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# SYLVESTER EQUATIONS FOR LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS ON THE REAL LINE

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ABSTRACT: Matrix Sylvester differential equations are introduced in the study of Laguerre-Hahn orthogonal polynomials. Matrix Sylvester differential systems are shown to yield representations for the Laguerre-Hahn orthogonal polynomials. Lax pairs are given, formed from the differential system and the recurrence relation, that yield discrete non-linear equations for the three term recurrence relation coefficients of the Laguerre-Hahn orthogonal polynomials.

KEYWORDS: Orthogonal polynomials on the real line, Stieltjes functions, Riccati differential equations, matrix Sylvester differential equations.
AMS SUBJECT CLASSIFICATION (2000): 33C47, 42C05.

## 1. Introduction

Laguerre-Hahn orthogonal polynomials arise in several subjects of mathematics, such as probability theory [9], differential equations [8, 14, 17], measure perturbation theory [6], constructive theory of approximation [11].

From an analytical point of view, Laguerre-Hahn orthogonal polynomials can be regarded as a generalization of semi-classical orthogonal polynomials, since they are related to Stieltjes functions that are solutions of Riccati differential equations

$$AS' = BS^2 + CS + D \,, \tag{1}$$

where $A - D$ are polynomials (note that in the semi-classical case there holds $B \equiv 0$) [1, 7, 15].

In this paper we present a study of the Laguerre-Hahn families of orthogonal polynomials in terms of matrix Sylvester differential equations. Such a connection is established by means of the equivalence between (1) and the matrix Sylvester differential equations

$$AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C} \,, \quad n \geq 0 \,, \tag{2}$$

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where $Y_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}$, with $\{P_n\}$ the sequence of monic orthogonal polynomials related to $S$ and $\{P_n^{(1)}\}$ the corresponding sequence of associated polynomials (also known as the numerator polynomials), and $\mathcal{B}_n, \mathcal{C}$ matrices with polynomial entries. The theory of matrix Sylvester differential equations [16] provides a representation for the solutions of (2) in terms of the solutions of two first order linear systems, which we will relate to the semi-classical orthogonal polynomials.

Our first main result is contained in Theorem 1, where we show the equivalence between (1), (2),

$$AQ_n' = (\mathcal{B}_n + (BS + C/2)I)Q_n\,, \quad Q_n = \begin{bmatrix} q_{n+1} \\ q_n \end{bmatrix}, \quad n \geq 0\,, \tag{3}$$

and

$$A\mathcal{A}_n' = \mathcal{B}_n\mathcal{A}_n - \mathcal{A}_n\mathcal{B}_{n-1}\,, \quad \mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}, \quad n \geq 1\,, \tag{4}$$

with $q_n$ the functions of the second kind, $I$ the identity matrix, and where $\beta_n, \gamma_n$ are the recurrence relation coefficients of $\{P_n\}$.

As a consequence of the previous equivalence, we obtain that a necessary and sufficient condition for a sequence of monic polynomials $\{P_n\}$, orthogonal with respect to a weight $w$, to be semi-classical is that the following differential system holds (cf. Theorem 2):

$$A\widetilde{Y}_n' = (\mathcal{B}_n - C/2\,I)\,\widetilde{Y}_n\,, \quad \widetilde{Y}_n = \begin{bmatrix} P_{n+1} & q_{n+1}/w \\ P_n & q_n/w \end{bmatrix}, \quad n \geq 1\,,$$

where $C$ is a polynomial (similar differential systems had been studied, for example, in [13, Section 3]). Such a characterization for semi-classical orthogonal polynomials allows us to deduce a representation for the solutions of (2) as $Y_n = \widetilde{\mathcal{P}}_n\mathcal{L}^{-1}$, where $\mathcal{L}$ satisfies $A\mathcal{L}' = \mathcal{C}\mathcal{L}$ and $\widetilde{\mathcal{P}}_n$ is defined in terms of a semi-classical family, say $\{\tilde{P}_n\}$ (cf. Lemma 3 and Theorem 4). Furthermore, it is shown that the Stieltjes functions related to $\{P_n\}$ and $\{\tilde{P}_n\}$ are a fractional linear transformation of each other (cf. Lemma 3).

The above referred results will be applied to the study of the Laguerre-Hahn family of class zero, i.e., $\max\{\deg(A), \deg(B)\} \leq 2$ and $\deg(C) = 1$ in (1): in the Theorem 5 we show the solutions of the equation (4) in closed form expressions for the recurrence relation coefficients; in the Theorem 6 we show a representation of a given sequence of Laguerre-Hahn orthogonal polynomials.

Let us emphasize that the equations (4), enclosing nonlinear difference equations for the recurrence coefficients $\beta_n$ and $\gamma_n$, are given by the Lax pair

$$Y_n = \mathcal{A}_n Y_{n-1}\,, \quad AY_n' = \mathcal{B}_n Y_n - Y_n\mathcal{C}\,.$$

The equations (4) can be viewed as comparable to the so-called Laguerre-Freud's equations that hold for semi-classical families [12]. There are studies on the recurrence coefficients of (semi-classical) orthogonal polynomials showing its relevance in the theory of integrable systems and Painlevé equations, and we refer the reader to [13] and the references therein (see also [4]).

This paper is organized as follows. In Section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In Section 3 we establish the equivalence between (1), (2), (3), (4), and we deduce the characterization for semi-classical orthogonal polynomials in terms of first order differential systems. In Section 4 we study the representation of Laguerre-Hahn orthogonal polynomials. The Section 5 is devoted to the study of Laguerre-Hahn orthogonal polynomials of class zero.

## 2. Preliminary Results

Let $\mathbb{P} = \text{span}\{z^k : k \in \mathbb{N}_0\}$ be the space of polynomials with complex coefficients, and let $\mathbb{P}'$ be its algebraic dual space. We will denote by $\langle u, f\rangle$ the action of $u \in \mathbb{P}'$ on $f \in \mathbb{P}$.

Given the moments of $u$, $u_n = \langle u, x^n\rangle$, $n \geq 0$, where we take $u_0 = 1$, the principal minors of the corresponding Hankel matrix are defined by $H_n = \det((u_{i+j})_{i,j=0}^n), n \geq 0$, where, by convention, $H_{-1} = 1$. $u$ is said to be *quasi-definite* (respectively, *positive-definite*) if $H_n \neq 0$ (respectively, $H_n > 0$), for all $n \geq 0$.

**Definition 1.** (see [18]) Let $u \in \mathbb{P}'$ and let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials such that $\deg(P_n) = n\,,\ n \geq 0\,.$ $\{P_n\}$ is said to be a sequence of orthogonal polynomials with respect to $u$ if

$$\langle u, P_n P_m\rangle = h_n \delta_{n,m}\,,\ h_n = \langle u, P_n^2\rangle \neq 0\,, \quad n, m \geq 0\,. \tag{5}$$

Throughout the paper we shall take each $P_n$ monic, that is, $P_n(z) = z^n+$ lower degree terms, and we will denote $\{P_n\}$ by SMOP.

The equivalence between the quasi-definiteness of $u \in \mathbb{P}'$ and the existence of a SMOP with respect to $u$ is well-known in the literature of orthogonal polynomials (see [5, 18]). Furthermore, if $u$ is positive-definite, then it has an integral representation in terms of a positive Borel measure, $\mu$, supported on an infinite set of points of the real line, $I$, such that

$$\langle u, x^n \rangle = \int_I x^n \, d\mu(x) \, , \quad n \geq 0 \, , \tag{6}$$

and the orthogonality condition (5) becomes

$$\int_I P_n(x) P_m(x) d\mu(x) = h_n \delta_{n,m} \, , \quad h_n > 0 \, , \qquad n, m \geq 0 \, .$$

Further, if $\mu$ is defined in terms of a weight $w$, $d\mu(x) = w(x)dx$, then we will also say that $\{P_n\}$ is orthogonal with respect to $w$.

Monic orthogonal polynomials satisfy a three term recurrence relation [18]

$$P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x) \, , \qquad n = 1, 2, \ldots \tag{7}$$

with $P_0(x) = 1, \; P_1(x) = x - \beta_0$ and $\gamma_n \neq 0, \; n \geq 1, \; \gamma_0 = 1$.

**Definition 2.** Let $\{P_n\}$ be the SMOP with respect to a linear functional $u$. The sequence of *associated polynomials* is defined by

$$P_n^{(1)}(x) = \langle u_t, \frac{P_{n+1}(x) - P_{n+1}(t)}{x - t} \rangle \, , \qquad n \geq 0 \, ,$$

where $u_t$ denotes the action of $u$ on the variable $t$.

**Definition 3.** The *Stieltjes function* of $u \in \mathbb{P}'$ is defined in terms of the moments, $(u_n)$, of $u$ by $S(x) = -\displaystyle\sum_{n=0}^{+\infty} \frac{u_n}{x^{n+1}}$.

$S$ has an expansion in terms of a continued fraction, given by

$$S(x) = \cfrac{1}{x - \beta_0 - \cfrac{\gamma_1}{x - \beta_1 - \cfrac{\gamma_2}{\ddots}}} \tag{8}$$

where the $\gamma$'s and the $\beta$'s are the three term recurrence relation coefficients of the corresponding SMOP. Note that if $u$ is positive-definite, defined by (6), then $S$ is given by $S(x) = \displaystyle\int_I \frac{d\mu(t)}{x - t} \, , \quad x \in \mathbb{C} \setminus I$.

The sequence of *functions of the second kind* corresponding to $\{P_n\}$ is defined as follows:

$$q_{n+1} = P_{n+1} S - P_n^{(1)} \, , \qquad n \geq 0 \, , \; q_0 = S \, .$$

**Definition 4.** (see [15]) A Stieltjes function, $S$, is said to be *Laguerre-Hahn* if there exist polynomials $A, B, C, D$, with $A \neq 0$, such that it satisfies a Riccati differential equation

$$AS' = BS^2 + CS + D. \tag{9}$$

The corresponding sequence of orthogonal polynomials is called *Laguerre-Hahn*. If $B = 0$, then $S$ is said to be *Laguerre-Hahn affine* or *semi-classical*.

If $S$ is related to a positive-definite linear functional defined in terms of a weight, $w$, then the semi-classical character of $S$ means $w'/w = C/A$, with $A, C$ polynomials, and such a differential equation is equivalent to $AS' = CS + D$, where $D$ is a polynomial given in terms of $A, C$ (see [15]).

Note that (9) is equivalent to the distributional equation [7, 15]

$$\mathcal{D}(Au) = \psi u + B(x^{-1}u^2)\,, \quad \psi = A' + C\,.$$

In the sequel we will use the following matrices:

$$Y_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}, \; \widetilde{Y}_n = \begin{bmatrix} P_{n+1} & q_{n+1}/w \\ P_n & q_n/w \end{bmatrix}, \; Q_n = \begin{bmatrix} q_{n+1} \\ q_n \end{bmatrix}, \; n \geq 0\,. \tag{10}$$

**Lemma 1.** *Let $\{P_n\}$ be a SMOP and let $\beta_n, \gamma_n$ be the coefficients of the three term recurrence relation (7). Let $\{Y_n\}$, $\{\widetilde{Y}_n\}$, $\{Q_n\}$ be the sequences defined in (10). Then,*
*(a) $Y_n$ and $\widetilde{Y}_n$ satisfy the difference equation*

$$X_n = \mathcal{A}_n X_{n-1}\,, \;\; \mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}, \quad n \geq 1\,, \tag{11}$$

*with initial conditions* $Y_0 = \begin{bmatrix} x-\beta_0 & 1 \\ 1 & 0 \end{bmatrix}$, $\widetilde{Y}_0 = \begin{bmatrix} x-\beta_0 & q_1/w \\ 1 & q_0/w \end{bmatrix}$;
*(b) $Q_n$ satisfies*

$$Q_n = \mathcal{A}_n Q_{n-1}\,, \quad n \geq 1\,, \tag{12}$$

*with $\mathcal{A}_n$ given in (11) and initial conditions* $Q_0 = \begin{bmatrix} (x-\beta_0)S - 1 \\ S \end{bmatrix}$.

Throughout the paper $I$ denotes the $2 \times 2$ identity matrix. The $(i, j)$ entry of a matrix $X$ will be denoted by $X^{(i,j)}$.

## 3. Characterization in terms of matrix Sylvester differential equations

**Theorem 1.** *Let $S$ be a Stieltjes function, let $\{Y_n\}$ and $\{Q_n\}$ be the corresponding sequences defined in (10), and let $\beta_n, \gamma_n$ be the corresponding recurrence relation coefficients. The following statements are equivalent:*
*(a) $S$ satisfies*

$$AS' = BS^2 + CS + D\,, \quad A, B, C, D \in \mathbb{P}\,;$$

*(b) $Y_n$ satisfies the matrix Sylvester equation*

$$AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}\,, \quad n \geq 0\,, \tag{13}$$

*where*

$$\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & l_{n-1} + (x-\beta_n)\Theta_{n-1}/\gamma_n \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix}$$

*with $l_n, \Theta_n$ polynomials of uniformly bounded degrees, satisfying the initial conditions*

$$A = (l_0 - C/2)(x-\beta_0) - B + \Theta_0\,,\ 0 = D(x-\beta_0) + l_0 + C/2\,,$$
$$\Theta_{-1} = D,\ l_{-1} = C/2\,; \tag{14}$$

*(c) the matrices defined in (11), $\mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}$, satisfy*

$$A\mathcal{A}_n' = \mathcal{B}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{B}_{n-1}\,, \quad n \geq 1\,; \tag{15}$$

*(d) $Q_n$ satisfies*

$$AQ_n' = (\mathcal{B}_n + (BS + C/2)I)Q_n\,, \quad n \geq 0\,.$$

*Proof*: Taking into account [3, Theorem 2] we only need to prove $(b) \Leftrightarrow (c)$.
$(b) \Rightarrow (c)$. To obtain (15) we take derivatives on $Y_n = \mathcal{A}_n Y_{n-1}$ (cf. (11)), and substitute it in (13), thus obtaining

$$A\mathcal{A}_n' Y_{n-1} + A\mathcal{A}_n Y_{n-1}' = \mathcal{B}_n Y_n - Y_n \mathcal{C}\,.$$

Using (13) for $n-1$ in the previous equation we get

$$A\mathcal{A}_n' Y_{n-1} + \mathcal{A}_n\left(\mathcal{B}_{n-1} Y_{n-1} - Y_{n-1}\mathcal{C}\right) = \mathcal{B}_n Y_n - Y_n \mathcal{C}\,.$$

Using the recurrence relation (11) for $Y_n$ we obtain

$$A\mathcal{A}_n' Y_{n-1} + \mathcal{A}_n\left(\mathcal{B}_{n-1} Y_{n-1} - Y_{n-1}\mathcal{C}\right) = \mathcal{B}_n \mathcal{A}_n Y_{n-1} - \mathcal{A}_n Y_{n-1}\mathcal{C}\,,$$

that is,

$$A\mathcal{A}_n' Y_{n-1} = (\mathcal{B}_n\mathcal{A}_n - \mathcal{A}_n\mathcal{B}_{n-1})\, Y_{n-1}\,.$$

Since $Y_n$ is nonsingular, for all $n \geq 0$, there follows (15).
$(c) \Rightarrow (b)$. If we multiply (15) by $Y_{n-1}$, we obtain

$$A(\mathcal{A}_n Y_{n-1})' - A\mathcal{A}_n Y_{n-1}' = \mathcal{B}_n\mathcal{A}_n Y_{n-1} - \mathcal{A}_n\mathcal{B}_{n-1}Y_{n-1}\,.$$

Taking into account the recurrence relation (11) for $Y_n$ we get

$$AY_n' - \mathcal{B}_n Y_n = \mathcal{A}_n(AY_{n-1}' - \mathcal{B}_{n-1}Y_{n-1})\,,$$

thus

$$AY_n' - \mathcal{B}_n Y_n = \mathcal{A}_n \cdots \mathcal{A}_2(AY_1' - \mathcal{B}_1 Y_1)\,.$$

The use of $\mathcal{A}_n \cdots \mathcal{A}_2 = Y_n Y_1^{-1}$ in the preceding equation yields an equation for $Y_n$ of the Sylvester type (13), $AY_n' = \mathcal{B}_n Y_n - Y_n \widetilde{C}$, with $\widetilde{C} = -Y_1^{-1}(AY_1' - \mathcal{B}_1 Y_1)$. ■

**Corollary 1.** *The following relations hold:*

$$\operatorname{tr}(\mathcal{B}_n) = 0\,, \quad n \geq 0\,, \tag{16}$$

$$\det(\mathcal{B}_n) = \det(\mathcal{B}_0) + A\sum_{k=1}^{n} \frac{\Theta_{k-1}}{\gamma_k}\,,\ n \geq 1\,, \tag{17}$$

*and* $\det(\mathcal{B}_0) = D(A+B) - (C/2)^2\,.$

*Proof*: The formulas for the trace and determinant of the matrices $\mathcal{B}_n$ involved in equations of the same type as (15) were deduced in [2, Lemma 2.5]. The use of the initial conditions (14) yields (16) and (17). ■

*Remark* 1. Since $\operatorname{tr}(\mathcal{B}_n) = 0$, henceforth we parameterize the matrix $\mathcal{B}_n$ in terms of the two functions $l_n$ and $\Theta_n$ as $\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & -l_n \end{bmatrix}$.

*Remark* 2. Equation (15) reads as

$$\begin{cases} (x-\beta_n)(l_n - l_{n-1}) = A - \Theta_n + \frac{\gamma_n}{\gamma_{n-1}}\Theta_{n-2} \\ l_n - l_{n-2} = -\frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1} + \frac{(x-\beta_{n-1})}{\gamma_{n-1}}\Theta_{n-2}\,, \quad n \geq 1\,, \end{cases}$$

which are comparable with the so-called Laguerre-Freud's equations [12, 13].

**3.1. Characterization of semi-classical orthogonal polynomials.** As a consequence of the previous theorem we deduce the characterization that follows.

**Theorem 2.** *Let* $\{P_n\}$ *be a SMOP with respect to a weight* $w$*, and let* $\{q_n\}$ *be the corresponding sequence of functions of the second kind. The weight* $w$ *is semi-classical and satisfies* $w'/w = C/A$ *if, and only if,* $\widetilde{Y}_n = \begin{bmatrix} P_{n+1} & q_{n+1}/w \\ P_n & q_n/w \end{bmatrix}$ *satisfies the matrix differential equation*

$$A\widetilde{Y}_n' = \left(\mathcal{B}_n - \frac{C}{2}I\right)\widetilde{Y}_n\,, \quad n \geq 1\,, \tag{18}$$

*where* $\mathcal{B}_n$ *is the matrix associated with the equation* $AS' = CS + D$ *for the Stieltjes function of* $w$*.*

*Proof*: Note that $w'/w = C/A$ implies $AS' = CS + D$, $D \in \mathbb{P}$, for the corresponding Stieltjes function (see [15]). Taking into account the Theorem 1 we get

$$\begin{cases} AP_{n+1}' = (l_n - C/2)P_{n+1} + \Theta_n P_n \\ Aq_{n+1}' = (l_n + C/2)q_{n+1} + \Theta_n q_n\,, \quad n \geq 0\,. \end{cases} \tag{19}$$

Therefore, using the three term recurrence relation (7) for $\{P_n\}$ we obtain

$$AP_n' = \left(l_{n-1} - C/2 + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\right)P_n - \frac{\Theta_{n-1}}{\gamma_n}P_{n+1}\,, \quad n \geq 1\,. \tag{20}$$

From (19) there follows

$$A\left(\frac{q_{n+1}}{w}\right)' = \Theta_n\frac{q_n}{w} + (l_n - C/2)\,\frac{q_{n+1}}{w}\,, \tag{21}$$

where we used $w'/w = C/A$. Furthermore, using the three term recurrence relation for $\{q_n\}$ (cf. (12)) we obtain

$$A\left(\frac{q_n}{w}\right)' = \left(l_{n-1} - C/2 + \frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1}\right)\frac{q_n}{w} - \frac{\Theta_{n-1}}{\gamma_n}\frac{q_{n+1}}{w}\,, \quad n \geq 1\,. \tag{22}$$

Equations. (19)-(22) yield

$$A\widetilde{Y}_n' = \widetilde{\mathcal{B}}_n\widetilde{Y}_n\,,\ \widetilde{\mathcal{B}}_n = \begin{bmatrix} l_n - C/2 & \Theta_n \\ -\Theta_{n-1}/\gamma_n & l_{n-1} + (x-\beta_n)\Theta_{n-1}/\gamma_n - C/2 \end{bmatrix},\ n \geq 1\,,$$

thus we get (18).

Conversely, if $\widetilde{Y}_n$ satisfies (18), then

$$(\det(\widetilde{Y}_n))' = \frac{\operatorname{tr}(\mathcal{B}_n - C/2\,I)}{A} \det(\widetilde{Y}_n)\,.$$

Since $\det(\widetilde{Y}_n) = (\gamma_1 \ldots \gamma_n)/w$ and $\operatorname{tr}(\mathcal{B}_n - C/2\,I) = -C$, there follows $w'/w = C/A$, thus $w$ is semi-classical. ■

*Remark* 3. The analogue result for orthonormal polynomials was established in [13].

## 4. Matrix Sylvester equations and Radon's Lemma

The theorem that follows is a particular case of the result known, in the literature of matrix Riccati equations, as Radon's Lemma [10, 16].

**Theorem 3.** *Let $A$ be a polynomial, let $\mathcal{B}_n/A$, $n \geq 1$, and $\mathcal{C}/A$ be matrices whose entries are integrable functions in a domain $G$ of the complex plane, and let $x_0 \in G$. If the matrices $\mathcal{P}_n$ and $\mathcal{L}$, $\mathcal{L}$ nonsingular, satisfy*

$$\begin{cases} A\mathcal{L}' = \mathcal{C}\mathcal{L}\,, \\ \mathcal{L}(x_0) = I\,, \end{cases} \tag{23}$$

*and*

$$\begin{cases} A\mathcal{P}_n' = \mathcal{B}_n\mathcal{P}_n\,, \quad n \geq 1\,, \\ \mathcal{P}_n(x_0) = Y_n(x_0)\,, \end{cases} \tag{24}$$

*then the solution of $AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}$, in $G$, is given by:*

$$Y_n = \mathcal{P}_n \mathcal{L}^{-1}\,, \quad n \geq 1\,.$$

Our aim is to find a representation for $Y_n$ satisfying (13), $AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}$, related to $AS' = BS^2 + CS + D$ (cf. Theorem 1). To that end we start with some remarks on the solution of the corresponding problem (24). Note that we are searching for matrices $\mathcal{P}_n$ of order two satisfying

$$A\mathcal{P}_n' = \mathcal{B}_n \mathcal{P}_n\,. \tag{25}$$

Hereafter we will consider $x_1 \in \mathbb{C}$ and $\tilde{C}$ a polynomial such that $\int_{x_1}^{x} \frac{\tilde{C}(t)}{2A(t)} dt$ is defined in suitable domains.

**Lemma 2.** *Let $\mathcal{B}_n$ be the matrices given in (13), and let $\tilde{C}$ be a polynomial. A matrix $\widetilde{Y}_n$ satisfies*

$$A\widetilde{Y}_n' = (\mathcal{B}_n - \tilde{C}/2\,I)\widetilde{Y}_n \tag{26}$$

*if, and only if, $\mathcal{P}_n = e^{\int_{x_1}^{x} \frac{\tilde{C}}{2A} dt}\, \widetilde{Y}_n$ satisfies (25).*

Taking into account the previous Lemma, we will solve (25) by considering

$$\mathcal{P}_n = e^{\int_{x_1}^{x} \frac{\tilde{C}}{2A} dt}\, \widetilde{Y}_n \,,$$

where $\widetilde{Y}_n$ satisfies (26). Furthermore, taking into account the Theorem 2, we will search for $\widetilde{Y}_n$ defined by

$$\widetilde{Y}_n = \begin{bmatrix} \tilde{P}_{n+1} & \tilde{q}_{n+1}/\tilde{w} \\ \tilde{P}_n & \tilde{q}_n/\tilde{w} \end{bmatrix}, \tag{27}$$

with $\{\tilde{P}_n\}$ a SMOP with respect to a weight function $\tilde{w}$ and $\{\tilde{q}_n\}$ the corresponding sequence of functions of the second kind.

*Remark* 4. Note that (26) implies $\det(\widetilde{Y}_n)' = \frac{\operatorname{tr}(\mathcal{B}_n - \tilde{C}/2\, I)}{A}\widetilde{Y}_n$, which combined with (27) yields $\tilde{w}'/\tilde{w} = \tilde{C}/A$, thus $\tilde{w} = e^{\int \tilde{C}/A}$.

**Lemma 3.** *Let $S$ be a Stieltjes function that satisfies the Riccati differential equation $AS' = BS^2 + CS + D$. Let $\{P_n\}$ be the corresponding SMOP such that the equations (13), $AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}$, hold, and let (24) be the corresponding system $A\mathcal{P}_n' = \mathcal{B}_n \mathcal{P}_n$. Let the following assumption hold:*

$$\exists \tilde{C} \in \mathbb{P},\ \exists n_0 \geq 1 : \mathcal{P}_n = e^{\int_{x_1}^{x} \frac{\tilde{C}(t)}{2A(t)}\, dt}\, \widetilde{Y}_{n-n_0} \,, \quad n \geq n_0 + 1 \,, \tag{28}$$

*with the $\widetilde{Y}_n$'s given as in (27), related to a SMOP $\{\tilde{P}_n\}$. Denote by $\tilde{S}$ the Stieltjes functions related to $\{\tilde{P}_n\}$. Then, the following statements hold:*
*(a) $\tilde{S}$ is a fractional linear transformation of $S$,*

$$\tilde{S} = \frac{a + bS}{c + dS}\,, \quad a, b, c, d \in \mathbb{P}\,; \tag{29}$$

*(b) $\tilde{S}$ satisfies*

$$A\tilde{S}' = \tilde{C}\tilde{S} + \tilde{D}\,, \quad \tilde{D} \in \mathbb{P}\,, \tag{30}$$

*the polynomials $\tilde{C}, \tilde{D}$ being related to $B, C, D$ by*

$$(bc - ad)B = (bd' - b'd)A + bd\tilde{C} + d^2\tilde{D}\,, \tag{31}$$

$$(bc - ad)C = (bc' + ad' - b'c - a'd)A + (ad + bc)\tilde{C} + 2cd\tilde{D}\,, \tag{32}$$

$$(bc - ad)D = (ac' - a'c)A + ac\,\tilde{C} + c^2\tilde{D}\,. \tag{33}$$

*Proof*: (a). Let us denote by $\tilde{\mathcal{A}}_n$ the matrices of the recurrence relation of $\{\widetilde{Y}_n\}$. From $\mathcal{P}_n = e^{\int_{z_1}^{z} \frac{\tilde{C}}{2A} dt}\widetilde{Y}_{n-n_0}$ and $\widetilde{Y}_{n-n_0} = \tilde{\mathcal{A}}_{n-n_0}\widetilde{Y}_{n-n_0-1}$ there follows

$$\mathcal{P}_n = \tilde{\mathcal{A}}_{n-n_0}\mathcal{P}_{n-1}\,, \quad n \geq n_0 + 1\,. \tag{34}$$

The substitution of (34) into $A\mathcal{P}_n' = \mathcal{B}_n\mathcal{P}_n$ yields

$$A(\tilde{\mathcal{A}}_{n-n_0}\mathcal{P}_{n-1})' = \mathcal{B}_n\tilde{\mathcal{A}}_{n-n_0}\mathcal{P}_{n-1}\,, \quad n \geq n_0+1\,,$$

from which we obtain

$$A\tilde{\mathcal{A}}_{n-n_0}' = \mathcal{B}_n\tilde{\mathcal{A}}_{n-n_0} - \tilde{\mathcal{A}}_{n-n_0}\mathcal{B}_{n-1}\,, \quad n \geq n_0+1\,. \tag{35}$$

Using the equations enclosed in the positions $(2,1)$ and $(2,2)$ of (35) we obtain

$$\tilde{\gamma}_{n-n_0} = \gamma_n, \quad \tilde{\beta}_{n-n_0} = \beta_n, \quad n \geq n_0+1\,.$$

Consequently, taking into account the representation of the Stieltjes function in terms of a continued fraction (8), (29) follows.
(b). From (28) we obtain that $\{\tilde{P}_n\}$ is semi-classical, and the corresponding $\tilde{w}$ satisfies $\tilde{w}'/\tilde{w} = \tilde{C}/A$ (note the remark 4). Thus, the first order differential equation (30) for the corresponding Stieltjes function, $\tilde{S}$, follows.

By substituting $\tilde{S}$ given by (29) in (30) we get

$$A(bc-ad)S' = B_1S^2 + C_1S + D_1 \tag{36}$$

with

$$\begin{aligned} B_1 &= (bd'-b'd)A + bd\tilde{C} + d^2\tilde{D}\,, \\ C_1 &= (bc'+ad'-b'c-a'd)A + (ad+bc)\tilde{C} + 2cd\tilde{D}\,, \\ D_1 &= (ac'-a'c)A + ac\tilde{C} + c^2\tilde{D}. \end{aligned}$$

Since $S$ satisfies $AS' = BS^2 + CS + D$ as well as (36), there follows, if $A-D$ and $B_1 - D_1$ are non vanishing, that

$$\varrho = \frac{B}{B_1} = \frac{C}{C_1} = \frac{D}{D_1}\,, \quad \varrho = 1/(bc-ad)\,.$$

Hence, we obtain (31)-(33). ∎

**Theorem 4.** *Let $S$ be a Stieltjes function satisfying $AS' = BS^2 + CS + D$. To $AS' = BS^2 + CS + D$ associate $AY_n' = \mathcal{B}_nY_n - Y_n\mathcal{C}$ and the corresponding systems (23) and (24), $A\mathcal{L}' = \mathcal{C}\mathcal{L}, A\mathcal{P}_n' = \mathcal{B}_n\mathcal{P}_n$. Let $G$ be a domain in the complex plane such that the entries of the matrices $\mathcal{B}_n/A$ and $\mathcal{C}/A$ are integrable in $G$. Assume that the assumption (28) holds. Then, there exists a polynomial $\tilde{C}$ (defined by (31)-(33)) and a weight function $\tilde{w} = e^{\int\frac{\tilde{C}}{A}}$ such the following representation holds in $G$:*

$$Y_n = \begin{bmatrix} \sqrt{\tilde{w}}\tilde{P}_{n-n_0+1} & \tilde{q}_{n-n_0+1}/\sqrt{\tilde{w}} \\ \sqrt{\tilde{w}}\tilde{P}_{n-n_0} & \tilde{q}_{n-n_0}/\sqrt{\tilde{w}} \end{bmatrix} E_n\mathcal{L}^{-1}\,, \quad n \geq n_0+1\,,$$

*with* $\{\tilde{P}_n\}$ *the SMOP with respect to* $\tilde{w}$, $\{\tilde{q}_n\}$ *the corresponding sequence of functions of the second kind, and* $E_n = \mathcal{P}_n(x_0)^{-1}Y_n(x_0)$, $x_0 \in G$.

*Proof*: The result follows from the Theorem 3 combined with the Lemma 3 (note the remark 4). ∎

## 5. Laguerre-Hahn orthogonal polynomials of class zero

Laguerre-Hahn orthogonal polynomials of class zero are related to Stieltjes functions, $S$, satisfying

$$AS' = BS^2 + CS + D\,, \quad \deg(C) = 1\,, \quad \max\{\deg(A), \deg(B)\} \leq 2\,.$$

There are, up to a linear change of variable, three canonical cases for the triples of polynomials satisfying the above Riccati differential equation, according to the degree of $A$ be zero, one or two [1].

In subsection 5.1 we will show the solutions of the corresponding difference equation (15) in the class zero: we give closed form expressions for the recurrence relation coefficients of the orthogonal polynomials. In subsection 5.2 we will use Radon's Lemma in order to obtain representations for a sequence of Laguerre-Hahn orthogonal polynomials.

### 5.1. The recurrence relation coefficients.

**Theorem 5.** *Let* $S$ *be a Stieltjes function satisfying* $AS' = BS^2 + CS + D$ *with* $\deg(C) = 1, \max\{\deg(A), \deg(B)\} \leq 2$. *Let* $A(x) = a_2x^2 + a_1x + a_0, B(x) = b_2x^2 + b_1x + b_0, C(x) = c_1x + c_0$. *The recurrence relation coefficients of the SMOP* $\{P_n\}$ *related to* $S$ *are given by*

$$\beta_n = \frac{(r_1^2 - a_2^2)\beta_1 + 2a_1(n-1)(r_1 - (n-1)a_2)}{(r_1 - (2n-1)a_2)(r_1 - (2n-3)a_2)}\,, \quad n \geq 2\,, \tag{37}$$

$$\gamma_{n+1} = \frac{r_1(r_1 - 2a_2)\gamma_2 + \sum_{k=2}^{n} A(\beta_k)(r_1 - 2(k-1)a_2)}{(r_1 - 2na_2)(r_1 - 2(n-1)a_2)}\,, \quad n \geq 2\,, \tag{38}$$

*where*

$$\beta_1 = \frac{a_1 + 2\beta_0 D - c_0}{r_1 - a_2}\,, \quad r_1 = \Theta_0/\gamma_1\,, \quad \Theta_0 = A + B + (x-\beta_0)^2 D + (x-\beta_0)C\,,$$

$$\gamma_1 = \frac{\Theta_0}{2D + c_1 - a_2}\,, \quad \gamma_2 = \frac{A(\beta_1) + \gamma_1 D}{r_1 - 2a_2}\,. \tag{39}$$

*Proof*: The corresponding equations (15) read as

$$\begin{cases} (x-\beta_n)(l_n-l_{n-1}) = A - \Theta_n + \frac{\gamma_n}{\gamma_{n-1}}\Theta_{n-2} \\ l_n - l_{n-2} = -\frac{(x-\beta_n)}{\gamma_n}\Theta_{n-1} + \frac{(x-\beta_{n-1})}{\gamma_{n-1}}\Theta_{n-2}\,, \quad n \geq 1\,, \end{cases} \tag{40}$$

where the initial conditions hold (cf. (14)):

$$l_{-1}(x) = C(x)/2\,,\ l_0(x) = -(x-\beta_0)D - C(x)/2\,,$$
$$\Theta_{-1} = D\,,\ \Theta_0 = A(x) + B(x) + (x-\beta_0)^2 D + (x-\beta_0)C(x)\,.$$

Note that under the stated conditions on the degrees of $A, B, C$, one has

$$l_n(x) = \ell_{n,1}x + \ell_{n,0}\,,\quad \Theta_n, D \text{ constants.} \tag{41}$$

By substituting (41) into (40) and equating the coefficients of $x^2, x, x^0$ we obtain, for all $n \geq 1$,

$$\ell_{n,1} - \ell_{n-1,1} = a_2\,, \tag{42}$$
$$\ell_{n,0} - \ell_{n-1,0} = \beta_n(\ell_{n,1} - \ell_{n-1,1}) + a_1\,, \tag{43}$$
$$-\beta_n(\ell_{n,0} - \ell_{n-1,0}) = a_0 - \Theta_n + \frac{\gamma_n}{\gamma_{n-1}}\Theta_{n-2}\,,$$
$$\ell_{n,1} - \ell_{n-2,1} = -\frac{\Theta_{n-1}}{\gamma_n} + \frac{\Theta_{n-2}}{\gamma_{n-1}}\,, \tag{44}$$
$$\ell_{n,0} - \ell_{n-2,0} = \beta_n\frac{\Theta_{n-1}}{\gamma_n} - \beta_{n-1}\frac{\Theta_{n-2}}{\gamma_{n-1}}\,. \tag{45}$$

From (42) and (43) there follows

$$\ell_{n,1} = \ell_{0,1} + na_2\,,\quad n \geq 1\,, \tag{46}$$
$$\ell_{n,0} = \ell_{0,0} + a_2\sum_{k=1}^{n}\beta_k + na_1\,,\quad n \geq 1\,. \tag{47}$$

Note that (46) and (47) are also valid for $n = 0$, using the convention $\sum_{k=i}^{j} \cdot = 0$ whenever $i > j$.

The use of (46) for $n$ and $n-2$ in (44) yields

$$\frac{\Theta_{n-1}}{\gamma_n} = \frac{\Theta_{n-2}}{\gamma_{n-1}} - 2a_2\,,\quad n \geq 2\,,$$

thus

$$\frac{\Theta_{n-1}}{\gamma_n} = \frac{\Theta_0}{\gamma_1} - 2(n-1)a_2\,,\ n \geq 2.$$

Note that the above equality also holds for $n = 1$, thus we get

$$\Theta_{n-1} = (r_1 - 2(n-1)a_2)\gamma_n\,, \quad r_1 = \frac{\Theta_0}{\gamma_1}\,, \quad n \geq 1\,. \tag{48}$$

The use of (47) for $n$ and $n-2$ as well as (48) for $n$ and $n-1$ in (45) yields $a_2(\beta_{n-1} + \beta_n) + 2a_1 = \beta_n(r_1 - 2(n-1)a_2) - \beta_{n-1}(r_1 - 2(n-2)a_2)\,, \quad n \geq 2\,,$ that is,

$$\beta_n(r_1 - (2n-1)a_2) = \beta_{n-1}(r_1 - (2n-5)a_2) + 2a_1\,, \quad n \geq 2\,.$$

If we multiply the above equation by $r_1 - (2n-3)a_2$ we get

$$L_{n+1} = L_n + 2a_1(r_1 - (2n-3)a_2)\,,$$
$$L_n = \beta_{n-1}(r_1 - (2n-3)a_2)(r_1 - (2n-5)a_2)\,.$$

Therefore, we obtain

$$L_{n+1} = L_2 + 2a_1\sum_{k=2}^{n}(r_1 - (2k-3)a_2)\,, \quad n \geq 2\,,$$

thus for $n \geq 2$

$$\beta_n(r_1 - (2n-1)a_2)(r_1 - (2n-3)a_2) = \beta_1(r_1^2 - a_2^2) + 2a_1(n-1)(r_1 - (n-1)a_2)\,,$$

thus (37) follows.

To obtain an equation for the $\gamma$'s we start by evaluating the first equation of (40) at $\beta_n$, thus obtaining

$$\Theta_n - \frac{\gamma_n}{\gamma_{n-1}}\Theta_{n-2} = A(\beta_n)\,, \quad n \geq 1\,.$$

Using (48) for $n+1$ and $n-1$ in the above equation we get

$$\gamma_{n+1}(r_1 - 2na_2) - \gamma_n(r_1 - 2(n-2)a_2) = A(\beta_n)\,, \quad n \geq 2\,.$$

If we multiply the above equation by $r_1 - 2(n-1)a_2$ we get for $n \geq 2$

$$T_{n+1} = T_n + A(\beta_n)(r_1 - 2(n-1)a_2)\,, \quad T_n = \gamma_n(r_1 - 2(n-2)a_2)(r_1 - 2(n-1)a_2)\,.$$

Therefore, we obtain

$$T_{n+1} = T_2 + \sum_{k=2}^{n} A(\beta_k)(r_1 - 2(k-1)a_2)\,, \quad n \geq 2\,,$$

thus (38) follows.

(39) follows from the initial conditions. ∎

### 5.2. Representations of Laguerre-Hahn orthogonal polynomials of class zero via Radon's Lemma: an example.

**Lemma 4.** *Let $\{P_n\}$ be a SMOP related to $AS' = BS^2 + CS + D$, $A(x) = a_2x^2 + a_1x + a_0, B(x) = b_2x^2 + b_1x + b_0, C(x) = c_1x + c_0$ . Then, the matrices $\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & -l_n \end{bmatrix}$ involved in the corresponding equations (13) are defined by $l_n(x) = \ell_{n,1}x + \ell_{n,0}$ and $\Theta_n$ constant, where, for all $n \geq 1$,*

$$\begin{aligned} \ell_{n,1} &= c_1/2 + (n+1)a_2 + b_2 \,, && (49) \\ \ell_{n,0} &= c_0/2 + (n+1)a_1 + b_1 + \beta_0(b_2 + a_2) + \alpha_n a_2 \,, && (50) \\ \Theta_n &= (\ell_{n,0} + c_0/2 - (n-1)a_1)\alpha_n - (\ell_{n,1} + c_1/2 - (n-2)a_2)\nu_n \\ &\quad -(\nu_n + \beta_0\alpha_n - \gamma_1)D + na_0 \,, && (51) \end{aligned}$$

*and*

$$\ell_{0,1} = -D - c_1/2 \,, \;\; \ell_{0,0} = \beta_0 D - c_0/2 \,,$$
$$\Theta_0 = A + B + (x-\beta_0)^2 D + (x-\beta_0)C \,, \;\; \Theta_{-1} = D \,, \quad (52)$$

*with*

$$D = -a_2 - c_1 - b_2 \,, \;\; \alpha_n = \sum_{k=1}^{n} \beta_k \,, \;\; \nu_n = \sum_{1\leq i<j\leq n} \beta_i\beta_j - \sum_{k=2}^{n} \gamma_k \,, \quad n \geq 1 \,.$$

*Proof*: Write

$$\begin{aligned} P_n^{(1)}(x) &= x^n - \alpha_n x^{n-1} + \nu_n x^{n-2} + \cdots \\ P_{n+1}(x) &= x^{n+1} - (\alpha_n + \beta_0)x^n + (\nu_n + \beta_0\alpha_n - \gamma_1)x^{n-1} + \cdots \end{aligned}$$

and compare the coefficients of the corresponding monomials. To obtain $l_0(x), \Theta_0$, and $\Theta_{-1}$, use the initial conditions (14). ■

For later purposes we show some results concerning the Hermite polynomials.

**Lemma 5.** *Let $\{H_n\}$ denote the sequence of monic Hermite polynomials, and let $\{\tilde{q}_n\}$ be the corresponding sequence of functions of the second kind. The following statements hold:*
*(a) $\{H_n\}$ is related to the Stieltjes function $\tilde{S}$ satisfying*

$$A\tilde{S}' = \tilde{C}\tilde{S} + \tilde{D} \,, \quad A = 1, \; \tilde{C} = -2x, \; \tilde{D} = 2 \,; \quad (53)$$

*(b) the recurrence relation coefficients of* $\{H_n\}$ *are given by*

$$\tilde{\beta}_n = 0\,,\ \tilde{\gamma}_{n+1} = (n+1)/2\,,\quad n \geq 0\,; \tag{54}$$

*(c)* $\widetilde{H}_n = \begin{bmatrix} H_{n+1} & \tilde{q}_{n+1}/\tilde{w} \\ H_n & \tilde{q}_n/\tilde{w} \end{bmatrix}$, *with* $\tilde{w} = e^{-x^2}$, *satisfies the differential system*

$$\widetilde{H}_n' = \left(\tilde{\mathcal{B}}_n - \frac{\tilde{C}}{2} I\right) \widetilde{H}_n\,,\quad \tilde{\mathcal{B}}_n = \begin{bmatrix} -x & n+1 \\ -2 & x \end{bmatrix}\,,\quad n \geq 1\,. \tag{55}$$

*Proof*: (53) is given in [15]. (54) follows from (37) and (38). To obtain (55) we use the Theorem 2 (cf. (18)), where the entries of $\tilde{\mathcal{B}}_n$ can be obtained using (49)-(51). ∎

**Lemma 6.** *Let* $\{P_n\}$ *be the SMOP related to the Stieltjes function* $S$ *satisfying* $AS' = BS^2 + CS + D$, *where*

$$A = 1\,,\ B = -2x^2 + 2\lambda x + \rho - 1\,,\ C = 2x\,,\ D = 0\,,$$

*with* $\rho = 2\gamma_1, \lambda = \beta_0$, *and* $\rho\lambda \neq 0$. *Let* $\{Y_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}\}$ *be the sequence associated with* $\{P_n\}$ *defined in (10). The following statements hold:*
*(a) the recurrence relation coefficients of* $\{P_n\}$ *are given by*

$$\beta_n = 0\,,\ \gamma_{n+1} = n/2\,,\quad n \geq 1\,; \tag{56}$$

*(b)* $Y_n$ *satisfies the matrix Sylvester equations* $Y_n' = \mathcal{B}_n Y_n - \mathcal{C} Y_n$, $n \geq 0$, *with*

$$\mathcal{B}_0 = \begin{bmatrix} -x & \rho \\ 0 & x \end{bmatrix}\,,\quad \mathcal{B}_n = \begin{bmatrix} -x & n \\ -2 & x \end{bmatrix}\,,\ n \geq 1\,,\ \mathcal{C} = \begin{bmatrix} x & 0 \\ B & -x \end{bmatrix}\,. \tag{57}$$

*(c) let* $x_0$ *be an arbitrary point in the complex plane. The solution of the initial value problem*

$$\begin{cases} \mathcal{L}'(x) = \mathcal{C}(x)\mathcal{L}(x)\,, \\ \mathcal{L}(x_0) = I\,, \end{cases} \tag{58}$$

*with $\mathcal{C}$ given in (57), is*

$$\mathcal{L}^{(1,1)}(x) = e^{-x_0^2/2}\, e^{x^2/2}\,, \tag{59}$$

$$\mathcal{L}^{(1,2)}(x) \equiv 0\,, \tag{60}$$

$$\mathcal{L}^{(2,1)}(x) = e^{-x_0^2/2} e^{-x^2/2} \int_{x_0}^{x} B(s) e^{s^2} ds\,, \tag{61}$$

$$\mathcal{L}^{(2,2)}(x) = e^{x_0^2/2}\, e^{-x^2/2}\,, \tag{62}$$

*and the solution of the initial value problem*

$$\begin{cases} \mathcal{P}_n'(x) = \mathcal{B}_n(x)\mathcal{P}_n(x)\,, & n \geq 1\,, \\ \mathcal{P}_n(x_0) = Y_n(x_0)\,, \end{cases} \tag{63}$$

*with $\mathcal{B}_n$ given in (57), is for $n \geq 2$*

$$\mathcal{P}_n(x) = e^{-x^2/2} \widetilde{H}_{n-1}(x)\mathcal{K}_n\,, \quad \mathcal{K}_n = e^{x_0^2/2} \left(\widetilde{H}_{n-1}(x_0)\right)^{-1} Y_n(x_0)\,. \tag{64}$$

*Proof*: (56) follows from (37) and (38). To obtain the entries of $\mathcal{B}_n$ in (57) use (49)-(52).

Let us solve (63). Since $\widetilde{H}_n = \begin{bmatrix} H_{n+1} & \tilde{q}_{n+1}/\tilde{w} \\ H_n & \tilde{q}_n/\tilde{w} \end{bmatrix}$ satisfies (55),

$$\widetilde{H}_n' = \left(\tilde{\mathcal{B}}_n - \frac{\tilde{C}}{2} I\right) \widetilde{Y}_n\,, \quad \tilde{\mathcal{B}}_n = \begin{bmatrix} -x & n+1 \\ -2 & x \end{bmatrix}, \quad \tilde{C} = -2x\,, \quad n \geq 1\,,$$

then, taking into account the Lemma 2,

$$\mathcal{P}_n = e^{-x^2/2} \widetilde{H}_{n-1} \tag{65}$$

satisfies $\mathcal{P}_n' = \tilde{\mathcal{B}}_{n-1}\mathcal{P}_n, \quad n \geq 2$, that is, $\mathcal{P}_n' = \begin{bmatrix} -x & n \\ -2 & x \end{bmatrix} \mathcal{P}_n, \quad n \geq 2$. Hence, $\mathcal{P}_n$ defined by (65) is a fundamental matrix of the differential system in (63). Thus, a solution of the initial value problem (63) is given by (64). ■

*Remark* 5. Taking into account (54) and (56), one has $P_n^{(1)} = H_n,\ n \geq 1$. Consequently, the matrix $\mathcal{K}_n$ given in (64) is such that

$$\mathcal{K}_n^{(1,2)} = e^{x_0^2/2},\ \mathcal{K}_n^{(2,2)} = 0\,. \tag{66}$$

Furthermore, using $Y_{n+1}(x_0) = \mathcal{A}_{n+1}(x_0)Y_n(x_0)$ (cf. (11)), we obtain

$$\widetilde{H}_{n-1}(x_0)\mathcal{K}_n = \mathcal{A}_{n+1}^{-1}(x_0)\widetilde{H}_n(x_0)\mathcal{K}_{n+1}\,, \quad n \geq 2\,,$$

from which we get

$$\mathcal{K}_n^{(1,1)} = \mathcal{K}_{n+1}^{(1,1)}\,, \quad \mathcal{K}_n^{(2,1)} = \mathcal{K}_{n+1}^{(2,1)}\,, \quad n \geq 2\,. \tag{67}$$

**Theorem 6.** *Let the conditions and the notations of the two previous Lemmas hold. Then,*
*(a) the following representation holds, for all $n \geq 2$:*

$$P_{n+1}(x) = e^{-x^2}\left(e^{x_0^2/2}\mathcal{K}_n^{(1,1)} - \int_{x_0}^{x} B(s)e^{s^2}ds\right) H_n(x) + e^{x_0^2/2}\mathcal{K}_n^{(2,1)}\,\tilde{q}_n(x)\,, \tag{68}$$

*where $x_0$ is an arbitrary complex number;*
*(b) the Stieltjes functions $S$ and $\tilde{S}$ are related through*

$$\tilde{S} = \frac{a+bS}{c+dS}\,, \tag{69}$$

*where*

$$a = -1/d\,,\ b(x) = 1/d\,x - \lambda/d\,,\ c = 0\,,\ d = \pm\sqrt{\rho/2}\,. \tag{70}$$

*Proof*: (a). Taking into account the Theorem 3, one has $Y_n = \mathcal{P}_n\mathcal{L}^{-1}$, where $\mathcal{L}, \mathcal{P}_n$, the solutions of the corresponding initial value problems (58) and (63), are given by (59)-(62) and (64), respectively. Therefore, $Y_n = \mathcal{P}_n\mathcal{L}^{-1}$ yields, in the position $(1,1)$, (68), where we used (66). The position $(2,1)$ gives us (68) for $n-1$, where we used (67). The positions $(1,2)$ and $(2,2)$ of $Y_n = \mathcal{P}_n\mathcal{L}^{-1}$ yield $P_n^{(1)} = H_n$ and $P_{n-1}^{(1)} = H_{n-1}$, respectively.
(b). Note that $\tilde{S}$ satisfies (53),

$$A\tilde{S}' = \tilde{C}\tilde{S} + \tilde{D}\,, \quad A = 1, \quad \tilde{C} = -2x\,, \quad \tilde{D} = 2\,.$$

Taking into account the Lemma 3, there holds a relation of the type (69), where the polynomials $a-d$ and the polynomials $A, \tilde{C}, \tilde{D}$ are related through the equations (31)-(33). Thus, (70) follows. ■

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