) \bullet \alpha^2(D_\gamma)$$

$$= \frac{1}{8}(D_\xi D_\eta \alpha(D_\theta) \alpha^2(D_\gamma) + \epsilon(D_\xi, D_\eta)D_\eta D_\xi \alpha(D_\theta) \alpha^2(D_\gamma) + \epsilon(D_\xi + D_\eta, D_\theta)\alpha(D_\theta)D_\xi D_\eta \alpha^2(D_\gamma)$$

$$+ \epsilon(D_\xi + D_\eta, D_\theta)\epsilon(D_\xi, D_\eta)\alpha(D_\theta)D_\eta D_\xi \alpha^2(D_\gamma) + \epsilon(D_\xi + D_\eta + D_\theta, D_\gamma)\alpha^2(D_\gamma)D_\xi D_\eta \alpha(D_\theta)$$

$$+ \epsilon(D_\xi, D_\eta)\epsilon(D_\xi + D_\eta + D_\theta, D_\gamma)\alpha^2(D_\gamma)D_\eta D_\xi \alpha(D_\theta) + \epsilon(D_\xi + D_\eta, D_\theta)\epsilon(D_\xi + D_\eta + D_\theta, D_\gamma)\alpha^2(D_\gamma)\alpha(D_\theta)D_\xi D_\eta$$

$$+ \epsilon(D_\xi + D_\eta, D_\theta)\epsilon(D_\xi, D_\eta)\epsilon(D_\xi + D_\eta + D_\theta, D_\gamma)\alpha^2(D_\gamma)\alpha(D_\theta)D_\eta D_\xi).$$

On the other hand, we have

$$\alpha(D_\xi \bullet D_\eta) \bullet (\alpha(D_\theta) \bullet \alpha(D_\gamma))$$

$$= \frac{1}{4}\alpha(D_\xi D_\eta + \epsilon(D_\xi, D_\eta)D_\eta D_\xi) \bullet (\alpha(D_\theta)\alpha(D_\gamma) + \epsilon(D_\theta, D_\gamma)\alpha(D_\gamma)\alpha(D_\theta))$$

$$= \frac{1}{8}(\alpha(D_\xi D_\eta)\alpha(D_\theta)\alpha(D_\gamma) + \epsilon(D_\xi + D_\eta, D_\theta + D_\gamma)\alpha(D_\theta)\alpha(D_\gamma)\alpha(D_\xi D_\eta)$$

$$+ \epsilon(D_\theta, D_\gamma)\alpha(D_\xi D_\eta)\alpha(D_\gamma)\alpha(D_\theta) + \epsilon(D_\theta, D_\gamma)\epsilon(D_\xi + D_\eta, D_\theta + D_\gamma)\alpha(D_\gamma)\alpha(D_\theta)\alpha(D_\xi D_\eta)$$

$$+ \epsilon(D_\xi, D_\eta)\alpha(D_\eta D_\xi)\alpha(D_\theta)\alpha(D_\gamma) + \epsilon(D_\xi, D_\eta)\epsilon(D_\xi + D_\eta, D_\theta + D_\gamma)\alpha(D_\theta)\alpha(D_\gamma)\alpha(D_\eta D_\xi)$$

$$+ \epsilon(D_\xi, D_\eta)\epsilon(D_\theta, D_\gamma)\alpha(D_\eta D_\xi)\alpha(D_\gamma)\alpha(D_\theta) + \epsilon(D_\xi, D_\eta)\epsilon(D_\theta, D_\gamma)\epsilon(D_\xi + D_\eta, D_\gamma + D_\theta)\alpha(D_\gamma)\alpha(D_\theta)\alpha(D_\eta D_\xi)).$$

So,

$$\epsilon(D_\gamma, D_\xi + D_\theta)as_\alpha(D_\xi \bullet D_\eta, \alpha(D_\theta), \alpha(D_\gamma))$$

$$= \frac{1}{8}\epsilon(D_\gamma, D_\xi + D_\theta)(\epsilon(D_\xi + D_\eta, D_\theta)\alpha(D_\theta)D_\xi D_\eta \alpha^2(D_\gamma) + \epsilon(D_\xi, D_\eta)\epsilon(D_\xi + D_\eta, D_\theta)\alpha(D_\theta)D_\eta D_\xi \alpha^2(D_\gamma)$$

$$+ \epsilon(D_\xi + D_\eta + D_\theta, D_\gamma)\alpha^2(D_\gamma)D_\xi D_\eta \alpha(D_\theta) + \epsilon(D_\xi, D_\eta)\epsilon(D_\xi + D_\eta + D_\theta, D_\gamma)\alpha^2(D_\gamma)D_\eta D_\xi \alpha(D_\theta)$$

$$- \epsilon(D_\xi + D_\eta, D_\theta + D_\gamma)\alpha(D_\theta)\alpha(D_\gamma)\alpha(D_\xi D_\eta) - \epsilon(D_\theta, D_\gamma)\alpha(D_\xi D_\eta)\alpha(D_\gamma)\alpha(D_\theta)$$

$$- \epsilon(D_\xi, D_\eta)\epsilon(D_\xi + D_\eta, D_\theta + D_\gamma)\alpha(D_\theta)\alpha(D_\gamma)\alpha(D_\eta D_\xi) - \epsilon(D_\xi, D_\eta)\epsilon(D_\theta, D_\gamma)\alpha(D_\eta D_\xi)\alpha(D_\gamma)\alpha(D_\theta)).$$

Hence

$$\epsilon(D_\gamma, D_\xi + D_\theta)as_\alpha(D_\xi \bullet D_\eta, \alpha(D_\theta), \alpha(D_\gamma)) + \epsilon(D_\xi, D_\eta + D_\theta)as_\alpha(D_\eta \bullet D_\gamma, \alpha(D_\theta), \alpha(D_\xi))$$

$$+ \epsilon(D_\eta, D_\gamma + D_\theta)as_\alpha(D_\gamma \bullet D_\xi, \alpha(D_\theta), \alpha(D_\eta)) = 0.$$

(2) We only need to show that for arbitrary  $D_\xi, D_\eta \in \text{QC}(T)$ ,  $D_\xi \bullet D_\eta \in \text{QC}(T)$ . We have

$$\begin{aligned} & [D_\xi \bullet D_\eta(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= \frac{1}{2}[D_\xi D_\eta(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] + \frac{1}{2}\epsilon(D_\xi, D_\eta)[D_\eta D_\xi(x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= \frac{1}{2}\epsilon(D_\xi, D_\eta + X_{i-1})[D_\eta(\alpha^k(x_1)), \alpha^{k+s}(x_2), \dots, D_\xi(x_i), \dots, \alpha^{k+s}(x_n)] \\ &\quad + \frac{1}{2}\epsilon(D_\eta, X_{i-1})[D_\xi(\alpha^s(x_1)), \alpha^{k+s}(x_2), \dots, D_\eta(x_i), \dots, \alpha^{k+s}(x_n)] \\ &= \frac{1}{2}\epsilon(D_\xi, D_\eta + X_{i-1})\epsilon(D_\eta, X_{i-1})[\alpha^{k+s}(x_1), \dots, D_\eta D_\xi(x_i), \dots, \alpha^{k+s}(x_n)] \\ &\quad + \frac{1}{2}\epsilon(D_\xi + D_\eta, X_{i-1})[\alpha^{k+s}(x_1), \dots, D_\xi D_\eta(x_i), \dots, \alpha^{k+s}(x_n)] \\ &= \epsilon(D_\xi + D_\eta, X_{i-1})[\alpha^{k+s}(x_1), \dots, D_\xi \bullet D_\eta(x_i), \dots, \alpha^{k+s}(x_n)]. \end{aligned}$$

Then  $D_\xi \bullet D_\eta \in \text{QC}(T)$  and  $\text{QC}(T)$  is a color *Hom*-Jordan algebra.  $\square$

**Theorem 23.** Let  $T$  be a multiplicative  $n$ -ary *Hom*- $\Omega$  color algebra. Then the following statements hold:

(1)  $\text{QC}(T)$  is a color *Hom*-Lie algebra with  $[D_\xi, D_\eta] = D_\xi D_\eta - \epsilon(D_\xi, D_\eta) D_\eta D_\xi$  if and only if  $\text{QC}(T)$  is a color *Hom*-associative algebra with respect to the usual composition of operators;

(2) If  $\text{char } \mathbb{F}$  is not 2,  $\alpha$  is a surjective map and  $\text{Ann}(T) = \{0\}$ , then  $\text{QC}(T)$  is a color *Hom*-Lie algebra if and only if  $\{\text{QC}(T), \text{QC}(T)\} = \{0\}$ .

*Proof.* (1) ( $\Leftarrow$ ) For arbitrary  $D_\xi \in \text{QC}_{\alpha^k}(T)$ ,  $D_\eta \in \text{QC}_{\alpha^s}(T)$ , we have  $D_\xi D_\eta \in \text{QC}_{\alpha^{k+s}}(T)$  and  $D_\eta D_\xi \in \text{QC}_{\alpha^{k+s}}(T)$ . So,  $[D_\xi, D_\eta] = D_\xi D_\eta - \epsilon(D_\xi, D_\eta) D_\eta D_\xi \in \text{QC}_{\alpha^{k+s}}(T)$ . Hence,  $\text{QC}(T)$  is a color *Hom*-Lie algebra.

( $\Rightarrow$ ) Note that  $D_\xi D_\eta = D_\xi \bullet D_\eta + \frac{[D_\xi, D_\eta]}{2}$ . By Lemma 22, we have  $D_\xi \bullet D_\eta \in \text{QC}(T)$ ,  $[D_\xi, D_\eta] \in \text{QC}(T)$ . It follows that  $D_\xi D_\eta \in \text{QC}(T)$ , as desired.

(2) ( $\Rightarrow$ ) Let  $D_\xi \in \text{QC}_{\alpha^k}(T)$ ,  $D_\eta \in \text{QC}_{\alpha^s}(T)$ . Now we have

$$\begin{aligned} & [[D_\xi, D_\eta](x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] \\ &= \epsilon(D_\xi, D_\eta + X_{i-1})\epsilon(D_\eta, X_{i-1})[\alpha^{k+s}(x_1), \dots, D_\eta D_\xi(x_i), \dots, \alpha^{k+s}(x_n)] \\ &\quad - \epsilon(D_\xi + D_\eta, X_{i-1})[\alpha^{k+s}(x_1), \dots, D_\xi D_\eta(x_i), \dots, \alpha^{k+s}(x_n)] = \\ &\quad - \epsilon(D_\xi + D_\eta, X_{i-1})[\alpha^{k+s}(x_1), \dots, [D_\xi, D_\eta](x_i), \dots, \alpha^{k+s}(x_n)]. \end{aligned}$$

On the other hand, since  $\text{QC}(T)$  is a color *Hom*-Lie algebra, then, for arbitrary  $x_1, \dots, x_n \in T$ , we have

$$[[D_\xi, D_\eta](x_1), \alpha^{k+s}(x_2), \dots, \alpha^{k+s}(x_n)] = \epsilon(D_\xi + D_\eta, X_{i-1})[\alpha^{k+s}(x_1), \dots, [D_\xi, D_\eta](x_i), \dots, \alpha^{k+s}(x_n)].$$

Therefore

$$[\alpha^{k+s}(x_1), \dots, [D_\xi, D_\eta](x_i), \dots, \alpha^{k+s}(x_n)] = 0, \text{ hence } [x'_1, \dots, [D_\xi, D_\eta](x_i), \dots, x'_n] = 0.$$

Hence,  $[D_\xi, D_\eta] = 0$ .

( $\Leftarrow$ ) Is clear.  $\square$

### 3. QUASIDERIVATIONS OF MULTIPLICATIVE $n$ -ARY *Hom*- $\Omega$ COLOR ALGEBRAS

In this section we investigate the quasiderivations of the multiplicative  $n$ -ary *Hom*- $\Omega$  color algebra  $T$ . We prove that  $Q\text{Der}(T)$  can be embedded in the derivation algebra of a larger multiplicative  $n$ -ary *Hom*- $\Omega$  color algebra  $\check{T}$  for the same variety of polynomial *Hom*-identities  $\Omega$ . Moreover, we conclude that  $\text{Der}(\check{T})$  has a direct sum decomposition when  $\text{Ann}(T) = \{0\}$ .

**Lemma 24.** Let  $T$  be a multiplicative  $n$ -ary *Hom*- $\Omega$  color algebra over  $\mathbb{F}$  and  $t$  be an indeterminate. Define

$$\check{T} := \{\Sigma(x \otimes t + y \otimes t^n) \mid x, y \in T\} \text{ and } \check{\alpha}(\check{T}) = \{\Sigma(\alpha(x) \otimes t + \alpha(y) \otimes t^n) \mid x, y \in T\}.$$

Endow  $(\check{T}, \check{\alpha})$  with the multiplication

$$[x_1 \otimes t^{i_1}, x_2 \otimes t^{i_2}, \dots, x_n \otimes t^{i_n}] = [x_1, x_2, \dots, x_n] \otimes t^{\sum i_j},$$

for  $i_1, \dots, i_n \in \{1, n\}$  (we put  $t^k = 0$  if  $k > n$ ).

Then  $(\check{T}, \check{\alpha})$  is a multiplicative  $n$ -ary *Hom*- $\Omega$  color algebra.

*Proof.* Let the class of  $n$ -ary *Hom*- $\Omega$  color algebras be defined by the family  $\{f_k\}$  of color multilinear *Hom*-identities. Then, for arbitrary  $x_1, x_2, \dots, x_m \in T$  and  $i_j \in \{1, n\}$  we have

$$f_j(x_1 \otimes t^{i_1}, x_2 \otimes t^{i_2}, \dots, x_m \otimes t^{i_m}) = f_j(x_1, x_2, \dots, x_m) \otimes t^{\sum i_i} = 0.$$

Therefore  $\check{T}$  is a  $n$ -ary *Hom*- $\Omega$  color algebra.  $\square$

For the sake of convenience, we write  $xt$  ( $xt^n$ ) instead of  $x \otimes t$  ( $x \otimes t^n$ ).

If  $U$  is a  $\mathbb{G}$ -graded subspace of  $T$  such that  $T = U \oplus [T, \dots, T]$ , then

$$\check{T} = Tt + Tt^n = Tt + Ut^n + [T, \dots, T]t^n.$$

Now define a map  $\varphi : Q\text{Der}(T) \rightarrow \text{End}(\check{T})$  by

$$\varphi(D)(at + ut^n + bt^n) = D(a)t + D'(b)t^n,$$

where  $D \in Q\text{Der}(T)$ ,  $D'$  is a map related to  $D$  by the definition of quasiderivation,  $a \in T, u \in U, b \in [T, \dots, T]$ .

**Theorem 25.** Let  $T, \check{T}, \varphi$  be as above. Then

- (1)  $\varphi$  is even;
- (2)  $\varphi$  is injective and  $\varphi(D)$  does not depend on the choice of  $D'$ ;
- (3)  $\varphi(\text{QDer}(T)) \subseteq \text{Der}(\check{T})$ .

*Proof.* (1) Follows from the definition of  $\varphi$ .

- (2) If  $\varphi(D_1) = \varphi(D_2)$ , then for all  $a \in T, b \in [T, \dots, T]$  and  $u \in U$  we have

$$\varphi(D_1)(at + ut^n + bt^n) = \varphi(D_2)(at + ut^n + bt^n),$$

or, in terms of  $D_1, D_2$ ,

$$D_1(a)t + D'_1(b)t^n = D_2(a)t + D'_2(b)t^n,$$

so  $D_1(a) = D_2(a)$ . Hence  $D_1 = D_2$ , and  $\varphi$  is injective.

Suppose that there exists  $D''$  such that

$$\varphi(D)(at + ut^n + bt^n) = D(a)t + D''(b)t^n,$$

and

$$\sum \epsilon(D, X_{i-1})[\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)] = D''([x_1, \dots, x_n]),$$

then we have

$$D'([x_1, \dots, x_n]) = D''([x_1, \dots, x_n]),$$

thus  $D'(b) = D''(b)$ . Hence

$$\varphi(D)(at + ut^n + bt^n) = D(a)t + D'(b)t^n = D(a)t + D''(b)t^n,$$

which implies that  $\varphi(D)$  is determined only by  $D$ .

- (3) We have  $[x_1 t^{i_1}, \dots, x_n t^{i_n}] = [x_1, \dots, x_n] t^{\sum i_j} = 0$  for all  $\sum i_j \geq n+1$ . Thus, to show that  $\varphi(D) \in \text{Der}(\check{T})$ , we only need to check that the following equality holds:

$$\varphi(D)([x_1 t, \dots, x_n t]) = \sum \epsilon(D, X_{i-1})[\check{\alpha}^k(x_1 t), \dots, \varphi(D)(x_i t), \dots, \check{\alpha}^k(x_n t)].$$

For arbitrary  $x_1, \dots, x_n \in T$  we have

$$\begin{aligned} \varphi(D)([x_1 t, \dots, x_i t, \dots, x_n t]) &= \varphi(D)([x_1, \dots, x_n] t^n) = D'([x_1, \dots, x_n]) t^n \\ &= \sum \epsilon(D, X_{i-1})[\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)] t^n \\ &= \sum \epsilon(D, X_{i-1})[\alpha^k(x_1 t), \dots, D(x_i t), \dots, \alpha^k(x_n t)] \\ &= \sum \epsilon(D, X_{i-1})[\check{\alpha}^k(x_1 t), \dots, \varphi(D)(x_i t), \dots, \check{\alpha}^k(x_n t)]. \end{aligned}$$

Therefore, for all  $D \in \text{QDer}(T)$  we have  $\varphi(D) \in \text{Der}(\check{T})$ . □

**Lemma 26.** Let  $T$  be a multiplicative  $n$ -ary Hom- $\Omega$  color algebra such that  $\text{Ann}(T) = \{0\}$  and let  $\check{T}, \varphi$  be as previously defined. Then

$$\text{Der}(\check{T}) = \varphi(\text{QDer}(T)) \oplus \text{ZDer}(\check{T}).$$

*Proof.* Since  $\text{Ann}(T) = \{0\}$ , we have  $\text{Ann}(\check{T}) = Tt^n$ . For  $g \in \text{Der}(\check{T})$  we have  $g(\text{Ann}(\check{T})) \subseteq \text{Ann}(\check{T})$ , hence  $g(Ut^n) \subseteq g(\text{Ann}(\check{T})) \subseteq \text{Ann}(\check{T}) = Tt^n$ . Now define a map  $f : Tt + Ut^n + [T, \dots, T]t^n \rightarrow Tt^n$  by

$$f(x) = \begin{cases} g(x) \cap Tt^n, & x \in Tt; \\ g(x), & x \in Ut^n; \\ 0, & x \in [T, \dots, T]t^n. \end{cases}$$

It is clear that  $f$  is linear. Note that

$$f([\check{T}, \dots, \check{T}]) = f([T, \dots, T]t^n) = 0,$$

$$[\check{\alpha}^k(\check{T}), \dots, f(\check{T}), \dots, \check{\alpha}^k(\check{T})] \subseteq [\alpha^k(T)t + \alpha^k(T)t^n, \dots, Tt^n, \dots, \alpha^k(T)t + \alpha^k(T)t^n] = 0,$$

hence  $f \in \text{ZDer}(\check{T})$ . Since

$$(g - f)(Tt) = g(Tt) - g(Tt) \cap Tt^n = g(Tt) - Tt^n \subseteq Tt, \quad (g - f)(Ut^n) = 0,$$

and

$$(g - f)([T, \dots, T]t^n) = g([\check{T}, \dots, \check{T}]) \subseteq [\check{T}, \dots, \check{T}] = [T, \dots, T]t^n,$$

there exist  $D, D' \in \text{End}(T)$  such that for all  $a \in T, b \in [T, \dots, T]$ ,

$$(g - f)(at) = D(a)t, \quad (g - f)(bt^n) = D'(b)t^n.$$

Since  $g - f \in \text{Der}(\check{T})$ , by the definition of  $\text{Der}(\check{T})$  we have

$$\sum \epsilon(g - f, A_{i-1})[\check{\alpha}^k(a_1 t), \dots, (g - f)(a_i t), \dots, \check{\alpha}^k(a_n t)] = (g - f)([a_1 t, \dots, a_n t]),$$



for all  $a_1, \dots, a_n \in T$ . Hence

$$\sum \epsilon(D, A_{i-1})[\alpha^k(a_1), \dots, D(a_i), \dots, \alpha^k(a_n)] = D'([a_1, \dots, a_n]).$$

Thus  $D \in \text{QDer}(T)$ . Therefore,  $g - f = \varphi(D) \in \varphi(\text{QDer}(T))$ , so  $\text{Der}(\check{T}) \subseteq \varphi(\text{QDer}(T)) + \text{ZDer}(\check{T})$ . By Lemma 25 (3) we have  $\text{Der}(\check{T}) = \varphi(\text{QDer}(T)) + \text{ZDer}(\check{T})$ .

For any  $f \in \varphi(\text{QDer}(T)) \cap \text{ZDer}(\check{T})$  there exists an element  $D \in \text{QDer}(T)$  such that  $f = \varphi(D)$ . Then

$$f(at + ut^n + bt^n) = \varphi(D)(at + ut^n + bt^n) = D(a)t + D'(b)t^n,$$

where  $a \in T, b \in [T, \dots, T]$ .

On the other hand, since  $f \in \text{ZDer}(\check{T})$ , we have

$$f(at + bt^n + ut^n) \in \text{Ann}(\check{T}) = Tt^n.$$

That is,  $D(a) = 0$ , for all  $a \in T$  and so  $D = 0$ . Hence  $f = 0$ .

Therefore  $\text{Der}(\check{T}) = \varphi(\text{QDer}(T)) \oplus \text{ZDer}(\check{T})$  as desired. □

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