Statistical properties of dynamical systems via induced Weak Gibbs Markov maps

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Abstract

This thesis addresses the statistical properties of dynamical systems that admit an induced Weak Gibbs Markov map (not necessarily full branch). More precisely, it provides estimates on the decay of correlations, a Central Limit Theorem and Large Deviations results for such dynamical systems. These estimates for the decay of correlations and the Central Limit Theorem are obtained for Hölder observables as well as larger classes of observables with weaker regularity than Hölder. The results on Large Deviations are a consequence of the estimates on the decay of correlations in this thesis.

To deal with the estimates on the decay of correlations and the Central Limit Theorem for Hölder observables, our approach generalizes L.-S. Young's coupling arguments. Initially, we discuss how to ensure that the invariant probability measure for the tower system of an induced Weak Gibbs Markov map, is mixing. Then, we estimate the decay of correlations in terms of the tail of the return time function and derive the Central Limit Theorem for the tower system. Finally, using a semiconjugacy, we obtain results on the decay of correlations and Central Limit Theorem for Hölder observables.

The results for larger classes of observables follow a similar strategy. Specifically, we extend the decay of correlation estimates, the Central Limit Theorem and the Large Deviations estimates for Hölder observables to larger classes of observables.

Keywords

Central Limit Theorem; Decay of Correlations; Large Deviations; Weak Gibbs Markov map; Young Tower.

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Resumo Alargado

Esta tese aborda propriedades estatísticas de sistemas dinâmicos que admitem uma transformação induzida Gibbs-Markov fraca (ou seja, sem a exigência de retornos completos). Para isso, consideramos sistemas dinâmicos $f : M \to M$, definidos num espaço métrico M, para os quais assumimos que existe uma transformação induzida Gibbs-Markov fraca $f^R : \Delta_0 \to \Delta_0$, com $\Delta_0 \subset M$. Em particular, obtemos estimativas para o decaimento de correlações, um Teorema do Limite Central e resultados de Grandes Desvios para estes sistemas dinâmicos. As estimativas para o decaimento de correlações e para o Teorema do Limite Central são obtidas quer para observáveis Hölder, quer para classes mais amplas de observáveis, com regularidade mais fraca do que Höder. Os resultados sobre Grandes Desvios derivam das estimativas sobre o decaimento das correlações obtidas nesta tese.

Para lidar com estimativas sobre sobre o decaimento de correlações e o Teorema do Limite Central para observáveis Hölder, introduzimos uma transformação numa torre, $T : \Delta \to \Delta$, associada a uma transformação induzida Gibbs-Markov fraca $f^R : \Delta_0 \to \Delta_0$, e generalizamos os argumentos de acoplamento (*coupling*) de L.-S. Young. Inicialmente, discutimos as propriedades de mistura das medida de probabilidade invariantes pela transformação na torre T. Uma vez que, em geral, estas medidas não são misturadoras por T, consideramos hipóteses adicionais sobre f^R, como a aperiodicidade e a existência de um *bloco coprimo*, e demonstramos que, nesse caso, a única medida de probabilidade equivalente à medida de referência e invariante por T é misturadora. Em seguida, estimamos o decaimento de correlações em termos da cauda dos tempos de retorno e o Teorema do Limite Central para a transformação na torre. Finalmente, usando uma semi-conjugação, obtemos conclusões semelhantes sobre o decaimento de correlações e o Teorema do Limite Central para observáveis Hölder relativos à dinâmica f, permitindo assim formular resultados abstratos sobre estas propriedades estatísticas para sistemas dinâmicos que admitem transformações induzidas Gibbs-Markov fracas.

Os resultados para outras classes de observáveis são obtidos com uma estratégia semelhante. Mais concretamente, estendemos os resultados obtidos anteriormente sobre decaimento de correlações, Teorema do Limite Central e Grandes Desvios para observáveis Hölder, considerando agora classes mais gerais de observáveis.

Palavras-chave

Decaimento de correlações; Grandes Desvios; Teorema do Limite Central; Torre de Young; Transformação Gibbs-Markov fraca.

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Initialisms

a.e	almost every(where)
CLT	Central Limit Theorem
GM	Gibbs Markov
gcd	Greatest Common Divisor
WGM	Weak Gibbs Markov
CMA	Centro de Matemática e Aplicações
FCT	Fundação para a Ciência e a Tecnologia
UBI	Universidade da Beira Interior

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List of Symbols

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$(M, \mathcal{B}, m)$	Measure space
Δ_0	Subset of M
$(\Delta_0, \mathcal{A}, m)$	Measure space
R	Measurable function $R : \Delta_0 \to \mathbb{N}$ (Return time)
f^R	Induced map $f^R : \Delta_0 \to \Delta_0$ for $f : M \to M$
Δ	Tower set
$L^\infty(M, m)$	Space of essentially-bounded functions defined on (M, m)
$L^1(M, m)$	Space of integrable functions on (M, m)
$R \in L^1(m)$	$R \in L^1(\Delta_0, m)$

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Chapter 1

Introduction

The induced scheme developed by L.-S. Young in [70, 71] is certainly among the most powerful tools for studying the statistical properties of nonuniformly hyperbolic dynamical systems. Roughly speaking, it involves considering a region of the phase space partitioned into subdomains where a return time is defined (not necessarily the first one) and establishing an induced map within that region that satisfies several good properties. In this approach, there is an explicit connection between the tail of the return times and the decay of correlations for the dynamical system, at least for some specific rates. In [70], the existence of a hyperbolic product structure for a given dynamical system was assumed, from which a mixing rate at exponential speed was obtained, based on studying the spectrum properties of the transfer operator of a tower system constructed from the induced map of the given dynamical system. The structure considered in [70] was called the 'horseshoe with infinitely many branches and variable return times'; we refer to it here as the *Young structure*. On the other hand, in [71] an abstract model was considered allowing different decay rates. More precisely, it was introduced a suitable dynamical system, nowadays known as Young towers, built over an induced full branch map, referred as induced Gibbs Markov (GM) map, and some statistical properties for this tower system were derived based on a coupling argument. A Central Limit Theorem (CLT) was also obtained in both works. Those results were directly transferred from the tower system to the dynamical system with such an induced map.

The induced scheme in [70, 71] is widely used and applied to investigate various other statistical properties, such as large deviations [4, 55, 57, 61], almost sure invariance principles [48, 56], Berry-Esseen theorem [39], weak invariance principle [59], concentration inequalities [22, 30], Poisson distribution [20, 45, 46] and local limit theorems [36].

The scheme in [70] has been implemented successfully for a number of explicit examples, including the logistic map [70], piecewise hyperbolic maps [23, 70], nonuniformly expanding maps of arbitrary dimension [10], dispersing billiards [24, 70], and the Hénon map [16]. We refer to [7, 9, 26, 27, 66, 73] for more applications.

There are also many works devoted to the construction of the induced scheme considered in [71] for various classes of maps, including one-dimensional maps [17, 28, 29, 32, 47, 50] and two-dimensional maps [34, 35]. Alves, Luzzatto and Pinheiro, in [10], constructed an induced GM map for nonuniformly expanding maps. As a consequence, the polynomial decay of correlations and CLT for those maps were obtained. Later in [14], Alves and Pinheiro had the parallel result for partially hyperbolic attractors with mostly expanding direction. In [40], a new construction of the induced GM map for nonuniformly expanding maps was developed by Goüezel. This construction provides more precise estimates than those presented in [10], yielding also a (stretched) exponential result. In [6], Alves, Dias and Luzzatto, provided an improved construction of the induced GM map, as compared to [10], that made it possible to established the integrability of inducing times under general conditions.

Other works explore the consequences of having induced GM maps, as in [2, 25, 43, 51, 56, 57]. In addition, Alves, Oliveira and Tahzibi, discussed the continuous variation of metric entropy for certain dynamical systems admitting an induced GM map in [12], while in [11], Alves together with Mesquita obtained entropy formula for this type of dynamical systems. Chazottes and Goüezel in [21], proved almost sure versions of distributional limit theorems, and these results were applied to dynamical systems with an induced GM map. It was shown the link between hitting time statistics and extreme value theory by Freitas and Todd in [37]. In particular, the obtained results were applied to dynamical system considered in [71]. Moreover, Bruin and Terhesiu in [18] obtained upper and lower bounds for the decay of correlations for nonuniformly expanding maps, having induced GM map with a general return. In the opposite direction, it was shown that for a large class of maps, having an induced GM map is a necessary condition for some statistical properties, by Alves, Dias and Luzzatto in [6], and by Alves, Freitas, Luzzatto and Vaienti in [8].

There are numerous works focused on extending [70, 71]. For instance, Alves and Pinheiro, in [13], extended the results of [70] by weakening some assumptions, obtaining, in addition to the exponential decay of correlations and CLT, a polynomial decay of correlations assuming a polynomial return time. Mixing rates were obtained with uniform control on various constants for nonuniformly expanding/hyperbolic maps similar to those considered in [70, 71] by Korepanov, Kosloff and Melbourne in [49]. The estimates obtained in [71] were shown to be optimal using the method of operator renewal theory by Sarig in [63]. Later on, in [38], the method of operator renewal theory was substantially extended by Goüezel and obtained slightly better estimates. Melbourne and Terhesiu, in [58], obtained results on the decay of correlations for nonuniformly expanding systems with a *good inducing scheme* using operator renewal argument. In particular, these results applied to the system considered in [70, 71].

The aim of this thesis is to obtain similar results to those in [71] under weaker assumptions, where the induced map is not necessarily full branch, which we refer to as a *Weak Gibbs Markov* (WGM) map (see Definition 2.1.1). This work provides abstract results to be applied to dynamical system that admit an induced WGM map, even when the existence of an induced GM map is not guaranteed.

Constructing an induced scheme with the desired properties is often a difficult task. Furthermore, if an induced WGM map is obtained, transitioning to a full branch scenario is not always straightforward. There are some examples in the literature where an induced WGM map is constructed. In some cases, as for instance in L.-S. Young [71, §6], Goüezel [40] and Eslami [34], used the induced WGM map to get a GM map in order to apply the abstract results from [71]. In other situations, the step to get a full branch map seems extremely laborious or may not be possible; *c.f.* [41].

There are some previous works that have considered the WGM map. In [38] Goüezel considered a topologically mixing Markov map, preserving the reference probability measure and an induced WGM map was defined with respect to this reference measure. An operator renewal theory was used as a method to obtain sharp polynomial bounds for decay of correlations and also a CLT, both for observables supported in the inducing region. In contrast, we do not assume that the given dynamical system is a topological mixing Markov map and that the reference measure is invariant. We also do not assume that the observables are supported in the induced region. As method we follow L.-S. Young's coupling arguments to get estimates on decay of correlations, CLT and large deviations. The exponential decay of correlations for uniform expanding Markov maps defined on a compact metric space with a 'finite images' condition was obtained by Zweimüller in [74]. In particular, the 'finite images' condition provides a positive lower bound for the density of the invariant measure, which simplifies matters considerably. The compactness assumption and 'finite images' condition are not required in present work. In [54] Maume-Deschamps addressed the decay of correlations for a mixing tower map associated with an induced WGM map via Birkhoff's cones and projective metrics. This work was extended together with Buzzi in [19] allowing the mixing tower map to have a non-Hölder Jacobian. In particular the Gibbs property (bounded distortion) for the tower map is replaced by a summable variation condition. The setting considered in the present work is different than [19, 54] as we provide sufficient conditions on the induced map to obtain a mixing measure for a tower system. Moreover, due to the different techniques, the estimates on the decay of correlations we obtain are slightly better.

In this thesis, we also discus statistical properties for larger classes of observables. In particular, we obtain estimates on the decay of correlations, CLT and large deviations for dynamical systems having an induced WGM map, for observables with weaker regularity than Hölder. Often, these statistical properties for Hölder observables are well known for a large variety of dynamical systems. For instance, as said before, L.-S. Young in [70, 71] obtained the decay of correlations and CLT for nonuniformly hyperbolic systems. The large deviations and almost sure invariance principles for nonuniformly expanding systems were discussed by Melbourne and Nicol in [56, 57]. We also refer to [16, 17, 24, 32, 40] where these statistical properties for distinct classes of dynamical systems were analysed. Note that in the above references, observables were always assumed to be Hölder.

There are several works dealing with classes of observables strictly larger than Hölder. Pollicott snd Yuri in [60] considered a class of observables which contains all piecewise Lipschitz functions, and obtained the decay of correlation for certain nonuniformly expanding systems. Estimates for the rates of mixing for observables with weaker regularity than Hölder were given by Díaz-Ordaz in [31] for one-dimensional expanding Lorenz-like map. We refer to [62, 72] for invertible maps with non-Hölder observables. More generally, Lynch in [53], obtained the decay of correlations and CLT for dynamical system that admit an induced GM map, for much larger classes of observables. More precisely, the results obtained in [53] were an extension of [71] in the sense of considering larger classes of observables. This work inspired us to further explore our results for Hölder observables in the context of larger classes of observables considered in [53].

In this thesis, we assume the existence of an induced WGM map. With additional hypothesis on the induced map, we build a tower map that has a mixing measure. We obtain results on the decay of correlations in terms of the tail of the return time function and CLT for this tower map. The results then go back to the given dynamical system admitting an induced WGM map through a semi-conjugacy. This provides an abstract results for dynamical systems admitting induced WGM maps. We also obtain results on the large deviations as an immediate corollary for those dynamical systems.

This thesis is organized as follows. The aim of Chapter 2 is to present the main results of this thesis, which provide estimates on the decay of correlations, CLT and large deviations for given dynamical system admitting induced WGM map. In Section 2.1, we introduce the necessary definitions that are important for presenting the main results. We sate the main results in Section 2.2 and Section 2.3. In Section 2.4 we give an example of a dynamical system that full fills our assumptions and, consequently, for which our main conclusions hold. Chapter 3 is concerned with mixing measures for a WGM map, a tower map, and the given dynamical system with induced WGM map. In Section 3.1 we ensure the existence of a mixing invariant probability measure for Weak Gibbs Markov maps. In Section 3.2 we introduce the tower map, a tool that will be essential in the proofs of the main results, and making use of our hypothesis we obtain a mixing invariant probability measure for the tower map. In Section 3.2.3 we introduce semi-conjugacy as a tool to retrieve relevant statistical properties of the given dynamical system from the tower map. In particular, we get a mixing invariant probability measure for given dynamical system in this section. Chapter 4 is dedicated to the proofs of main results for Hölder observables. In Section 4.1, we obtain the decay of correlations for the tower map and then we conclude the results for the given dynamical system. In Section 4.2, we obtain the CLT for the tower map and then apply it to the given dynamical system. Chapter 5 is dealing with the proof of the main theorem on decay of correlations for larger classes of observables. In Section 5.1 we discuss the decay of correlations for the tower map for larger classes of observables, and transfer them to the given dynamical system. In Section 5.2, we conclude the CLT for non-Hölder observables. Chapter 6 summarises the achieved goals and lists the future perspectives of the work carried out throughout this thesis.

We would like to conclude this introduction by mentioning that the main results of this thesis are included in the preprints [67, 68].

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Chapter 2

Statement of main results

In this chapter, we state our main results concerning statistical properties for Hölder and non-Hölder observables. These include the decay of correlations, CLT and large deviations. Moreover, we recall a known example from the literature where we apply our results.

2.1 Preliminaries

We start this section with definitions that are important for stating the main results. We recall elementary material of ergodic theory in Section A.1.

Consider a measure space $(\Delta_0, \mathcal{A}, m)$, with $0 < m(\Delta_0) < \infty$, a measurable map $F : \Delta_0 \to \Delta_0$ and an (m mod 0) countable partition $\mathcal{P}_0$ of Δ_0 into invertibility domains of F, that is F is a bijection from each $\omega \in \mathcal{P}_0$ to $F(\omega)$, with measurable inverse.

For each $n \geq 0$ we set

$$F^{-n}(\mathcal{P}_0) = \{F^{-n}(\omega) : \omega \in \mathcal{P}_0\}.$$

Assuming $F_* m \ll m$, we have that $F^{-i}(\mathcal{P}_0)$ is an (m mod 0) partition of Δ_0, for all $i \geq 0$. In this case we define a sequence $(\mathcal{P}_0^n)_n$ of (m mod 0) partitions given by

$$\mathcal{P}_0^n = \bigvee_{i=0}^{n-1} F^{-i}(\mathcal{P}_0) = \{\omega_0 \cap F^{-1}(\omega_1) \cap ... \cap F^{-n+1}(\omega_{n-1}) : \omega_0, \ldots, \omega_{n-1} \in \mathcal{P}_0\},$$

for $n \geq 1$, and the (m mod 0) partition

$$\mathcal{P}_0^\infty = \bigvee_{i=0}^{\infty} F^{-i}(\mathcal{P}_0) = \{\omega_0 \cap F^{-1}(\omega_1) \cap \cdots : \omega_n \in \mathcal{P}_0 \text{ for all } n \geq 0\}.$$

The sequence $(\mathcal{P}_0^n)_n$ is *a basis of* Δ_0 if it generates $\mathcal{A}$ (m mod 0) and $\mathcal{P}_0^\infty$ is the partition into single points.

Definition 2.1.1. *We say that $F : \Delta_0 \to \Delta_0$ is a Weak Gibbs Markov (WGM) map, with respect to the (m mod 0) countable partition $\mathcal{P}_0$, if the following hold:*

W1) *Markov: F maps each $\omega \in \mathcal{P}_0$ bijectively to an (m mod 0) union of elements of $\mathcal{P}_0$.*

W2) *Separability: the sequence $(\mathcal{P}_0^n)_n$ is a basis of Δ_0.*

W3) *Nonsingular: there exists a strictly positive measurable function J_F, defined on Δ_0, such that for each $A \subset \omega \in \mathcal{P}_0$,*

$$m(F(A)) = \int_A J_F dm.$$

W4) *Gibbs: there exist $C_F > 0$ and $0 < \beta < 1$ such that for all $\omega \in \mathcal{P}_0$ and $x, y \in \omega$,*

$$\log \frac{J_F(x)}{J_F(y)} \leq C_F \beta^{s(F(x),F(y))},$$

where $s(x,x) = 0$ and, for $x \neq y$,

$$s(x,y) = min\{n \geq 0 : F^n(x) \text{ and } F^n(y) \text{ lie in distinct elements of } \mathcal{P}_0\}.$$

W5) *Long branches: there exists $\delta_0 > 0$ such that $m(F(\omega)) \geq \delta_0$, for all $\omega \in \mathcal{P}_0$.*

The term *weak* in the above definition means that in $W1$) we do not require a *full branch* property (that is, $F(\omega) = \Delta_0$ for all $\omega \in \mathcal{P}_0$). In the case of full branch, F is called a *Gibbs Markov* (GM) map. This terminology was introduced in [3].

Consider a measure space $(M, \mathcal{B}, m)$, a measurable map $f : M \to M$ and $\Delta_0 \subset M$ with $m(\Delta_0) > 0$. For simplicity we denote the restriction of m to Δ_0 also by m. We say that $F : \Delta_0 \to \Delta_0$ is an *induced map for f* if there exists a countable (m mod 0) partition $\mathcal{P}_0$ of Δ_0 and a measurable function $R : \Delta_0 \to \mathbb{N}$ (*return time* function), constant on each element of $\mathcal{P}_0$, such that $F|_\omega = f^{R(\omega)}|_\omega$. We formally denote the induced map F by f^R.

An induced map $f^R \colon \Delta_0 \to \Delta_0$ is *aperiodic* if for all $\omega_1, \omega_2 \in \mathcal{P}_0$ there exists $k_0 \in \mathbb{N}$ such that for all $n \geq k_0$,

$$m(\omega_1 \cap (f^R)^{-n}(\omega_2)) > 0.$$

We say that an induced map $f^R \colon \Delta_0 \to \Delta_0$ has a *coprime block* if there exist $N \geq 2$ and $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathcal{P}_0$ such that $\gcd\{R(\alpha_i)\}_{i=1}^N = 1$ and for all $i = 1, \dots, N$,

$$f^R(\alpha_i) \supseteq \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_N \quad (m \text{ mod } 0).$$

Assume that M is a metric space with metric d. An induced map $f^R\colon \Delta_0 \to \Delta_0$ is *expanding* if there are $C > 0$ and $0 < \beta < 1$ such that, for all $\omega \in \mathcal{P}_0$ and $x, y \in \omega$,

i) $d(f^R(x), f^R(y)) \leq C\beta^{s(x,y)}$,

ii) $d(f^j(x), f^j(y)) \leq Cd(f^R(x), f^R(y))$, for all $0 \leq j \leq R$.

The correlation sequence of two observable functions $\varphi \in \mathcal{H}$ and $\psi \in L^\infty(M, m)$, for some Banach space $\mathcal{H} \subset L^\infty(M, m)$, with respect to an f-invariant probability μ, is defined for all $n \in \mathbb{N}$ by

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \left| \int \varphi(\psi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$

Given a sequence $(\gamma_n)_n$ of positive numbers converging to zero, we say that (f, μ) *has decay of correlations with rate* $(\gamma_n)_n$ if for every $\varphi \in \mathcal{H}$ and $\psi \in L^\infty(M, m)$ there exists a constant $k(\varphi, \psi) > 0$ such that

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq k(\varphi, \psi)\gamma_n, \text{ for every } n \geq 1.$$

Remark 2.1.2. *We will see decay of correlations with the following rates in main results:*

i) $\gamma_n = n^{-a}$, *with* $a > 0$ *(polynomial decay);*

ii) $\gamma_n = e^{-cn}$, *with* $c > 0$ *(exponential decay);*

iii) $\gamma_n = e^{-cn^a}$, *with* $c > 0$ *and* $0 < a < 1$ *(stretched exponential decay);*

iv) $\gamma_n = n^{1-a}\log(n)$, *with* $a > 1$*;*

v) $\gamma_n = e^{-(\log(n))^c}$, *with* $c > 0$.

Let $f : M \to M$ be a dynamical system with an ergodic invariant probability measure μ. We say that an observable $\varphi\colon M \to \mathbb{R}$, with $\int \varphi d\mu = 0$, satisfies the *Central Limit Theorem* (CLT) if $\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ f^i$ converges in law (or in distribution) to a normal distribution $\mathcal{N}(0, \sigma)$, for some $\sigma > 0$. We may also consider observables of non-zero expectation by replacing φ with $\varphi - \int \varphi d\mu$. In this situation, an observable φ satisfies the CLT if there exists $\sigma > 0$ such that for every interval $J \subset \mathbb{R}$,

$$\mu\left\{x : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left(\varphi(f^i(x)) - \int \varphi d\mu\right) \in J\right\} \to \frac{1}{\sigma\sqrt{2\pi}} \int_J e^{-\frac{t^2}{2\sigma^2}} dt, \text{ as } n \to \infty.$$

An observable $\varphi : M \to \mathbb{R}$ is *coboundary* if there exists a measurable function $g : M \to \mathbb{R}$ such that $\varphi \circ f = g \circ f - g$.

Given $\varepsilon > 0$ we define the *large deviation at time* n, of the time average of an observable $\varphi \colon M \to \mathbb{R}$ from the spatial average, as

$$LD_\mu(\varphi, \varepsilon, n) := \mu\left(\left|\frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ f^i - \int \varphi d\mu\right| > \varepsilon\right).$$

Birkhoff's ergodic theorem guarantees that time averages converges to spatial average μ-a.e., and so $LD_\mu(\varphi, \varepsilon, n) \to 0$, as $n \to \infty$. We are interested in the rate of this decay.

2.2 Statistical properties for Hölder observables

In this section, we state our main results for Hölder observables.

For a given $\eta > 0$, a function $\varphi \colon M \to \mathbb{R}$ is η*-Hölder continuous* if

$$|\varphi|_\eta = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\eta} < \infty.$$

We set $\mathcal{H}_\eta = \{\varphi \colon M \to \mathbb{R} \colon \varphi \text{ is } \eta\text{-Hölder continuous}\}$.

Theorem A. *Consider a measure space* $(M, \mathcal{B}, m)$*, endowed with some metric, and a measurable map* $f : M \to M$ *satisfying* $f_* m \ll m$*. Let* $f^R : \Delta_0 \to \Delta_0$ *be an aperiodic induced WGM expanding map with a coprime block and* $R \in L^1(m)$*. Then there exists a unique mixing* f*-invariant probability measure* μ *such that* $\mu \ll m$ *and* $\mu(\Delta_0) > 0$*. Moreover,*

1. *if* $m\{R > n\} \leq Cn^{-a}$ *for some* $C > 0$ *and* $a > 1$*, then for all* $\varphi \in \mathcal{H}_\eta$ *and* $\psi \in L^\infty(M, m)$*, there is some* $C' > 0$ *such that*

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' n^{-a+1};$$

2. *if* $m\{R > n\} \leq Ce^{-cn^a}$ *for some* $C, c > 0$ *and* $0 < a \leq 1$*, given* $\eta > 0$ *there is* $c' > 0$ *such that, for all* $\varphi \in \mathcal{H}_\eta$ *and* $\psi \in L^\infty(M, m)$*, there is some* $C' > 0$ *such that*

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' e^{-c' n^a}.$$

The existence of such measure μ will be discussed in Chapter 3. We remark that the coprime block is used to get μ to be mixing. We will explore the significance of the coprime block in Section 3.2. The remaining proof of Theorem A is given in Chapter 4.

Corollary B (Central Limit Theorem)**.** *Consider a measure space $(M, \mathcal{B}, m)$, endowed with some metric, and a measurable map $f : M \to M$ satisfying $f_*m \ll m$. Let $f^R : \Delta_0 \to \Delta_0$ be an aperiodic induced WGM expanding map with a coprime block such that $R \in L^1(m)$, and let μ be the unique mixing f-invariant probability measure μ such that $\mu \ll m$ and $\mu(\Delta_0) > 0$. If $m\{R > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 2$, then the CLT is satisfied for all $\varphi \in \mathcal{H}_\eta$ if and only if φ is not coboundary.*

The proof of Corollary B is given in Section 4.2. The next corollary follows from Theorem A together with Proposition A.2.4.

Corollary C (Large Deviations)**.** *Consider a measure space $(M, \mathcal{B}, m)$, endowed with some metric, and a measurable map $f : M \to M$ satisfying $f_*m \ll m$. Let $f^R : \Delta_0 \to \Delta_0$ be an aperiodic induced WGM expanding map with a coprime block such that $R \in L^1(m)$, and let μ be the unique mixing f-invariant probability measure μ such that $\mu \ll m$ and $\mu(\Delta_0) > 0$. Then,*

1. *if $m\{R > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 1$, then for all $\varepsilon > 0$ and $\varphi \in \mathcal{H}_\eta$ there is some $C' = C'(\varepsilon, \varphi) > 0$ such that*
$$LD_\mu(\varphi, \varepsilon, n) \leq C' n^{-a+1};$$
2. *if $m\{R > n\} \leq Ce^{-cn^a}$ for some $C, c > 0$ and $0 < a \leq 1$, then for all $\varepsilon > 0$ and $\varphi \in \mathcal{H}_\eta$ there is some $C' = C'(\varepsilon, \varphi) > 0$ and $c' = c'(c, \varphi, \varepsilon, \eta)$ such that*
$$LD_\mu(\varphi, \varepsilon, n) \leq C' e^{-c' n^{\frac{a}{a+2}}}.$$

2.3 Statistical properties for non-Hölder observables

In this section, we state our main results for larger classes of observables with weaker regularity than Hölder. First we define such classes of observables in terms of the *modulus of continuity.* Define modulus of continuity for a continuous function $\varphi : M \to \mathbb{R}$ and $\varepsilon > 0$ as
$$\mathcal{R}_\varepsilon(\varphi) := \sup\{|\varphi(x) - \varphi(y)| : d(x, y) \leq \varepsilon\}.$$

Note that $\mathcal{R}_\varepsilon(\varphi) \to 0$ as $\varepsilon \to 0$ if and only if φ is uniformly continuous. We consider the following classes of observables, which were defined in [53]:

i) $(R1, \tau) = \{\varphi : \mathcal{R}_\varepsilon(\varphi) = \mathcal{O}(\varepsilon^\tau)\}$, $\tau \in (0, 1)$.

ii) $(R2, \tau) = \{\varphi : \mathcal{R}_\varepsilon(\varphi) = \mathcal{O}(e^{-|\log \varepsilon|^\tau})\}$, $\tau \in (0, 1)$.

iii) $(R3, \tau) = \{\varphi : \mathcal{R}_\varepsilon(\varphi) = \mathcal{O}(e^{-(\log|\log\varepsilon|)^\tau})\}$, $\tau > 1$.

iv) $(R4, \tau) = \{\varphi : \mathcal{R}_\varepsilon(\varphi) = \mathcal{O}(|\log\varepsilon|^{-\tau})\}$, $\tau > 1$.

Theorem D. *Consider a measure space $(M, \mathcal{B}, m)$, endowed with some metric, and a measurable map $f : M \to M$ satisfying $f_* m \ll m$. Let $f^R : \Delta_0 \to \Delta_0$ be an aperiodic induced WGM expanding map with a coprime block, $R \in L^1(m)$ and let μ be the unique mixing f-invariant probability measure μ such that $\mu \ll m$ and $\mu(\Delta_0) > 0$. Then,*

1. *if $m\{R > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 1$, given $\kappa > 0$, there is $0 < \zeta < 1$ such that for all $\varphi \in (R4, \tau)$, with $\tau > \frac{2}{\zeta}$ and $\mathcal{R}_\infty(\varphi) < \kappa$, and for all $\psi \in L^\infty(M, m)$ we have*

   i) if $\tau = \frac{a+1}{\zeta}$, there is some $C' > 0$ such that

   $$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' \left(n^{1-a} \log n\right);$$

   ii) otherwise, there is some $C' > 0$ such that

   $$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' \max\left\{n^{1-a}, n^{2-\zeta\tau}\right\};$$

2. *if $m\{R > n\} \leq Ce^{-cn}$ for some $C, c > 0$, then*

   i) for all $\varphi \in (R1, \tau)$ and $\psi \in L^\infty(M, m)$ there are $C', c' > 0$ such that

   $$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' e^{-c' n};$$

   ii) for all $\varphi \in (R2, \tau)$ and $\psi \in L^\infty(M, m)$ there is $C' > 0$ such that, for every $0 < \tau' < \tau$,

   $$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' e^{-n^{\tau'}};$$

   iii) for all $\varphi \in (R3, \tau)$ and $\psi \in L^\infty(M, m)$ there is $C' > 0$ such that, for every $0 < \tau' < \tau$,

   $$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' e^{-(\log n)^{\tau'}};$$

   iv) for given $\kappa > 0$ there is $0 < \zeta < 1$ such that for all $\varphi \in (R4, \tau)$, with $\tau > \frac{1}{\zeta}$ and $\mathcal{R}_\infty(\varphi) < \kappa$, and for all $\psi \in L^\infty(M, m)$, there is $C' > 0$ for which

   $$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' n^{1-\zeta\tau}.$$

The proof of Theorem D can be deduced from the proof of Proposition 5.1.1. This is done in Chapter 5. Note that the existence of such measure μ was already discussed in Theorem A.

Corollary E (Central Limit Theorem)**.** *Consider a measure space $(M, \mathcal{B}, m)$, endowed with some metric, and a measurable map $f : M \to M$ satisfying $f_* m \ll m$. Let $f^R : \Delta_0 \to \Delta_0$ be an aperiodic induced WGM expanding map with a coprime block such that $R \in L^1(m)$, and let μ be the unique mixing f-invariant probability measure μ such that $\mu \ll m$ and $\mu(\Delta_0) > 0$. If $m\{R > n\} \le Cn^{-a}$ for some $C > 0$ and $a > 2$, then the CLT is satisfied for all $\varphi \in (R4, \tau)$, some $\tau > 0$ sufficiently large if and only if φ is not coboundary.*

The proof of Corollary E will be discussed in Section 5.2. By combining Theorem D and Proposition A.2.4, we have the following immediate corollary.

Corollary F (Large Deviations)**.** *Consider a measure space $(M, \mathcal{B}, m)$, endowed with some metric, and a measurable map $f : M \to M$ satisfying $f_* m \ll m$. Let $f^R : \Delta_0 \to \Delta_0$ be an aperiodic induced WGM expanding map with a coprime block such that $R \in L^1(m)$, and let μ be the unique mixing f-invariant probability measure μ such that $\mu \ll m$ and $\mu(\Delta_0) > 0$. Then,*

1. *if $m\{R > n\} \le Cn^{-a}$ for some $C > 0$ and $a > 1$, given $\kappa > 0$ there is $0 < \zeta < 1$, such that for all $\varepsilon > 0$ and $\varphi \in (R4, \tau)$, with $\frac{2}{\zeta} < \tau \neq \frac{a+1}{\zeta}$ and $\mathcal{R}_\infty(\varphi) < \kappa$, there is some $C' = C'(\varepsilon, \varphi) > 0$ such that*
$$LD_\mu(\varphi, \varepsilon, n) \le C' \max\{n^{-a+1}, n^{2-\zeta\tau}\};$$
2. *if $m\{R > n\} \le Ce^{-cn}$ for some $C, c > 0$, then*
   i) for all $\varepsilon > 0$ and $\varphi \in (R1, \tau)$, there is some $C' = C'(\varepsilon, \varphi) > 0$ and $c' = c'(c, \varphi, \varepsilon, \tau)$ such that
   $$LD_\mu(\varphi, \varepsilon, n) \le C' e^{-c' n^{\frac{1}{3}}};$$
   ii) for all $\varepsilon > 0$ and $\varphi \in (R2, \tau)$, there is some $C' = C'(\varepsilon, \varphi) > 0$ and $c' = c'(\varphi, \varepsilon)$ such that, for every $0 < \tau' < \tau$,
   $$LD_\mu(\varphi, \varepsilon, n) \le C' e^{-c' n^{\frac{\tau'}{\tau'+2}}};$$
   iii) for given $\kappa > 0$ there is $0 < \zeta < 1$ such that for all $\varepsilon > 0$ and $\varphi \in (R4, \tau)$, with $\tau > \frac{1}{\zeta}$ and $\mathcal{R}_\infty(\varphi) < \kappa$, there is some $C' = C'(\varepsilon, \varphi) > 0$, for which
   $$LD_\mu(\varphi, \varepsilon, n) \le C' n^{1-\zeta\tau}.$$

We remark that the results in this section extend those of Section 2.2 by considering larger classes of observables. Notice that the estimates for observables in $(R1, \tau)$ are the same as for Hölder observables.

2.4 Example

In this section, we apply our main results to the following example with polynomial return time decay, introduced in [41].

Let $M = S^1 \times [0,1]$ and let m denote the Lebesgue measure on M. We consider the map $f : M \to M$ defined by

$$f(\theta, x) = \left(g(\theta), f_{\alpha(\theta)}(x)\right), \tag{2.4.1}$$

where $g(\theta) = 4\theta \pmod 1$, $\alpha : S^1 \to (0,1)$ is a C^1 map that has minimum $\alpha_{\min}$ and maximum $\alpha_{\max}$, with $\alpha_{\min} < \alpha_{\max}$, and

$$f_{\alpha(\theta)}(x) = \begin{cases} x\left(1 + 2^{\alpha(\theta)}x^{\alpha(\theta)}\right) & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} < x \leqslant 1. \end{cases}$$

In [41] a partition $\mathcal{P}_0$ of $\Delta_0 = S^1 \times (\frac{1}{2}, 1]$ with a certain return time $R : \Delta_0 \to \mathbb{N}$ was given in such a way that f^R is an aperiodic induced WGM map (no full branch property) with a coprime block. Moreover, we have $m\{R > n\} \leq Cn^{-a}$ for some $C > 0$, with $a = 1/\alpha_{\max}$. Since $\alpha_{\max} < 1$, we have $R \in L^1(m)$. We provide details on the construction of the induced map f^R in Section A.4.

By Theorem A, there exists a unique mixing f-invariant probability measure μ such that $\mu \ll m$, $\mu(\Delta_0) > 0$ and for all $\varphi \in \mathcal{H}_\eta$ and $\psi \in L^\infty(M, m)$, there is some $C' > 0$ such that

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' n^{-a+1}.$$

By Corollary B, if $\alpha_{max} < 1/2$ (thus $a > 2$) then CLT is satisfied for all $\varphi \in \mathcal{H}_\eta$ which is not coboundary, and by Corollary C we also have

$$LD_\mu(\varphi, \varepsilon, n) \leq C'' n^{-a+1}$$

for all $\varphi \in \mathcal{H}_\eta$ and some $C'' = C''(\varepsilon, \varphi) > 0$.

Now we are going to apply non-Hölder case results. By Theorem D, for given $\kappa > 0$, there is $0 < \zeta < 1$ such that for all $\varphi \in (R4, \tau)$ with $\tau > \frac{2}{\zeta}$ and $\mathcal{R}_\infty(\varphi) < \kappa$, and for all $\psi \in L^\infty(M, m)$ we have

- if $\tau = \frac{a+1}{\zeta}$, there is some $C' > 0$ such that

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' \left(n^{-a+1} \log n\right);$$

- otherwise, there is some $C' > 0$ such that

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' \max\left\{n^{-a+1}, n^{2-\zeta\tau}\right\}.$$

By Corollary E, if $\alpha_{max} < 1/2$ (thus $a > 2$) then CLT is satisfied for all $\varphi \in (R4, \tau)$, that is not coboundary with some $\tau > 0$ sufficiently large, and by Corollary F we also have

$$LD_\mu(\varphi, \varepsilon, n) \leq C'' \max\left\{n^{-a+1}, n^{2-\zeta\tau}\right\}$$

for all $\varphi \in (R4, \tau)$, with $\tau \neq \frac{a+1}{\zeta}$ and some $C'' = C''(\varepsilon, \varphi) > 0$.

Remark 2.4.1. *It is worth to mention that for $\alpha_{min} < 1/2$, a CLT for the map in* (2.4.1) *for Hölder observables was obtained in [41], under additional hypothesis on α. Moreover, a polynomial decay of correlations for Hölder observables supported on Δ_0 was pointed out in [42].*

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Chapter 3

Mixing measures for certain dynamical systems

In this chapter, we discuss mixing invariant probability measures for WGM maps and associated tower maps. The most vital part of this chapter is to prove that the invariant probability measure for such a tower map is mixing. As a consequence, we obtain a mixing invariant probability measure for the given dynamical system $f\colon M \to M$.

3.1 Mixing measures for Weak Gibbs Markov maps

In the present section, we are going to discuss the existence of a mixing invariant probability measure for a WGM map $F : \Delta_0 \to \Delta_0$. First, we present some results on WGM maps, that will be used later in this chapter. Let us start to introduce linear spaces which will contain the densities of the invariant measures.

Let $F : \Delta_0 \to \Delta_0$ be a WGM map. Given $0 < \beta < 1$ as in Definition 2.1.1, we define the following *spaces of densities*:

$$\mathcal{F}_\beta(\Delta_0) = \{\varphi\colon \Delta_0 \to \mathbb{R}\colon \exists C_\varphi > 0 \text{ s.t. } |\varphi(x) - \varphi(y)| \le C_\varphi \beta^{s(x,y)}, \forall x, y \in \Delta_0\},$$

and

$$\mathcal{F}_\beta^+(\Delta_0) = \{\varphi \in \mathcal{F}_\beta(\Delta_0)\colon \exists C'_\varphi > 0 \text{ s.t. } \varphi(x) > 0 \text{ and } \left|\frac{\varphi(x)}{\varphi(y)} - 1\right| \le C'_\varphi \beta^{s(x,y)}, \forall x, y \in \omega \in \mathcal{P}_0\}.$$

Given $\varphi \in \mathcal{F}_\beta(\Delta_0)$, set

$$|\varphi|_\beta = \sup_{x \ne y} \frac{|\varphi(x) - \varphi(y)|}{\beta^{s(x,y)}}.$$

Since $\mathcal{F}_\beta(\Delta_0) \subset L^\infty(\Delta_0, m) \subset L^1(\Delta_0, m)$, we may define a norm on $\mathcal{F}_\beta(\Delta_0)$ as

$$\|\varphi\|_\beta = |\varphi|_\beta + \|\varphi\|_\infty.$$

Set, for each $n \in \mathbb{N}$ and $\omega \in \mathcal{P}_0^n$,

$$\rho_n = \frac{dF_*^n m}{dm} \quad \text{and} \quad \rho_{n,\omega} = \frac{dF_*^n (m|\omega)}{dm}.$$

Lemma 3.1.1. *[3] Let $F : \Delta_0 \to \Delta_0$ be a WGM map. Then,*

i) there is some $\bar{C_F} > 0$ such that, for all $n \geq 1$ and $x, y \in \omega \in \mathcal{P}_0^n$,

$$\log \frac{J_{F^n}(x)}{J_{F^n}(y)} \leq \bar{C_F} \beta^{s(F^n(x), F^n(y))}.$$

ii) there is some $C > 0$ such that, for all $n \geq 1$, $\omega \in \mathcal{P}_0^n$ and measurable sets $A_1, A_2 \subset \omega$,

$$\frac{m(F^n(A_1))}{m(F^n(A_2))} \leq C \frac{m(A_1)}{m(A_2)}.$$

iii) there is some $C > 0$ such that $||\rho_n||_\beta \leq C$, for all $n \geq 0$.

The following corollary is useful for getting information about the density of the invariant probability measure (see Remark 3.1.4).

Corollary 3.1.2. *If $F : \Delta_0 \to \Delta_0$ is a WGM map, then there is some $C > 0$ such that, for all $\omega \in \mathcal{P}_0$ and $x, y \in F(\omega)$,*

$$\rho_n(x) \leq C \rho_n(y) \textit{ for every } n \geq 1.$$

Proof. We know that for all $n \geq 1$,

$$\rho_n(x) = \sum_{\omega' \in \mathcal{P}_0^n} \rho_{n,\omega'} \mathbb{1}_{F^n(\omega')}(x).$$

Take $\omega' \in \mathcal{P}_0^n$ and $x', y' \in \omega'$ such that $F^n(x') = x, F^n(y') = y$. Since by Lemma A.1.9, $\rho_{n,\omega'}|_{F^n(\omega')} = \frac{1}{J_{F^n}}|_{\omega'}$, then we have

$$\rho_n(x) = \sum_{\substack{\omega' \in \mathcal{P}_0^n \\ x' \in \omega', F^n(x') = x}} \frac{1}{J_{F^n}(x')} = \sum_{\substack{\omega' \in \mathcal{P}_0^n \\ x', y' \in \omega', \\ F^n(x') = x, F^n(y') = y}} \frac{J_{F^n}(y')}{J_{F^n}(x')} \frac{1}{J_{F^n}(y')}.$$

Together with the first item of Lemma 3.1.1, we have

$$\rho_n(x) \leq C \sum_{\substack{\omega' \in \mathcal{P}_0^n \\ y' \in \omega', F^n(y') = y}} \frac{1}{J_{F^n}(y')} = C \rho_n(y),$$

where $C = e^{\bar{C_F}}$. □

The following result provides an invariant probability measure for the WGM map. The proof can be found in [3], and we include it here just for completeness since it provides useful informations for future use (see Remark 3.1.4).

Proposition 3.1.3. *If $F : \Delta_0 \to \Delta_0$ is a WGM map, then F has an invariant probability measure $\nu_0 \ll m$ with $d\nu_0/dm \in \mathcal{F}_\beta(\Delta_0)$.*

Proof. By *iii*) of Lemma 3.1.1, there exists $C > 0$ such that, for all $n \geq 0$,

$$\rho_n \leq C \text{ and } |\rho_n|_\beta \leq C.$$

Then by Lemma A.2.1, there exist $\rho_0 \in L^1(\Delta_0, m)$ and a subsequence $(\rho_{n_k})_k$ converging pointwise to ρ_0 and in $L^1(\Delta_0, m)$. So we have

$$\rho_0 = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \rho_j.$$

Let $\tilde{\nu_0}$ be a finite measure such that $d\tilde{\nu_0}/dm = \rho_0$. Since

$$\int \rho_n dm = \int dF_*^n m = m(\Delta_0), \quad for\ all\ n \geq 0,$$

and $(\rho_{n_k})_k$ converges in $L^1(m)$ to ρ_0, we have $m(\Delta_0) = \int \rho_{n_k} dm$, which implies that

$$m(\Delta_0) = \lim_{k\to 0} \int \rho_{n_k} dm = \int \rho_0 dm = \int \frac{d\tilde{\nu_0}}{dm} dm = \tilde{\nu_0}(\Delta_0) > 0.$$

The pointwise convergence gives $\rho_0 \leq C$ and $|\rho_0|_\beta \leq C$, and so $\rho_0 \in \mathcal{F}_\beta(\Delta_0)$. It remains to prove the invariance of $\tilde{\nu_0}$. We have

$$\left| \int \varphi_1 \psi_1 dm - \int \varphi_2 \psi_2 dm \right| \leq \|\varphi_1\|_\infty \|\psi_1 - \psi_2\|_{L^1} + \|\psi_2\|_{L^1} \|\varphi_1 - \varphi_2\|_\infty,$$

where $\varphi_i \in L^\infty(\Delta_0, m)$ and $\psi_i \in L^1(\Delta_0, m)$, for $i = 1, 2$. Therefore $(\varphi, \psi) \mapsto \int \varphi\psi\ dm$ is continuous in $L^\infty(\Delta_0, m) \times L^1(\Delta_0, m)$. Thus,

$$\begin{aligned}
\int (\varphi \circ F)\rho_0 dm &= \int \varphi \circ F \left(\lim_{k\to\infty} \frac{1}{n_k} \sum_{n=0}^{n_k-1} \rho_n \right) dm \\
&= \lim_{k\to\infty} \frac{1}{n_k} \sum_{n=0}^{n_k-1} \int (\varphi \circ F)\rho_n dm = \lim_{k\to\infty} \frac{1}{n_k} \sum_{n=0}^{n_k-1} \int \varphi \circ F dF_*^n m \\
&= \lim_{k\to\infty} \frac{1}{n_k} \sum_{n=0}^{n_k-1} \int \varphi dF_*^{n+1} m = \lim_{k\to\infty} \frac{1}{n_k} \sum_{n=0}^{n_k-1} \int \varphi \rho_{n+1} dm \\
&= \lim_{k\to\infty} \left(\frac{1}{n_k} \sum_{n=0}^{n_k-1} \int \varphi \rho_n dm - \frac{1}{n_k} \int \varphi \rho_0 dm + \frac{1}{n_k} \int \varphi \rho_{n+1} dm \right) \\
&= \lim_{k\to\infty} \frac{1}{n_k} \sum_{n=0}^{n_k-1} \int \varphi \rho_n dm = \int \varphi \left(\lim_{k\to\infty} \frac{1}{n_k} \sum_{n=0}^{n_k-1} \rho_n \right) dm = \int \varphi \rho_0 dm.
\end{aligned}$$

This gives that

$$\int \varphi dF_*\tilde{\nu_0} = \int (\varphi \circ F)\rho_0 dm = \int \varphi\rho_0 dm = \int \varphi d\tilde{\nu_0},$$

for all $\varphi \in L^\infty(m)$, and this implies that $F_*\tilde{\nu_0} = \tilde{\nu_0}$. Therefore $\tilde{\nu_0}$ is F-invariant. Hence $\nu_0 = \frac{\tilde{\nu_0}}{\tilde{\nu_0}(\Delta_0)}$ is the required F-invariant probability measure $\nu_0 \ll m$, with $d\nu_0/dm \in \mathcal{F}_\beta(\Delta_0)$. $\square$

Remark 3.1.4. *We can see that, in the proof of Proposition 3.1.3, the density $d\nu_0/dm$ is the pointwise limit of a subsequence of ρ_n. Similar to Corollary 3.1.2, we can write for each $\omega \in \mathcal{P}_0$, and $x, y \in F(\omega)$,*

$$\rho_n(x) \le e^{\bar{C}_F \beta^{s(x,y)}} \rho_n(y) \text{ for every } n \ge 1,$$

which implies

$$\text{either } \frac{d\nu_0}{dm}(x) = 0 \text{ or } \frac{d\nu_0}{dm}(x) > 0 \text{ and } \left| \log \frac{\frac{d\nu_0}{dm}(x)}{\frac{d\nu_0}{dm}(y)} \right| \le \bar{C}_F \beta^{s(x,y)}. \tag{3.1.1}$$

Note that $\left| \log \frac{\frac{d\nu_0}{dm}(x)}{\frac{d\nu_0}{dm}(y)} \right| \le \bar{C}_F \beta^{s(x,y)}$ implies that there exists $C > 0$ such that

$$\left| \frac{\frac{d\nu_0}{dm}(x)}{\frac{d\nu_0}{dm}(y)} - 1 \right| \le C\beta^{s(x,y)}.$$

Remark 3.1.5. *Note that in the WGM map setting, the invariant probability measure ν_0 is not necessarily mixing (not even ergodic). This forces us to impose some extra assumptions. Moreover, the density $\frac{d\nu_0}{dm}$ does not necessarily have a positive lower bound (see Example A.3.1). However, ν_0 to be mixing and $\frac{d\nu_0}{dm}$ to have a positive lower bound are commonly known for the GM maps.*

We say that a WGM map $F : \Delta_0 \to \Delta_0$ is *irreducible*, with respect to the partition $\mathcal{P}_0$, if for all $\omega_1, \omega_2 \in \mathcal{P}_0$ there exists $n \in \mathbb{N}$ such that $m(\omega_1 \cap F^{-n}(\omega_2)) > 0$.

Theorem 3.1.6. *If $F : \Delta_0 \to \Delta_0$ is an irreducible WGM map, then the F-invariant probability measure $\nu_0 \ll m$ is unique, ergodic, $\frac{d\nu_0}{dm} \in \mathcal{F}_\beta^+(\Delta_0)$ and there is C_0 such that $0 < \frac{d\nu_0}{dm} \le C_0$. Moreover, if F is aperiodic then ν_0 is exact.*

Proof. The ergodicity and exactness of the measure ν_0 are proved in [1]. The details on their proofs can be found in Section A.5. Let us prove that $\frac{d\nu_0}{dm} \in \mathcal{F}_\beta^+(\Delta_0)$. From Proposition 3.1.3, $d\nu_0/dm \in \mathcal{F}_\beta(\Delta_0)$. Let $\omega \in \mathcal{P}_0$ be an arbitrary element. Since F is an irreducible map, then ω must intersect $F(\omega_0)$ for some $\omega_0 \in \mathcal{P}_0$, $\omega_0 \ne \omega$, and this implies $\omega \subset F(\omega_0)$. By Remark 3.1.4, the expression (3.1.1) holds for each $x, y \in F(\omega_0)$, and in particular it holds for all $x, y \in \omega$. So, its enough to prove $\frac{d\nu_0}{dm} > 0$ on ω. Assume $\frac{d\nu_0}{dm} = 0$ on ω. By irreducibility, for any $\omega' \in \mathcal{P}_0$ there exists $n_0 \in \mathbb{N}$ such that $m(\omega' \cap F^{-n_0}(\omega)) > 0$. By the invariance of ν_0, we have

$$0 = \nu_0(\omega) = \nu_0(F^{-n_0}(\omega)) \geq \nu_0(\omega' \cap F^{-n_0}(\omega)),$$

and this means that $\nu_0(\omega' \cap F^{-n_0}(\omega)) = 0$. Then we must have $\frac{d\nu_0}{dm} = 0$ on $\omega' \cap F^{-n_0}(\omega) \subset \omega'$. From (3.1.1), we have $\frac{d\nu_0}{dm} = 0$ on ω'. This means that $\nu_0(\omega') = 0$ for any $\omega' \in \mathcal{P}_0$ and this implies $\nu_0(\Delta_0) = 0$, which is a contradiction. We conclude that $\frac{d\nu_0}{dm} > 0$ on each $\omega \in \mathcal{P}_0$. Since $d\nu_0/dm \in \mathcal{F}_\beta(\Delta_0)$ there is $C_0 > 0$ such that $0 < \frac{d\nu_0}{dm} \leq C_0$, which also implies the uniqueness of ν_0. □

Now we are able to get an ergodic invariant probability measure for the given dynamical system (that is $f : M \to M$). We will see in Section 3.2.3 that this measure is mixing, and indeed will be the measure μ in Theorem A.

Remark 3.1.7. *If $f^R : \Delta_0 \to \Delta_0$ is an irreducible induced WGM map for f then, by Theorem 3.1.6, f^R admits a unique ergodic invariant probability measure ν_0 which is equivalent to m. We can define a measure $\tilde{\mu}$ on M by*

$$\tilde{\mu} = \sum_{j=0}^{\infty} f_*^j(\nu_0|\{R > j\}).$$

If $R \in L^1(m)$ and $f_* m \ll m$, then by standard results in Ergodic Theory, $\mu = \frac{\tilde{\mu}}{\tilde{\mu}(M)}$ is the unique ergodic f-invariant probability measure such that $\mu \ll m$ and $\mu(\Delta_0) > 0$.

3.2 Mixing measures for tower maps

In this section we discuss how to obtain a mixing measure for a tower map of an induced WGM map.

3.2.1 Tower maps

We start by defining a tower map. In particular, a tower map built over an induced WGM map $f^R : \Delta_0 \to \Delta_0$ for $f : M \to M$. We also recall some important notions on this setting that will be useful. We define a *tower*

$$\Delta = \{(x, \ell) : x \in \Delta_0 \text{ and } R(x) > \ell \geq 0\}$$

and the tower map $T : \Delta \to \Delta$ given by

$$T(x,\ell) = \begin{cases} (x,\ell+1), & \text{if } R(x) > \ell+1 \\ (f^R(x),0), & \text{if } R(x) = \ell+1. \end{cases}$$

Note that we can naturally identify the set $\{(x,0)\colon x \in \Delta_0\} \subset \Delta$ with Δ_0, and the induced map $T^R : \Delta_0 \to \Delta_0$ with f^R. For each $\ell \geq 0$, we define the *ℓth level* of the tower

$$\Delta_\ell = \{(x,\ell) : x \in \Delta_0\}.$$

Moreover, the ℓth level of the tower is a copy of the set $\{R > \ell\} \subset \Delta_0$. In view of this, we may extend the σ-algebra $\mathcal{A}$ and the reference measure m on Δ_0 to a σ-algebra and a measure on Δ, that we still denote by $\mathcal{A}$ and m, respectively. We have

$$m(\Delta) = \sum_{\ell\geq 0} m(\Delta_\ell) = \sum_{\ell\geq 0} m\{R > \ell\} = \sum_{\ell\geq 1} \ell m\{R = \ell\} = \int_{\Delta_0} R dm,$$

which implies that measure m on Δ is finite if and only if $R \in L^1(m)$. The countable partition $\mathcal{P}_0$ of Δ_0 naturally induces an (m mod 0) partition of each level, that is, if $\mathcal{P}_0 = \{\Delta_{0,i}\}_{i\in\mathbb{N}}$ is the partition of Δ_0, then we can define a partition on each level of the tower Δ_ℓ, by defining its elements as

$$\Delta_{\ell,i} = \{(x,\ell) \in \Delta_\ell : (x,0) \in \Delta_{0,i}\},$$

and $\{\Delta_{\ell,i}\}_{i\in\mathbb{N}}$, forms a partition of Δ_ℓ. So, the set $\eta = \{\Delta_{\ell,i}\}_{\ell,i\in\mathbb{N}}$ is an (m mod 0) partition of Δ. We introduce a sequence of an (m mod 0) partitions η_n of Δ defined as follows

$$\eta_n = \bigvee_{i=0}^{n-1} T^{-i}\eta \quad \text{for each} \quad n \geq 1.$$

We can extend the separation time from $\Delta_0 \times \Delta_0$ to $\Delta \times \Delta$ in the following way: if $x, y \in \Delta_\ell$, then there are unique $x_0, y_0 \in \Delta_0$ such that $x = T^\ell(x_0, 0)$ and $y = T^\ell(y_0, 0)$; set $s(x, y) = s(x_0, y_0)$ and set $s(x, y) = 0$ for all other points $x, y \in \Delta$. It can be easily shown that J_T is

$$J_T(x,\ell) = \begin{cases} 1, & \text{if } R(x) > \ell+1 \\ J_{f^R}(x), & \text{if } R(x) = \ell+1. \end{cases}$$

Given $0 < \beta < 1$ we can now define the following spaces of densities for the tower:

$$\mathcal{F}_\beta(\Delta) = \{\varphi : \Delta \to \mathbb{R}\colon \exists C_\varphi > 0 : |\varphi(x) - \varphi(y)| \leq C_\varphi \beta^{s(x,y)}, \forall x, y \in \Delta\}$$

and

$$\mathcal{F}_\beta^+(\Delta) = \{\varphi \in \mathcal{F}_\beta(\Delta) \colon \exists C'_\varphi > 0 \text{ s.t.}$$
$$\varphi(x) > 0 \text{ and } \left|\frac{\varphi(x)}{\varphi(y)} - 1\right| \le C'_\varphi \beta^{s(x,y)}, \forall x, y \in \omega \in \eta\}.$$

Assuming $R \in L^1(m)$, we have $\mathcal{F}_\beta(\Delta) \subset L^\infty(\Delta, m) \subset L^1(\Delta, m)$. The next result is similar to [71, Theorem 1] (see also [3, Theorem 3.24]), and provides an ergodic invariant probability measure for the tower map $T : \Delta \to \Delta$. We notice that in our setting we do not have a positive lower bound for $d\nu/dm$.

Theorem 3.2.1. *Let $T : \Delta \to \Delta$ be the tower map of an irreducible induced WGM map $f^R : \Delta_0 \to \Delta_0$ with $R \in L^1(m)$. If ν_0 is the unique ergodic f^R-invariant probability measure such that $\frac{d\nu_0}{dm} \in \mathcal{F}_\beta^+(\Delta_0)$, then*

$$\nu = \frac{1}{\sum_{j=0}^\infty \nu_0\{R > j\}} \sum_{j=0}^\infty T_*^j(\nu_0|\{R > j\})$$

is the unique ergodic T-invariant probability measure such that $\nu \ll m$ with $\frac{d\nu}{dm} \in \mathcal{F}_\beta^+(\Delta)$, and there is $C_0 > 0$ such that $0 < \frac{d\nu}{dm} \le C_0$.

Proof. Since $T^R : \Delta_0 \to \Delta_0$ is identified with the irreducible WGM map f^R and $R \in L^1(m)$. Since $J_T > 0$, implies $T_*m \ll m$, then we conclude that ν is unique ergodic T-invariant probability measure. Now we are going to prove $\frac{d\nu}{dm} \in \mathcal{F}_\beta^+(\Delta)$. In fact $\nu|_{\Delta_0} = \nu(\Delta_0)\nu_0$, this implies that $\frac{d\nu}{dm}|_{\Delta_0} = \nu(\Delta_0)\frac{d\nu_0}{dm}$. Since $\frac{d\nu_0}{dm} \in \mathcal{F}_\beta^+(\Delta_0)$, so we must have

$$\frac{d\nu}{dm}|_{\Delta_0} \in \mathcal{F}_\beta^+(\Delta). \tag{3.2.1}$$

Let $x, y \in B \subset \Delta \setminus \Delta_0$. Since the measure m on the tower is the copies of m restricted to certain subset of Δ_0. So we can find $B' \subset \Delta_0$ and $k \ge 1$ such that $m(B) = km(B')$. Also by definition of measure ν, we have $\nu(B) = k\nu(\Delta_0)\nu_0(B')$. This means that $\frac{d\nu}{dm}|_B = k\nu(\Delta_0)\frac{d\nu_0}{dm}|_{B'}$. Moreover, the separation time $s(x, y) = s(x_0, y_0)$ for some $x_0, y_0 \in B'$. Hence from the fact $\frac{d\nu}{dm}|_B = k\nu(\Delta_0)\frac{d\nu_0}{dm}|_{B'}$ and $s(x, y) = s(x_0, y_0)$, we deduce that (3.2.1) hold on whole Δ. □

Remark 3.2.2. *From Remark 3.1.4 and the fact that the measure ν in an upper level of the tower is a normalized copy of the measure ν_0 restricted to some subset of the basis Δ_0, we conclude that there exists $C > 0$ such that for all $x, y \in \omega \in \eta$*

$$\left|\log \frac{\frac{d\nu}{dm}(x)}{\frac{d\nu}{dm}(y)}\right| \le C\beta^{s(x,y)}.$$

3.2.2 Mixing measures

In this subsection, we are going to prove that, under additional hypothesis, the ergodic T-invariant probability measure ν is mixing. To start, we would like to highlight the importance of having a coprime block.

In general, the ergodic T-invariant probability measure $\nu \ll m$ given by Theorem 3.2.1 is not mixing. For instance, if $\gcd(R) = d \geq 2$, then we have

$$T^{-dn+1}(\Delta_0) \cap \Delta_0 = \emptyset, \text{ for } n \geq 1,$$

but $\nu(\Delta_0) > 0$, which implies that the measure ν is not mixing. Hence, we conclude that $\gcd(R) = 1$ is a necessary condition for the measure ν to be mixing. If $f^R : \Delta_0 \to \Delta_0$ is an induced WGM map that satisfies the full branch property then $\gcd(R) = 1$ is a sufficient condition to conclude that ν is mixing (see [71]). However, in the absence of the full branch property, this condition is not enough to guarantee that the measure ν is mixing. For instance, consider $M = \Delta_0 = [0, 1]$, m to be the Lebesgue measure, $\mathcal{P}_0 = \{\omega_1, \omega_2, \omega_3\}$ an (m mod 0) partition of Δ_0, with $m(\omega_i) > 0$, $i = 1, 2, 3$, and $f \colon \Delta_0 \to \Delta_0$ such that (see Figure 3.1)

$$f(\omega_1) = \omega_2 \cup \omega_3, \ f(\omega_2) = \omega_1 \ \text{ and } \ f(\omega_3) = \omega_1.$$

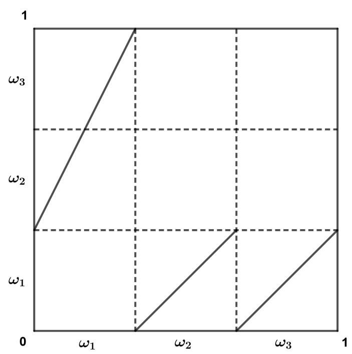

Figure 3.1: Map f.

We define $R : \Delta_0 \to \mathbb{N}$ as $R(\omega_1) = R(\omega_3) = 1$, and $R(\omega_2) = 2$. We have $(f^R)^2(\omega_1) = \Delta_0$, $(f^R)^2(\omega_2) = \Delta_0$ and $(f^R)^3(\omega_3) = \Delta_0$ (see Figure 3.2). This means that f^R is an aperiodic induced WGM map with $\gcd\{R(\omega_i)\}_{i=1}^3 = 1$. We can see that for the tower map T of f^R we have $T^k(\omega_1) \cap \omega_1 = \emptyset$, for $k = 2n + 1$ and all $n \in \mathbb{N}$, but $\nu(\omega_1) > 0$, so the measure ν is not mixing.

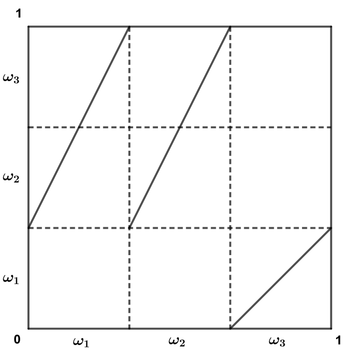

Figure 3.2: Induced map f^R.

In order to ensure that the measure ν is mixing with respect to a tower map of an aperiodic induced WGM map f^R we also need to assume that f^R has a coprime block. In fact, in this work, the coprime block is only used to get the measure ν (hence μ) to be mixing.

The next couple of lemmas are the core ingredient of proving the measure ν to be mixing.

Lemma 3.2.3. *Let $T : \Delta \to \Delta$ be the tower map of an aperiodic induced WGM map f^R with a coprime block. Given $\mathcal{P}_0' = \{\omega_1, \omega_2, \dots, \omega_k\} \subset \mathcal{P}_0$ there exists $t_0 \in \mathbb{N}$ such that for any $\omega_i \in \mathcal{P}_0'$, we have $T^t(\omega_i) \cap \Delta_0 \neq \emptyset, \forall t \geq t_0$.*

Proof. By the definition of a coprime block, we can find $\alpha_1, \alpha_2, ...\alpha_N \in \mathcal{P}_0$ such that, setting $R(\alpha_i) = R_i$, we have $\gcd\{R_i\}_{i=1}^N = 1$ and

$$f^R(\alpha_i) \supseteq \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_N \quad \text{(m mod 0)}$$

for all $i = 1, \dots, N$. Since $\gcd\{R_i\}_{i=1}^N = 1$, there exists $t_0' \in \mathbb{N}$ such that for all $t' \geq t_0'$ we can find integers $a_1, ..., a_N \geq 0$ for which $t' = a_1 R_1 + ... + a_N R_N$, which implies that

$$T^{t'}(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) \supseteq \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_N, \text{ for all } t' \geq t_0'. \tag{3.2.2}$$

By the aperiodicity of $(f^R, \mathcal{P}_0)$, there exists $n_0 \in \mathbb{N}$ such that for any $\omega_i \in \mathcal{P}_0'$ we have $(f^R)^n(\omega_i) \supseteq \alpha_1$ for all $n \geq n_0$. Then, we can find $\tilde{\omega}_i \subset \omega_i$

$$\tilde{\omega}_i = \omega_i \cap (f^R)^{-1}(\omega_{1,i}) \cap ... \cap (f^R)^{-n_0+1}(\omega_{n_0-1,i}) \in \mathcal{P}_0^{n_0}$$

for some $\omega_{1,i}, \omega_{2,i}, ..., \omega_{n_0-1,i} \in \mathcal{P}_0$ such that $(f^R)^{n_0}(\tilde{\omega}_i) = \alpha_1$. Taking

$$K_{\alpha_{1,i}} = R(\omega_i) + R(\omega_{1,i}) + ... + R(\omega_{n_0-1,i})$$

we can easily ensure that $T^{K_{\alpha_{1,i}}}(\omega_i) \supseteq \alpha_1$. By (3.2.2) and the fact that

$$T^{K_{\alpha_{1,i}}+R_1}(\omega_i) \supseteq \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_N,$$

we have $T^{t'}(T^{K_{\alpha_{1,i}}+R_1}(\omega_i)) \supseteq \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_N$, for all $t' \geq t'_0$. This implies that

$$T^t(\omega_i) \cap \Delta_0 \neq \emptyset, \qquad \text{for all } t \geq t'_0 + K_{\alpha_{1,i}} + R_1.$$

Hence by taking $t_0 = \max_{1 \leq i \leq k}\{t'_0 + K_{\alpha_{1,i}} + R_1\}$, we conclude that $T^t(\omega_i) \cap \Delta_0 \neq \emptyset$, for all $t \geq t_0$ and for any $\omega_i \in \mathcal{P}'_0$. □

Lemma 3.2.4. *Let $T : \Delta \to \Delta$ be the tower map of an aperiodic induced WGM map f^R with a coprime block. Given $\mathcal{P}'_0 = \{\omega_1, \omega_2, \ldots, \omega_k\} \subset \mathcal{P}_0$, $\varepsilon_0 > 0$ and $\ell_0 \geq 0$, there exists $t_0 \in \mathbb{N}$ such that for any $\omega_i \in \mathcal{P}'_0$ we have*

$$m(T^t(\omega_i)) \geq m\left(\bigcup_{j \leq \ell_0} \Delta_j\right) - \varepsilon_0 \text{ for all } t \geq t_0.$$

Proof. Let Z be a finite union of elements of $\mathcal{P}_0$ such that $m(Z) \geq m(\Delta_0) - \frac{\varepsilon_0}{\ell_0}$. By the aperiodicity of $(f^R, \mathcal{P}_0)$, there exists $n_0 \in \mathbb{N}$ such that for any $\omega_i \in \mathcal{P}'_0$ we have

$$m\big((f^R)^n(\omega_i)\big) \geq m(Z), \text{ for all } n \geq n_0.$$

As in the proof of Lemma 3.2.3, we have that there is $t'_0 \in \mathbb{N}$ such that

$$T^{t'}(T^{K_{\alpha_{1,i}}+R_1}(\omega_i)) \supseteq \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_N, \text{ for all } t' \geq t'_0. \tag{3.2.3}$$

Let $\bar{\alpha}$ be an arbitrary element of $\mathcal{P}_0$ such that $\bar{\alpha} \subset Z$. For any $\alpha_j \in \{\alpha_1, \alpha_2, ..., \alpha_N\}$, there exist $k_0 \in \mathbb{N}$ such that $(f^R)^n(\alpha_j) \supseteq \bar{\alpha}$ for all $n \geq k_0$. As in the proof of Lemma 3.2.3 we can find $\tilde{\alpha_j} \in \mathcal{P}_0^{k_0}$, $K_{\bar{\alpha}} \in \mathbb{N}$ such that $(f^R)^{k_0}(\tilde{\alpha_j}) = \bar{\alpha}$, and $T^{K_{\bar{\alpha}}}(\alpha_j) \supseteq \bar{\alpha}$. Since $T^{t'}(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) \supseteq \alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N$ for all $t' \geq t'_0$ and $T^{K_{\bar{\alpha}}}(\alpha_j) \supseteq \bar{\alpha}$, this implies that $T^{t'+K_{\bar{\alpha}}}(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) \supseteq \bar{\alpha}$ for all $t' \geq t'_0$ and hence we can find t'_1 such that $T^{t'}(\alpha_1 \cup \alpha_2 \cup ... \cup \alpha_N) \supseteq Z$ for all $t' \geq t'_1$, which, together with (3.2.3), implies that we can find t'_2 such that $T^{t'}(\omega_i) \supseteq Z$ for all $t' \geq t'_2$. Hence by using the above fact and Lemma 3.2.3, we can find $t_0 > 0$ such that for all $t \geq t_0$,

$$m(T^t(\omega_i)) \geq \sum_{j \leq \ell_0} \left(m(\Delta_j) - \frac{\varepsilon_0}{\ell_0}\right) = m\left(\bigcup_{j \leq \ell_0} \Delta_j\right) - \varepsilon_0.$$

□

Finally, we have the result about mixing measure for the tower map.

Theorem 3.2.5. *Let $T : \Delta \to \Delta$ be the tower map of an aperiodic induced WGM map f^R with a coprime block and $R \in L^1(m)$. Then the T-invariant probability measure ν is exact.*

Proof. Consider $A \in \bigcap_{n\in\mathbb{N}} T^{-n}(\mathcal{A})$, with $\nu(A) > 0$. It is enough to show that $\nu(A) > 1-\varepsilon$ for any $\varepsilon > 0$. Take an arbitrary $\varepsilon > 0$ and let $C_0 > 0$ be given by Theorem 3.2.1. Since $m(\Delta) < \infty$, we can find $\ell_0 \geq 0$ such that

$$m\left(\Delta \backslash \bigcup_{j\leq \ell_0} \Delta_j\right) < \frac{\varepsilon}{3C_0}. \tag{3.2.4}$$

From $W5$) of Definition 2.1.1, there exists $\mathcal{P}_0' = \{\omega_1, \omega_2, \ldots, \omega_k\} \subset \mathcal{P}_0$ such that

$$f^R(\omega) \cap \mathcal{P}_0' \neq \emptyset, \text{ for all } \omega \in \mathcal{P}_0. \tag{3.2.5}$$

By Lemma 3.2.4, there exists $t_0 \in \mathbb{N}$ for $1 \leq i \leq k$ we have

$$m(\bigcup_{j\leq \ell_0} \Delta_j \backslash T^t(A)) < m(T^t(\omega_i)\backslash T^t(A)) + \frac{\varepsilon}{3C_0} \text{ for all } t \geq t_0. \tag{3.2.6}$$

By using the fact that T is non-singular with respect to m, for $\varepsilon_0 = \varepsilon/3C_0$, t_0, taking $C > 0$ to be the constant given by ii) in Lemma 3.1.1, we can find $\delta > 0$ such that, for all $B \in \mathcal{A}$

$$m(B) < Cm(\Delta_0)\delta \Rightarrow m(T^{t_0}(B)) < \frac{\varepsilon}{3C_0}. \tag{3.2.7}$$

Since $\nu \ll m$, we have $m(A) > 0$, and then there exists $s \geq 0$ such that $m(T^s(A)\cap\Delta_0) > 0$. Applying Lemma A.1.7 to the WGM map $f^R : \Delta_0 \to \Delta_0$ and the set $T^s(A) \cap \Delta_0$, with $\delta' < \delta$ we find $n \in \mathbb{N}$ and $\omega \in \mathcal{P}_0^n$ such that

$$\frac{m(\omega\backslash T^s(A) \cap \Delta_0)}{m(\omega)} = \frac{m(\omega\backslash T^s(A))}{m(\omega)} < \delta'.$$

Applying ii) of Lemma 3.1.1 to $\omega \in \mathcal{P}_0^n$ we get

$$\frac{m((f^R)^n(\omega\backslash T^s(A)))}{m((f^R)^n(\omega))} \leq C\frac{m(\omega\backslash T^s(A))}{m(\omega)} < C\delta'.$$

From the above inequality, we have

$$m((f^R)^n(\omega\backslash T^s(A))) < C\delta' m((f^R)^n(\omega)) \leq C\delta m(\Delta_0). \tag{3.2.8}$$

Since $\omega \in \mathcal{P}_0^n$, the points in $\omega \in \mathcal{P}_0^n$ have the same first n-recurrence times to the base of the tower. This means that there are natural numbers $R_1, \ldots, R_n \in R(\Delta_0)$ such that

$$(f^R)^n|_\omega = T^{R_1+\cdots+R_n}|_\omega.$$

Set $r = R_1 + \cdots + R_n$. It follows from (3.2.8) that

$$m(T^r(\omega) \backslash T^{r+s}(A)) < C\delta m(\Delta_0).$$

By (3.2.7) we get

$$m(T^{t_0+r}(\omega) \backslash T^{t_0+r+s}(A)) < \frac{\varepsilon}{3C_0}. \tag{3.2.9}$$

It follows from (3.2.5) that for $\omega \in \mathcal{P}_0^n$, we have

$$T^r(\omega) = (f^R)^n(\omega) \supseteq \omega_i \text{ for some } \omega_i \in \mathcal{P}_0'. \tag{3.2.10}$$

By combining (3.2.6) and (3.2.10), we have

$$m(\bigcup_{j \le \ell_0} \Delta_j \backslash T^{t_0+r+s}(A)) < m(T^{t_0+r}(\omega) \backslash T^{t_0+r+s}(A)) + \frac{\varepsilon}{3C_0}. \tag{3.2.11}$$

Using (3.2.4), (3.2.11) and (3.2.9) we have

$$\begin{aligned} m(\Delta \backslash T^{t_0+r+s}(A)) &= m((\Delta \backslash \bigcup_{j \le \ell_0} \Delta_j) \backslash T^{t_0+r+s}(A)) + m(\bigcup_{j \le \ell_0} \Delta_j \backslash T^{t_0+r+s}(A)) \\ &< \frac{\varepsilon}{3C_0} + m(T^{t_0+r}(\omega) \backslash T^{t_0+r+s}(A)) + \frac{\varepsilon}{3C_0} \\ &< \frac{\varepsilon}{3C_0} + \frac{\varepsilon}{3C_0} + \frac{\varepsilon}{3C_0} = \frac{\varepsilon}{C_0}. \end{aligned}$$

This implies $\nu(\Delta \backslash T^{t_0+r+s}(A)) < \varepsilon$. Since $A \in \bigcap_{n \in \mathbb{N}} T^{-n}(\mathcal{A})$, then $A = T^{-(t_0+r+s)}(A')$, for some $A' \in \mathcal{A}$ with $T^{t_0+r+s}(A) = A'$. Since $\nu(A') = \nu(T^{t_0+r+s}(A)) > 1 - \varepsilon$, using the invariance of ν we have

$$\nu(A) = \nu(A') = \nu\left(T^{-(t_0+r+s)}(A')\right) > 1 - \varepsilon$$

Since ε is arbitrary, we have $\nu(A) = 1$, so ν is exact. $\square$

3.2.3 Tower Extension

In this subsection, we get the invariant probability measure μ, to be mixing through a semi-conjugacy. We will see the use of this semi-conjugacy in upcoming chapters as well.

Let us define a measurable semi-conjugacy $\pi : \Delta \to M$ between the tower map T and the map f, by $\pi(x, \ell) = f^{\ell}(x)$. Let ν be the measure from Theorem 3.2.5. It was shown in [3, Lemma 3.26] that $\pi \circ T = f \circ \pi$ and $\pi_* \nu$ coincides with the f-invariant measure μ given by Remark 3.1.7. In this case we can say that (T, ν) is an *extension* of (f, μ). Since ν is mixing (for T) it is an immediate consequence that the measure μ is mixing (for f).

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Chapter 4

Statistical properties for Hölder observables

In this chapter, we will prove the results of Section 2.2. Our strategy is to obtain the decay of correlations and CLT for the tower map, and then carry the results back to the given dynamical system $f : M \to M$ through a semi-conjugacy. More precisely, the proof of the Theorem A can be deduced from Proposition 4.1.4 (see Remark 4.1.7), and Corollary B follows from Proposition 4.2.2.

4.1 Decay of correlations for the tower map

The key result of this section is Proposition 4.1.4, which implies the decay of correlations for the tower map. We first reduce the problem of decay of correlations for the given dynamical system to the similar problem for a tower map, and provide some auxiliary results to deal with this problem.

Recall from Chapter 3 that (f, μ) and (T, ν) are semiconjugated systems via $\pi : \Delta \to M$. This allows us to relate the decay of correlations of the given dynamical system and the tower map, in the following way:, for all $\varphi, \psi : M \to \mathbb{R}$, we have

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \mathrm{Cor}_\nu(\varphi \circ \pi, \psi \circ \pi \circ T^n).$$

Note that to obtain a decay of correlations for (f, μ), it therefore suffices to obtain a decay of correlations for (T, ν), for the lifted observables.

Given observables $\varphi \in \mathcal{H}_\eta$ and $\psi \in L^\infty(M, m)$, we need that the lifted observables belong to some suitable spaces, namely $\varphi \circ \pi \in \mathcal{F}_\beta(\Delta)$ and $\psi \circ \pi \in L^\infty(\Delta, m)$. Obviously, if $\psi \in L^\infty(M, m)$ then $\psi \circ \pi \in L^\infty(\Delta, m)$. In order to prove $\varphi \circ \pi \in \mathcal{F}_\beta(\Delta)$ for $\varphi \in \mathcal{H}_\eta$ we have the following result.

Lemma 4.1.1. *[3, Lemma 3.53] If $T : \Delta \to \Delta$ is the tower map of an induced WGM expanding map $f^R : \Delta_0 \to \Delta_0$, then $\varphi \in \mathcal{H}_\eta \Rightarrow \varphi \circ \pi \in \mathcal{F}_{\beta^\eta}(\Delta)$.*

Given $\varphi \in L^\infty(\Delta, m)$ we define

$$\varphi^* = \frac{1}{\int (\varphi + 2||\varphi||_\infty + 1) d\nu} (\varphi + 2||\varphi||_\infty + 1).$$

Given a signed measure υ on Δ we define its *total variation* as

$$|\upsilon| = \int_\Delta \left|\frac{d\upsilon}{dm}\right| dm.$$

Lemma 4.1.2. *[3, Lemma 3.49] For all $\varphi \in \mathcal{F}_\beta(\Delta)$, with $\varphi \neq 0$, we have*

i) $\frac{1}{3} \leq \varphi^* \leq 3$ *and* $\varphi^* \in \mathcal{F}_\beta^+(\Delta)$;

ii) $\mathrm{Cor}_\nu(\varphi, \psi \circ T^n) \leq 3(||\varphi||_\infty + 1)||\psi||_\infty |T_*^n\lambda - \nu|$, *for all* $\psi \in L^\infty(\Delta, m)$, *where* λ *is the probability measure on* Δ *such that* $\frac{d\lambda}{dm} = \varphi^* \frac{d\nu}{dm}$.

Remark 4.1.3. *We notice that in [3, Lemma 3.53, Lemma 3,49], the author considered a tower map associated with an induced GM expanding map. However, the proofs do not require the full branch property. This means that we can derive Lemma 4.1.1 and Lemma 4.1.2 by considering an induced WGM expanding map.*

From *ii*) of Lemma 4.1.2 to obtain decay of correlation for the tower map it is enough to estimate $|T_*^n\lambda - \nu|$ with $\frac{d\lambda}{dm} = \varphi^* \frac{d\nu}{dm}$. We have the following result.

Proposition 4.1.4. *Let $T : \Delta \to \Delta$ be the tower map of an aperiodic induced WGM map f^R with a coprime block and $R \in L^1(m)$. If ν is the unique mixing T-invariant probability measure such that $\frac{d\nu}{dm} \in \mathcal{F}_\beta^+(\Delta)$, then*

1) if $m\{R > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 1$, then for any probability measure λ with $\frac{d\lambda}{dm} \in \mathcal{F}_\beta^+(\Delta)$, there exists C' such that

$$|T_*^n\lambda - \nu| \leq C'n^{-a+1};$$

2) if $m\{R > n\} \leq Ce^{-cn^a}$ for some $C, c > 0$ and $0 < a \leq 1$, given $0 < \beta < 1$ there is $c' > 0$ such that, for any probability measure λ with $\frac{d\lambda}{dm} \in \mathcal{F}_\beta^+(\Delta)$, there is some $C' > 0$ such that

$$|T_*^n\lambda - \nu| \leq C' e^{-c'n^a}.$$

We remark that in this result the coprime block and the aperiodicity are only used to ensure that the T-invariant probability measure ν is unique, mixing and that $d\nu/dm > 0$. Now, in order to apply Proposition 4.1.4 to prove Theorem A, given $\varphi \in \mathcal{F}_\beta(\Delta)$ we need to check the necessary regularity of $\frac{d\lambda}{dm} = \varphi^* \frac{d\nu}{dm}$.

Lemma 4.1.5. *If $\varphi \in \mathcal{F}_\beta(\Delta)$, then $\varphi^* \frac{d\nu}{dm} \in \mathcal{F}_\beta^+(\Delta)$.*

Proof. Set $\rho = \frac{d\nu}{dm}$. We notice that $\int(\varphi + 2||\varphi||_\infty + 1)d\nu \geq 1$, and since $0 < \rho \leq C_0$, we get

$$
\begin{aligned}
|\varphi^*(x)\rho(x) - \varphi^*(y)\rho(y)| &\leq |\varphi^*(x)(\rho(x) - \rho(y))| + |\rho(y)(\varphi^*(x) - \varphi^*(y))| \\
&\leq 3|\rho(x) - \rho(y)| + C_0|\varphi^*(x) - \varphi^*(y)| \\
&= 3|\rho(x) - \rho(y)| + \frac{C_0}{\int(\varphi + 2||\varphi||_\infty + 1)d\nu}|\varphi(x) - \varphi(y)| \\
&\leq 3|\rho(x) - \rho(y)| + C_0|\varphi(x) - \varphi(y)| \\
&\leq 3C_\rho \beta^{s(x,y)} + C_0 C_\varphi \beta^{s(x,y)} \\
&= C_{\varphi^*\rho}\beta^{s(x,y)},
\end{aligned}
$$

where $C_{\varphi^*\rho} = 3C_\rho + C_0 C_\varphi$. Thus $\varphi^*\rho \in \mathcal{F}_\beta(\Delta)$. From Lemma 4.1.2 we have

$$
\begin{aligned}
\left|\frac{\varphi^*(x)}{\varphi^*(y)} - 1\right| &= \frac{1}{\varphi^*(y)}\,|\varphi^*(x) - \varphi^*(y)| \\
&\leq 3\frac{|\varphi(x) - \varphi(y)|}{\int(\varphi + 2||\varphi||_\infty + 1)d\nu} \\
&\leq 3\,|\varphi(x) - \varphi(y)| \\
&\leq 3C_\varphi \beta^{s(x,y)}.
\end{aligned}
\tag{4.1.1}
$$

Since $\frac{1}{9} \leq \frac{\varphi^*(x)}{\varphi^*(y)} \leq 9$ for all $x, y \in \Delta$, then there exists $K_1' > 0$ such that

$$
\left|\log\frac{\varphi^*(x)}{\varphi^*(y)}\right| \leq K_1' \left|\frac{\varphi^*(x)}{\varphi^*(y)} - 1\right|. \tag{4.1.2}
$$

It follows from (4.1.1),(4.1.2) and Remark 3.2.2 that for all $x, y \in \omega \in \eta$,

$$
\begin{aligned}
\left|\log\frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)}\right| &\leq \left|\log\frac{\varphi^*(x)}{\varphi^*(y)}\right| + \left|\log\frac{\rho(x)}{\rho(y)}\right| \\
&\leq 3K_1' C_\varphi \beta^{s(x,y)} + C\beta^{s(x,y)} \\
&= (3K_1' C_\varphi + C)\beta^{s(x,y)}.
\end{aligned}
$$

Since $0 < \frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)} \leq 9e^C$, for all $x, y \in \omega \in \eta$, then we also have some uniform constant $K_2' > 0$ such that

$$
\left|\frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)} - 1\right| \leq K_2' \left|\log\frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)}\right| \leq K_2'(3K_1' C_\varphi + C)\beta^{s(x,y)}.
$$

Hence $\varphi^*\rho \in \mathcal{F}_\beta^+(\Delta)$. $\square$

We would like to point a remark about $\varphi^*\rho$, which is important because we will assume this kind of regularity in the upcoming sections.

Remark 4.1.6. *From the previous proof we can see that there exists $C''_{\varphi^*\rho} > 0$ such that for all $x, y \in \omega \in \eta$,*

$$\left|\log \frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)}\right| \leq C''_{\varphi^*\rho}\beta^{s(x,y)}.$$

Remark 4.1.7. *Our main Theorem A is direct consequence of Proposition 4.1.4, Lemma 4.1.1 and Lemma 4.1.2. So, in the remaining of this chapter we prove Proposition 4.1.4.*

Let λ_1 and λ_2 be probability measures on Δ with $\varphi_1 = \frac{d\lambda_1}{dm}, \varphi_2 = \frac{d\lambda_2}{dm} \in \mathcal{F}_\beta^+(\Delta)$. We start with some bounded distortion results for the tower map $T : \Delta \to \Delta$.

Lemma 4.1.8. *[3, Lemma 3.30] There exists $\bar{C}_T > 0$ such that for all $n \geq 1$, $\omega \in \eta_n$ and $x, y \in \omega$,*

$$\log \frac{J_{T^n}(x)}{J_{T^n}(y)} \leq \bar{C}_T\beta^{s(T^n(x),T^n(y))}.$$

Remark 4.1.9. *Note that in [3, Lemma 3.30], the tower map was considered associated with an induced GM map. However, the proof of [3, Lemma 3.30] relies on part i) of Lemma 3.1.1 which doest not require the full branch property.*

The next Lemma is the analogous of [3, Lemma 3.31]. Instead of taking $x, y \in \Delta_0$, we pick x, y in the image of the element of partition.

Lemma 4.1.10. *There exists $C_1 > 0$ depending on C'_{φ_1} such that for all $n \geq 1$, $\omega \in \eta_n$ and $x, y \in T^n(\omega)$, we have*

$$\frac{dT_*^n(\lambda_1|\omega)}{dm}(x) \bigg/ \frac{dT_*^n(\lambda_1|\omega)}{dm}(y) \leq C_1.$$

Moreover, the dependence of C_1 on C'_{φ_1} can be removed if the number of visits of ω to Δ_0 up to n is sufficiently large.

Proof. Let $x_0, y_0 \in \omega$ be such that $T^n(x_0) = x$ and $T^n(y_0) = y$. We may write

$$\frac{dT_*^n(\lambda_1|\omega)}{dm}(x) \bigg/ \frac{dT_*^n(\lambda_1|\omega)}{dm}(y) = \frac{J_{T^n}(y_0)}{J_{T^n}(x_0)}\frac{\varphi_1(x_0)}{\varphi_1(y_0)}. \tag{4.1.3}$$

By Lemma 4.1.8, there exists some $C'_T > 0$ such that

$$\frac{J_{T^n}(y_0)}{J_{T^n}(x_0)} \leq C'_T. \tag{4.1.4}$$

Considering i the number of visits of ω to Δ_0 until n, we have $s(x_0, y_0) \geq i$. Since $\varphi_1 \in \mathcal{F}_\beta^+(\Delta)$ we may write

$$\frac{\varphi_1(x_0)}{\varphi_1(y_0)} \leq 1 + \left|\frac{\varphi_1(x_0)}{\varphi_1(y_0)} - 1\right| \leq 1 + C'_{\varphi_1}\beta^i. \tag{4.1.5}$$

From (4.1.3), (4.1.4) and (4.1.5), we get

$$\frac{dT_*^n(\lambda_1|\omega)}{dm}(x) \Big/ \frac{dT_*^n(\lambda_1|\omega)}{dm}(y) \leq C_1,$$

where $C_1 = C'_T(1 + C'_{\varphi_1}\beta^i)$. Note that the constant C'_T does not depend on φ_1. So the dependence of C_1 on φ_1 can be removed if we take i sufficiently large. □

Remark 4.1.11. *If we replace λ_1 with λ_2 in Lemma 4.1.10, then the result remain true with $C_1 = C'_T(1 + C'_{\varphi_2}\beta^i)$.*

4.1.1 Coupling

In this section, we estimate $|T_*^n\lambda_1 - T_*^n\lambda_2|$ for a pair of measures λ_1, λ_2 whose densities with respect to m are of some given regularity. In order to conclude the proof of Proposition 4.1.4, we will take $\lambda_1 = \lambda$, where λ is coming from Lemma 4.1.2, and $\lambda_2 = \nu$ in Section 4.1.2 and Section 4.1.3. However, it is useful to work with these general pairs of measures λ_1, λ_2 because we need an estimate when $\lambda_2 \neq \nu$ for the CLT.

Our argument follows the line of approach of [71]. The key difference relies on the absence of the full branch property. We would like to share some remarks concerning the lack of full branch property:

- The density of the invariant measure ν for the tower map does not necessarily have a positive lower bound. The invariant measure behaves better in case of having a positive lower bound (this simplifies many situations, as seen in [3]).

- The choice of n_0 in the definition of the sequence of *stopping times* below is made by using the Long branch property $W5$) of Definition 2.1.1 in such a way that we obtain [71, Lemmas 1 and 2] without the full branch property.

- The *weak* hypothesis does not guarantee a symmetry as in [71, (1)] (see also [3, Lemmas 3.39 and 3.43]). This implies that the quantity I_2 in the proof of Proposition 4.1.25 does not vanish.

Let $P = \lambda_1 \times \lambda_2$ be the product measure on $\Delta \times \Delta$. We consider the product transformation $T \times T : \Delta \times \Delta \to \Delta \times \Delta$, and let $\pi_1, \pi_2 : \Delta \times \Delta \to \Delta$ be the projections onto the first and second coordinates, respectively. We use $\eta \times \eta$ to denote the product partition of $\Delta \times \Delta$ and $(\eta \times \eta)_n = \bigvee_{i=0}^{n-1} (T \times T)^{-i}(\eta \times \eta)$. Note that

$$T^n \circ \pi_1 = \pi_1 \circ (T \times T)^n \quad \text{and} \quad T^n \circ \pi_2 = \pi_2 \circ (T \times T)^n. \tag{4.1.6}$$

Let $\hat{R} : \Delta \to \mathbb{N}$ be the return time to the base Δ_0, defined for $x \in \Delta$ by

$$\hat{R}(x) = \min\{n \geq 0 : T^n(x) \in \Delta_0\}.$$

Note that

$$m\{\hat{R} > n\} = \sum_{\ell > n} m(\Delta_\ell) = \sum_{\ell > n} m\{R > \ell\}. \tag{4.1.7}$$

From $W5$) of Definition 2.1.1, there exists $\mathcal{P}_0' = \{\omega_1, \omega_2, \ldots, \omega_k\} \subset \mathcal{P}_0$ such that

$$f^R(\omega) \cap \mathcal{P}_0' \neq \emptyset, \quad \text{for all } \omega \in \mathcal{P}_0.$$

This implies that for any $\omega \in \mathcal{P}_0$, we have

$$f^R(\omega) \supseteq \omega_i \quad \text{for some } \omega_i \in \mathcal{P}_0'. \tag{4.1.8}$$

Since the measure $\nu \ll m$ is mixing, for any given $\varepsilon > 0$ small enough and any $\omega_i \in \mathcal{P}_0'$ there exists $n_i = n_i(\omega_i) \in \mathbb{N}$ such that $\nu(T^{-n}(\Delta_0) \cap \omega_i) \geq \nu(\Delta_0)\nu(\omega_i) - \varepsilon > 0$ for $n \geq n_i$. As $\frac{d\nu}{dm} \leq C_0$, then for $M_i = \frac{1}{C_0}(\nu(\Delta_0)\nu(\omega_i) - \varepsilon)$ we have

$$m(T^{-n}(\Delta_0) \cap \omega_i) \geq M_i \quad \text{for } n \geq n_i.$$

Let $n_0 = \max\{n_1, n_2, ..., n_k\}$ and $\gamma_0 = \min\{M_1, M_2, ..., M_k\}$. Then for all $1 \leq i \leq k$ we have

$$m(T^{-n}(\Delta_0) \cap \omega_i) \geq \gamma_0 \quad \text{for } n \geq n_0. \tag{4.1.9}$$

It follows from (4.1.8) and (4.1.9) that for any $\omega \in \mathcal{P}_0$, we have

$$m(T^{-n}(\Delta_0) \cap f^R(\omega)) \geq \gamma_0 \text{ for } n \geq n_0. \tag{4.1.10}$$

Let us now introduce a sequence of *stopping times* $0 = \tau_0 < \tau_1 < \tau_2 ...$ on $\Delta \times \Delta$,

$$\begin{aligned}
\tau_1 &= n_0 + \hat{R} \circ T^{n_0} \circ \pi_1 \\
\tau_2 &= n_0 + \tau_1 + \hat{R} \circ T^{n_0+\tau_1} \circ \pi_2 \\
\tau_3 &= n_0 + \tau_2 + \hat{R} \circ T^{n_0+\tau_2} \circ \pi_1 \\
\tau_4 &= n_0 + \tau_3 + \hat{R} \circ T^{n_0+\tau_3} \circ \pi_2 \\
&\vdots
\end{aligned}$$

We define the simultaneous return to the base $S : \Delta \times \Delta \to \mathbb{N}$ by

$$S(x,y) = \min_{i\geq 2}\{\tau_i(x,y) : (T^{\tau_i(x,y),}(x), T^{\tau_i(x,y),}(y)) \in \Delta_0 \times \Delta_0\}. \tag{4.1.11}$$

Let us check that S is well defined $m \times m$ almost everywhere. Indeed, (T, ν) is mixing, $(T \times T, \nu \times \nu)$ is mixing and in particular $(T \times T, \nu \times \nu)$ is ergodic.

Let $U = \{(x, y) \in \Delta \times \Delta : (T^{\tau_i(x,y),}(x), T^{\tau_i(x,y),}(y)) \in \Delta_0 \times \Delta_0 \text{ for some } \tau_i\}$. Since $U' = \{(x, x) \in \Delta \times \Delta : (T^{\tau_i(x,x),}(x), T^{\tau_i(x,x),}(x)) \in \Delta_0 \times \Delta_0\} \subset U$, with $(\nu \times \nu)(U') > 0$ and $(T \times T)^{-1}(U) = U$, we have that U has full $\nu \times \nu$ measure. Since $m \times m$ is equivalent to $\nu \times \nu$, U has full $m \times m$ measure and thus we conclude that S is well defined $m \times m$ almost everywhere.

Let $\xi_0 < \xi_1 < \xi_2 < \xi_3 ...$ be an increasing sequence of partitions on $\Delta \times \Delta$ defined as follows. As usual, given a partition ξ, we denote $\xi(x)$ the element of ξ containing x. First we take $\xi_0 = \eta \times \eta$. For each $\overline{x} = (x, y) \in \Delta \times \Delta$, we define

$$\xi_1(\overline{x}) = \bigvee_{i=0}^{\tau_1(\overline{x})-1} (T^{-i}(\eta))(x) \times \eta(y),$$

and set, $\xi_1 = \{\xi_1(\overline{x}) : \overline{x} \in \Delta \times \Delta\}$. Now we define $\overline{x} = (x, y) \in \Delta \times \Delta$,

$$\xi_2(\overline{x}) = \pi_1(\xi_1(\overline{x})) \times \bigvee_{i=0}^{\tau_2(\overline{x})-1} (T^{-i}(\eta))(y),$$

and set $\xi_2 = \{\xi_2(\overline{x}) : \overline{x} \in \Delta \times \Delta\}$. Similarly, we can define $\xi_3 = \{\xi_3(\overline{x}) : \overline{x} \in \Delta \times \Delta\}$, where

$$\xi_3(\overline{x}) = \bigvee_{i=0}^{\tau_3(\overline{x})-1} (T^{-i}(\eta))(\pi_1(\overline{x})) \times \pi_2(\xi_2(\overline{x})).$$

Now we describe the general inductive step in the construction of partitions ξ_k. Assume that ξ_j has been constructed for all $j < k$. The definition of ξ_k depends on whether k is odd or even. For definiteness we assume that k is odd. The construction for k even is the same apart from the change in the role of the first and second components.

We let $\xi_k = \{\xi_k(\overline{x}) : \overline{x} \in \Delta \times \Delta\}$, where

$$\xi_k(\overline{x}) = \bigvee_{i=0}^{\tau_k(\overline{x})-1} (T^{-i}(\eta))(\pi_1(\overline{x})) \times \pi_2(\xi_{k-1}(\overline{x})).$$

Remark 4.1.12. *From the construction of above sequence of partitions ξ_k, we have the following properties:*

i) the functions $\tau_1, \tau_2, ..., \tau_k$ are constant on elements of ξ_k;

ii) the sets $\{S = \tau_{k-1}\}$ and $\{S > \tau_{k-1}\}$ can be written as union of elements of ξ_k.

Remark 4.1.13. *For any $\Gamma \in \xi_k$, we can find some $\omega_1, \omega_2 \in \mathcal{P}_0$ such that*

i) if k is odd, $T^{\tau_k}(\pi_1(\Gamma)) = f^R(\omega_1) \subset \Delta_0$ and $T^{\tau_{k-1}}(\pi_2(\Gamma)) = f^R(\omega_2) \subset \Delta_0$;

ii) if k is even, $T^{\tau_{k-1}}(\pi_1(\Gamma)) = f^R(\omega_1) \subset \Delta_0$ and $T^{\tau_k}(\pi_2(\Gamma)) = f^R(\omega_2) \subset \Delta_0$.

4.1.1.1 Estimating $P\{S > n\}$

The following lemmas, which have a similar role of that [71, Lemma 1 and Lemma 2], are crucial to the proof of Proposition 4.1.4. More precisely, they will be helpful for estimating $P\{S > n\}$. We will see the final estimates of $P\{S > n\}$ in Section 4.1.2 and Section 4.1.3.

Lemma 4.1.14. *There exists $\varepsilon_0 > 0$, depending on $C'_{\varphi_1}, C'_{\varphi_2}$, such that for all $k \geq 1$ and $\Gamma \in \xi_k$ with $S|_\Gamma > \tau_{k-1}$, we have*

$$P(S = \tau_k | \Gamma) \geq \varepsilon_0.$$

Moreover, the dependence of ε_0 on $C'_{\varphi_1}, C'_{\varphi_2}$ can be removed if we take k sufficiently large.

Proof. Consider first that $k \geq 1$ is odd. For $\Gamma = \omega \times \omega' \in \xi_k$, we have T^{τ_k} mapping ω bijectively onto a union of elements of $\eta|_{\Delta_0}$ and $T^{\tau_{k-1}}$ mapping ω' bijectively onto a union of elements $\eta|_{\Delta_0}$. Let $\tilde{\omega} = \{z \in \omega' : T^{\tau_k}(z) \in \Delta_0\}$. We may write

$$P\left(S = \tau_k \mid \Gamma\right) = \frac{\lambda_2(\tilde{\omega})}{\lambda_2(\omega')} = \frac{T_*^{\tau_{k-1}}\left(\lambda_2 \mid \omega'\right)\left(T^{-(\tau_k - \tau_{k-1})}(\Delta_0) \cap T^{\tau_{k-1}}(\omega')\right)}{T_*^{\tau_{k-1}}\left(\lambda_2 \mid \omega'\right)\left(T^{\tau_{k-1}}(\omega')\right)}.$$

By Lemma 4.1.10, we get

$$P\left(S = \tau_k \mid \Gamma\right) \geq \frac{m\left(T^{-(\tau_k - \tau_{k-1})}(\Delta_0) \cap T^{\tau_{k-1}}(\omega')\right)}{C_1^2\, m\left(T^{\tau_{k-1}}(\omega')\right)} \geq \frac{m\left(T^{-(\tau_k - \tau_{k-1})}(\Delta_0) \cap T^{\tau_{k-1}}(\omega')\right)}{C_1^2\, m(\Delta_0)}.$$

By definition $\tau_k - \tau_{k-1} \geq n_0$ and, from Remark 4.1.13, $T^{\tau_{k-1}}(\omega') = f^R(\omega_2)$ for some $\omega_2 \in \mathcal{P}_0$. From (4.1.10) we get $m\left(T^{-(\tau_k-\tau_{k-1})}(\Delta_0) \cap T^{\tau_{k-1}}(\omega')\right) \geq \gamma_0$. Thus we conclude that

$$P(S = \tau_k|\Gamma) \geq \varepsilon_0,$$

with $\varepsilon_0 = \gamma_0/C_1^2 m(\Delta_0)$. The case k even can be proved similarly. Moreover, the dependence of C_1 (hence ε_0) on $C'_{\varphi_1}, C'_{\varphi_2}$ can be removed if we take k large enough. $\square$

In the proof of next lemma, we will use the fact that there exists some constant $M_0 > 0$ such that

$$\frac{dT_*^n m}{dm} \leq M_0 \text{ for all } n \geq 1. \tag{4.1.12}$$

Indeed, since $T_*^n m(\Delta) \leq m(\Delta) < \infty$ and for every $n \geq 1$, there is $k_n \geq 1$ such that $T_*^n m(A) = (f_*^R)^{k_n} m(A)$, for $A \subset \Delta_0$, then it follows from *iii)* of Lemma 3.1.1 that

$$\frac{dT_*^n m}{dm}|_{\Delta_0} = \frac{d(f_*^R)^{k_n} m}{dm} \leq \text{ some } M_0.$$

For $\ell \geq n$ and $A \subset \Delta_\ell$, we have $T_*^n m(A) = m(A)$, and hence $\frac{dT_*^n m}{dm}|_{\Delta_\ell} = 1$. Now For $\ell < n$ and $A \subset \Delta_\ell$, there exists $A' \subset \Delta_0$ such that $T^\ell(A') = A$, this means that in this case it is enough to prove $\frac{dT_*^{n-\ell} m}{dm} \leq M_0$. Since $T_*^{n-\ell} m(A') = (f_*^R)^{k_n} m(A')$, for some k_n and this implies that $\frac{dT_*^{n-\ell} m}{dm} = \frac{d(f_*^R)^{k_n} m}{dm} \leq$ some M_0.

Lemma 4.1.15. *There exists C_2, depending on $C'_{\varphi_1}, C'_{\varphi_2}$, such that for all $n, k \geq 0$ and for all $\Gamma \in \xi_k$, we have*

$$P(\tau_{k+1} - \tau_k > n + n_0|\Gamma) \leq C_2 m\{\hat{R} > n\}.$$

Moreover, the dependence of C_2 on $C'_{\varphi_1}, C'_{\varphi_2}$ can be removed if we take k sufficiently large.

Proof. First we consider $k = 0$, then we can write for all $n \geq 0$

$$\begin{aligned} P\{\tau_1 > n_0 + n\} &= P\left(\{\hat{R} \circ T^{n_0} > n\} \times \Delta\right) \\ &= (T_*^{n_0}\lambda_1)\{\hat{R} > n\} \\ &\leq \|\varphi_1\|_\infty (T_*^{n_0} m)\{\hat{R} > n\} \\ &\leq M_0 \|\varphi_1\|_\infty m\{\hat{R} > n\}. \end{aligned}$$

Now consider $k \geq 1$ odd. Since for $\Gamma = \omega \times \omega' \in \xi_k$, we have $T^{\tau_{k-1}}(\omega)$ contained in some element of η and $T^{\tau_{k-1}}$ mapping ω' bijectively onto a union of elements in $\eta|_{\Delta_0}$.

This implies that $\omega' \in \eta_{\tau_{k-1}}$ and

$$\frac{1}{P(\Gamma)} T_*^{\tau_{k-1}} \pi_{2*}(P \mid \Gamma) = \frac{T_*^{\tau_{k-1}} (\lambda_2 \mid \omega')}{T_*^{\tau_{k-1}} (\lambda_2 \mid \omega') (T^{\tau_{k-1}}(\omega'))}. \tag{4.1.13}$$

By definition, $\tau_{k+1} - \tau_k = n_0 + \hat{R} \circ T^{\tau_k + n_0} \circ \pi_2$, we can write

$$\begin{aligned} P(\tau_{k+1} - \tau_k > n_0 + n \mid \Gamma) &= P(\hat{R} \circ T^{\tau_k+n_0} \circ \pi_2 > n \mid \Gamma) \\ &= \frac{1}{P(\Gamma)} (P \mid \Gamma)\{\hat{R} \circ T^{\tau_k+n_0} \circ \pi_2 > n\} \\ &= \frac{1}{P(\Gamma)} T_*^{\tau_k+n_0} \pi_{2*}(P \mid \Gamma)\{\hat{R} > n\} \\ &= T_*^{\tau_k - \tau_{k-1} + n_0} \frac{1}{P(\Gamma)} T_*^{\tau_{k-1}} \pi_{2*}(P \mid \Gamma)\{\hat{R} > n\}. \end{aligned} \tag{4.1.14}$$

Combining (4.1.13) and (4.1.14), we have

$$\begin{aligned} P\left(\tau_{k+1} - \tau_k > n_0 + n \mid \Gamma\right) &= T_*^{\tau_k - \tau_{k-1} + n_0} \frac{T_*^{\tau_{k-1}}(\lambda_2 \mid \omega')\{\hat{R} > n\}}{T_*^{\tau_{k-1}}(\lambda_2 \mid \omega')(T^{\tau_{k-1}}(\omega'))} \\ &= T_*^{\tau_k - \tau_{k-1} + n_0} \frac{T_*^{\tau_{k-1}}(\lambda_2 \mid \omega')(\{\hat{R} > n\} \cap T^{\tau_{k-1}}(\omega'))}{T_*^{\tau_{k-1}} (\lambda_2 \mid \omega') (T^{\tau_{k-1}}(\omega'))}. \end{aligned} \tag{4.1.15}$$

By Lemma 4.1.10, we get

$$\frac{T_*^{\tau_{k-1}}(\lambda_2 \mid \omega')(\{\hat{R} > n\} \cap T^{\tau_{k-1}}(\omega'))}{T_*^{\tau_{k-1}} (\lambda_2 \mid \omega') (T^{\tau_{k-1}}(\omega'))} \le C_1^2 \frac{m(\{\hat{R} > n\} \cap T^{\tau_{k-1}}(\omega'))}{m(T^{\tau_{k-1}}(\omega'))} \tag{4.1.16}$$

By using (4.1.16), (4.1.12) and the fact that $m(T^{\tau_{k-1}}(\omega')) \ge \delta_0$ all in (4.1.15), we have

$$\begin{aligned} P\left(\tau_{k+1} - \tau_k > n_0 + n \mid \Gamma\right) &\le M_0 \frac{C_1^2}{\delta_0} m(\{\hat{R} > n\} \cap T^{\tau_{k-1}}(\omega')) \\ &\le C_2 m\{\hat{R} > n\}, \end{aligned}$$

with $C_2 = M_0 C_1^2/\delta_0$. The case k even is similar. Moreover, the dependence of C_1 (hence C_2) on $C'_{\varphi_1}, C'_{\varphi_2}$ can be removed if we take k large enough.

□

4.1.1.2 $|T_*^n \lambda_1 - T_*^n \lambda_2|$ vs $P\{S > n\}$

We are now going to estimate an upper bound of $|T_*^n \lambda_1 - T_*^n \lambda_2|$ in terms of $P\{S > n\}$ in this and the remaining subsections. In this context our key result is Proposition 4.1.25.

Consider the induced dynamical system

$$\widetilde{T} = (T \times T)^S : \Delta \times \Delta \to \Delta \times \Delta,$$

with S as in (4.1.11), and functions $0 = S_0 < S_1 < S_2 < ...$, given by

$$S_n = S_{n-1} + S \circ (T \times T)^{S_{n-1}} \quad \text{for all } n \geq 1.$$

From the definition, we have

$$\widetilde{T}^n = (T \times T)^{S_n}.$$

Let $\widetilde{\xi}$ be the partition of $\Delta \times \Delta$ into the *rectangles* Ω on which S is constant and $\widetilde{T}$ maps Ω bijectively onto a union of elements of $\eta \times \eta|_{\Delta_0 \times \Delta_0}$. Without loss of generality, we assume that for any $\Omega \in \widetilde{\xi}|_{\Delta_0 \times \Delta_0}$, there exists $\omega_j \times \omega_{j'} \in \eta \times \eta|_{\Delta_0 \times \Delta_0}$ such that $\Omega \subset \omega_j \times \omega_{j'}$.

For each $n \geq 1$, define

$$\tilde{\xi}_n = \bigvee_{j=0}^{n-1} \widetilde{T}^{-j}(\tilde{\xi}).$$

Each $\tilde{\xi}_n$ is a partition into sets $\Omega \subset \Delta \times \Delta$ on which S_n is constant and $\widetilde{T}^n$ maps Ω bijectively onto an $m \times m$ mod 0 union of elements of $\eta \times \eta|_{\Delta_0 \times \Delta_0}$. Let us introduce a separation time in $\Delta \times \Delta$, defining for each $u, v \in \Delta \times \Delta$

$$\tilde{s}(u,v) = \min \left\{ n \geq 0 : \widetilde{T}^n(u) \text{ and } \widetilde{T}^n(v) \text{ lie in distinct elements of } \tilde{\xi} \right\}.$$

Let $\Phi(x,y) = \varphi_1(x)\varphi_2(y)$ and C''_{φ_1} and C''_{φ_2} be constants such that

$$\left| \log \frac{\varphi_1(x)}{\varphi_1(y)} \right| \leq C''_{\varphi_1} \beta^{s(x,y)} \text{ and } \left| \log \frac{\varphi_2(x)}{\varphi_2(y)} \right| \leq C''_{\varphi_2} \beta^{s(x,y)} \text{ for all } x, y \in \omega \in \eta. \tag{4.1.17}$$

Before proceeding further we want to state some auxiliary results that will be useful later.

Lemma 4.1.16. *[3, Lemma 3.34] For all $u = (x, x')$ and $v = (y, y')$ in $\Delta \times \Delta$,*

$$s(x,y) \geq \tilde{s}(u,v) \text{ and } s(x',y') \geq \tilde{s}(u,v).$$

Lemma 4.1.17. *[3, Lemma 3.35] There exists $\bar{C}_{\widetilde{T}} > 0$ such that for all $n \geq 1$, $\Omega \in \tilde{\xi}_n$ and $u, v \in \Omega$ we have*

$$\log \frac{J_{\widetilde{T}^n}(u)}{J_{\widetilde{T}^n}(v)} \leq \bar{C}_{\widetilde{T}} \beta^{\tilde{s}\left(\widetilde{T}^n(u), \widetilde{T}^n(v)\right)}.$$

The next couple of lemmas will be helpful for proving Proposition 4.1.20.

Lemma 4.1.18. *There exists $C_\Phi > 0$ such that for all $u, v \in \Omega \in \tilde{\xi}$,*

$$\log \frac{\Phi(u)}{\Phi(v)} \leq C_\Phi \beta^{\tilde{s}(u,v)}.$$

Proof. Let $u = (x, x')$ and $v = (y, y')$, we may write

$$\begin{aligned}
\log \frac{\Phi(u)}{\Phi(v)} &= \log \frac{\varphi_1(x)}{\varphi_1(y)} + \log \frac{\varphi_2(x')}{\varphi_2(y')} \\
&\leq C''_{\varphi_1} \beta^{s(x,y)} + C''_{\varphi_2} \beta^{s(x',y')}, \text{ by (4.1.17)} \\
&\leq C''_{\varphi_1} \beta^{\tilde{s}(u,v)} + C''_{\varphi_2} \beta^{\tilde{s}(u,v)}, \text{ by Lemma 4.1.16} \\
&= C_\Phi \beta^{\tilde{s}(u,v)},
\end{aligned}$$

with $C_\Phi = C''_{\varphi_1} + C''_{\varphi_2}$. □

Lemma 4.1.19. *There exists $C_3 > 0$, depending on $C''_{\varphi_1}, C''_{\varphi_2}$, such that for all $n \geq 1, \Omega \in \tilde{\xi}_n$ and $u, v \in \widetilde{T}^n(\Omega)$ we have*

$$\frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m \times m)}(u) \Bigg/ \frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m \times m)}(v) \leq C_3.$$

Proof. Let $u_0, v_0 \in \Omega$ be such that $\widetilde{T}^n(u_0) = u$ and $\widetilde{T}^n(v_0) = v$. We have

$$\frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m \times m)}(u) \Bigg/ \frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m \times m)}(v) = \frac{J_{\widetilde{T}^n}(v_0)}{J_{\widetilde{T}^n}(u_0)} \cdot \frac{\Phi(u_0)}{\Phi(v_0)}. \tag{4.1.18}$$

Observing that for all $u_0, v_0 \in \Omega \in \tilde{\xi}_n$, we have $\tilde{s}(u_0, v_0) \geq n$, it follows from Lemma 4.1.17, that there is some $C'_{\widetilde{T}} > 0$ such that

$$\frac{J_{\widetilde{T}^n}(v_0)}{J_{\widetilde{T}^n}(u_0)} \leq C'_{\widetilde{T}}. \tag{4.1.19}$$

On the other hand, Lemma 4.1.18 gives $C_\Phi > 0$ such that

$$\frac{\Phi(u_0)}{\Phi(v_0)} \leq e^{C_\Phi}. \tag{4.1.20}$$

Using (4.1.18), (4.1.19) and (4.1.20), we get

$$\frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m\times m)}(u) \Bigg/ \frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m\times m)}(v) \leq C'_{\widetilde{T}} e^{C_\Phi},$$

which is the required conclusion with $C_3 = C'_{\widetilde{T}} e^{C_\Phi}$, depending on $C'''_{\varphi_1}, C'''_{\varphi_2}$. □

The following result will be used in Corollary 4.1.26.

Proposition 4.1.20. *There exists $C_4 > 0$, depending on $C'''_{\varphi_1}, C'''_{\varphi_2}$, such that for all $n, k \geq 0$, we have*

$$P\{S_{k+1} - S_k > n\} \leq C_4 (m\times m)\{S > n\}.$$

Proof. Consider first $k = 0$. Since the density of P with respect to $m \times m$ is given by $\Phi(x, x') = \varphi_1(x)\varphi_2(x')$, which is bounded from above by some constant $C'_0 > 0$, then we have $P\{S_1 = S > n\} \leq C'_0 (m\times m)\{S > n\}$.

Assume now $k \geq 1$ and take $\Omega \in \tilde{\xi}_k$. By definition, $\widetilde{T}^k$ maps Ω bijectively onto a union of elements of $\eta \times \eta|_{\Delta_0\times\Delta_0}$, and $S_{k+1} - S_k = S \circ \widetilde{T}^k$. Then, we can write

$$\begin{aligned} P(S_{k+1} - S_k > n \mid \Omega) =& \frac{P(\{S\circ \widetilde{T}^k > n\} \cap \Omega)}{P(\Omega)} \\ =& \frac{\widetilde{T}^k_*(P\mid\Omega)\{S > n\}}{\widetilde{T}^k_*(P\mid\Omega)(\widetilde{T}^k(\Omega))} \\ =& \frac{\widetilde{T}^k_*(P\mid\Omega)(\{S > n\} \cap \widetilde{T}^k(\Omega))}{\widetilde{T}^k_*(P\mid\Omega)(\widetilde{T}^k(\Omega))}. \end{aligned}$$

Using Lemma 4.1.19, and the fact that $(m\times m)(\widetilde{T}^k(\Omega)) \geq \delta_0^2$ (proceeding similarly as in the proof of Lemma 4.1.15), we have

$$P(S_{k+1} - S_k > n \mid \Omega) \leq \frac{C_3^2}{\delta_0^2}(m\times m)\{S > n\}.$$

It follows that

$$\begin{aligned} P\{S_{k+1} - S_k > n\} &= \sum_{\Omega\in\tilde{\xi}_k} P(S_{k+1} - S_k > n \mid \Omega) P(\Omega) \\ &\leq \frac{C_3^2}{\delta_0^2} \sum_{\Omega\in\tilde{\xi}_k} (m\times m)\{S > n\} P(\Omega) \\ &= \frac{C_3^2}{\delta_0^2}(m\times m)\{S > n\} P(\Delta\times\Delta), \end{aligned}$$

which gives us the required result with $C_4 = C_3^2/\delta_0^2$. □

4.1.1.3 Densities in Induced Times

We are going to define a sequence of densities $\tilde{\Phi}_0 \geq \tilde{\Phi}_1 \geq \tilde{\Phi}_2 \geq \cdots$ in $\Delta \times \Delta$. The densities $\tilde{\Phi}_i$ are those of the total measure remaining in the system after i iterations by the induced map $\widetilde{T}$. Recall $\bar{C}_{\widetilde{T}}$ and C_Φ from Lemma 4.1.17 and Lemma 4.1.18 receptively. Let $\varepsilon > 0$ be a small number to be given in Lemma 4.1.22, and $i_1 = i_1(C_\Phi) \geq 0$ be such that

$$C_\Phi \beta^{i_1} < \bar{C}_{\widetilde{T}}. \tag{4.1.21}$$

Given $u \in \Delta \times \Delta$, let $\Omega_k(u)$ be the element of $\tilde{\xi}_k$ that contains u. Define

$$\tilde{\Phi}_k(u) = \begin{cases} \Phi(u), & \text{if } k \leq i_1 \\ \tilde{\Phi}_{k-1}(u) - \varepsilon J_{\widetilde{T}^k}(u) \min_{v \in \Omega_k(u)} \frac{\tilde{\Phi}_{k-1}(v)}{J_{\widetilde{T}^k}(v)}, & \text{if } k > i_1. \end{cases} \tag{4.1.22}$$

The proof of the following lemma is valid in our setting (meaning that the full branch property is not required for the proof), and we are recalling the proof for the sake of completeness, which can be found in [71, Subsection 3.4].

Lemma 4.1.21. *There exists $0 < \varepsilon_1 < 1$ such that for all $k > i_1$ we have*

$$\tilde{\Phi}_k \leq (1 - \varepsilon_1)\tilde{\Phi}_{k-1} \text{ on } \Delta \times \Delta.$$

To prove the above lemma, we need to introduce some new notations. Given $u \in \Delta \times \Delta$, set for each $k > i_1$

$$\tilde{\Psi}_k(u) = \frac{\tilde{\Phi}_k(u)}{J_{\widetilde{T}^k}(u)}, \quad \Psi_k(u) = \frac{\tilde{\Phi}_{k-1}(u)}{J_{\widetilde{T}^k}(u)} \quad \text{and} \quad \mathcal{E}_k(u) = \varepsilon \min_{v \in \Omega_k(u)} \Psi_k(v). \tag{4.1.23}$$

We have the following observations:

$$\Psi_k(u) = \frac{\tilde{\Psi}_{k-1}(u)}{J_{\widetilde{T}}\left(\widetilde{T}^{k-1}(u)\right)} \quad \text{and} \quad \Psi_k(u) - \mathcal{E}_k(u) = \tilde{\Psi}_k(u). \tag{4.1.24}$$

Sublemma 4.1.22. *There exist $C_5 > 0$ and $\varepsilon > 0$ such that for all $k \geq i_1$ and u, v in the same element of $\tilde{\xi}_k$, we have*

$$\log \frac{\tilde{\Psi}_k(u)}{\tilde{\Psi}_k(v)} \leq C_5 \beta^{\tilde{s}\left(\widetilde{T}^k(u), \widetilde{T}^k(v)\right)}.$$

Proof. Fix $\delta > 0$ small so that $(1+\delta)\beta < 1$, and take

$$C_5 = \max\left\{2, \frac{(1+\delta)}{1-(1+\delta)\beta}\right\} \bar{C_{\widetilde{T}}}. \tag{4.1.25}$$

We will prove the result by induction method $k \geq i_1$. First we want to prove on $k = i_1$ that the result holds with this constant C_5. Given u, v in a same element of $\tilde{\xi}_{i_1}$, we have $\tilde{s}(u,v) = \tilde{s}\left(\widetilde{T}^{i_1}(u), \widetilde{T}^{i_1}(v)\right) + i_1$. By definition

$$\begin{aligned}
\log \frac{\tilde{\Psi}_{i_1}(u)}{\tilde{\Psi}_{i_1}(v)} &= \log \frac{\tilde{\Phi}_{i_1}(u)}{\tilde{\Phi}_{i_1}(v)} + \log \frac{J_{\widetilde{T}^{i_1}}(v)}{J_{\widetilde{T}^{i_1}}(u)} \\
&= \log \frac{\Phi(u)}{\Phi(v)} + \log \frac{J_{\widetilde{T}^{i_1}}(v)}{J_{\widetilde{T}^{i_1}}(u)}
\end{aligned}$$

By using Lemma 4.1.17, Lemma 4.1.18, (4.1.21) and (4.1.25) we have

$$\begin{aligned}
\log \frac{\tilde{\Psi}_{i_1}(u)}{\tilde{\Psi}_{i_1}(v)} &\leq C_\Phi \beta^{\tilde{s}(u,v)} + \bar{C_{\widetilde{T}}} \beta^{\tilde{s}\left(\widetilde{T}^{i_1}(u), \widetilde{T}^{i_1}(v)\right)} \\
&= C_\Phi \beta^{\tilde{s}\left(\widetilde{T}^{i_1}(u), \widetilde{T}^{i_1}(v)\right) + i_1} + \bar{C_{\widetilde{T}}} \beta^{\tilde{s}\left(\widetilde{T}^{i_1}(u), \widetilde{T}^{i_1}(v)\right)} \\
&\leq \bar{C_{\widetilde{T}}} \beta^{\tilde{s}\left(\widetilde{T}^{i_1}(u), \widetilde{T}^{i_1}(v)\right)} + \bar{C_{\widetilde{T}}} \beta^{\tilde{s}\left(\widetilde{T}^{i_1}(u), \widetilde{T}^{i_1}(v)\right)} \\
&= 2\bar{C_{\widetilde{T}}} \beta^{\tilde{s}\left(\widetilde{T}^{i_1}(u), \widetilde{T}^{i_1}(v)\right)} \\
&\leq C_5 \beta^{\tilde{s}\left(\widetilde{T}^{i_1}(u), \widetilde{T}^{i_1}(v)\right)}.
\end{aligned}$$

Thus the result is true for $k = i_1$. Now assume that the result holds for $k-1$ with $k > i_1$. Given u, v in the same element of $\tilde{\xi}_k$. From (4.1.24), we can write,

$$\log \frac{\Psi_k(u)}{\Psi_k(v)} = \log \frac{\tilde{\Psi}_{k-1}(u)}{\tilde{\Psi}_{k-1}(v)} + \log \frac{J_{\widetilde{T}}\left(\widetilde{T}^{k-1}(v)\right)}{J_{\widetilde{T}}\left(\widetilde{T}^{k-1}(u)\right)}. \tag{4.1.26}$$

It follows from (4.1.26), Lemma 4.1.17 and the induction step that

$$\log \frac{\Psi_k(u)}{\Psi_k(v)} \leq \left(C_5 + \beta^{-1}\bar{C_{\widetilde{T}}}\right) \beta^{\tilde{s}\left(\widetilde{T}^{k-1}(u), \widetilde{T}^{k-1}(v)\right)}. \tag{4.1.27}$$

Since u, v in a same element of $\tilde{\xi}_k$, then $\mathcal{E}_k(u) = \mathcal{E}_k(v)$, and we denote $\varepsilon_k = \mathcal{E}_k(u) = \mathcal{E}_k(v)$.

From (4.1.24) we can write

$$\left|\log\frac{\tilde{\Psi}_k(u)}{\tilde{\Psi}_k(v)} - \log\frac{\Psi_k(u)}{\Psi_k(v)}\right| = \left|\log\left(\frac{\Psi_k(u)-\varepsilon_k}{\Psi_k(u)}\cdot\frac{\Psi_k(v)}{\Psi_k(v)-\varepsilon_k}\right)\right|$$
$$\begin{aligned} &= \log\left(1+\frac{\frac{\varepsilon_k}{\Psi_k(v)}-\frac{\varepsilon_k}{\Psi_k(u)}}{1-\frac{\varepsilon_k}{\Psi_k(v)}}\right) \\ &\le 2\left|\frac{\frac{\varepsilon_k}{\Psi_k(v)}-\frac{\varepsilon_k}{\Psi_k(u)}}{1-\frac{\varepsilon_k}{\Psi_k(v)}}\right| \\ &= 2\frac{\varepsilon_k}{\Psi_k(u)}\cdot\frac{1}{1-\frac{\varepsilon_k}{\Psi_k(v)}}\cdot\left|\frac{\Psi_k(u)}{\Psi_k(v)}-1\right|. \end{aligned} \tag{4.1.28}$$

Since for all $z \in \Omega$

$$\frac{\varepsilon_k}{\Psi_k(z)} = \varepsilon\frac{\min_{w\in\Omega}\Psi_k(w)}{\Psi_k(z)} \le \varepsilon, \text{ this implies that } 1-\frac{\varepsilon_k}{\Psi_k(z)} \ge 1-\varepsilon. \tag{4.1.29}$$

Also from (4.1.27), we can find K_1 only depend on C_5 and $\bar{C_{\widetilde{T}}}$ such that

$$\left|\frac{\Psi_k(u)}{\Psi_k(v)}-1\right| \le K_1\left|\log\frac{\Psi_k(u)}{\Psi_k(v)}\right|. \tag{4.1.30}$$

By using (4.1.29) and (4.1.30), we have the estimate for the expression (4.1.28),

$$\left|\log\frac{\tilde{\Psi}_k(u)}{\tilde{\Psi}_k(v)} - \log\frac{\Psi_k(u)}{\Psi_k(v)}\right| \le 2\frac{\varepsilon}{1-\varepsilon}K_1\left|\log\frac{\Psi_k(u)}{\Psi_k(v)}\right|. \tag{4.1.31}$$

Let us now obtained the final conclusion. Since

$$\left|\log\frac{\tilde{\Psi}_k(u)}{\tilde{\Psi}_k(v)}\right| \le \left|\log\frac{\tilde{\Psi}_k(u)}{\tilde{\Psi}_k(v)} - \log\frac{\Psi_k(u)}{\Psi_k(v)}\right| + \left|\log\frac{\Psi_k(u)}{\Psi_k(v)}\right|,$$

which together with (4.1.31) and (4.1.27), we get

$$\begin{aligned} \left|\log\frac{\tilde{\Psi}_k(u)}{\tilde{\Psi}_k(v)}\right| &\le 2\frac{\varepsilon}{1-\varepsilon}K_1\left|\log\frac{\Psi_k(u)}{\Psi_k(v)}\right| + \left|\log\frac{\Psi_k(u)}{\Psi_k(v)}\right| \\ &= \left(1+\frac{2K_1\varepsilon}{1-\varepsilon}\right)\left|\log\frac{\Psi_k(u)}{\Psi_k(v)}\right| \\ &\le \left(1+\frac{2K_1\varepsilon}{1-\varepsilon}\right)\left(C_5+\beta^{-1}\bar{C_{\widetilde{T}}}\right)\beta^{\tilde{s}\left(\widetilde{T}^{k-1}(u),\widetilde{T}^{k-1}(v)\right)}. \end{aligned} \tag{4.1.32}$$

Let $\varepsilon = \frac{\delta}{2K_1+\delta}$, this implies that $\delta = \frac{2K_1\varepsilon}{1-\varepsilon}$.

Since $\tilde{s}\left(\widetilde{T}^{k-1}(u), \widetilde{T}^{k-1}(v)\right) = \tilde{s}\left(\widetilde{T}^{k}(u), \widetilde{T}^{k}(v)\right) + 1$. Hence, (4.1.32) become

$$\left|\log \frac{\tilde{\Psi}_k(u)}{\tilde{\Psi}_k(v)}\right| \leq (1+\delta)\beta(C_5 + \beta^{-1}\bar{C}_{\widetilde{T}})\beta^{\tilde{s}\left(\widetilde{T}^k(u),\widetilde{T}^k(v)\right)} \leq C_5\beta^{\tilde{s}\left(\widetilde{T}^k(u),\widetilde{T}^k(v)\right)}.$$

Thus we proved the required induction step. □

Proof of Lemma 4.1.21. We need to prove that there exists $0 < \varepsilon_1 < 1$ such that for $k > i_1$ and any $v \in \Delta \times \Delta$, we have

$$\tilde{\Phi}_{k-1}(v) - \tilde{\Phi}_k(v) \geq \varepsilon_1 \tilde{\Phi}_{k-1}(v).$$

It follows from (4.1.23) and (4.1.24) that for any $u, v \in \Omega \in \tilde{\xi}_k$, we have

$$\left|\log \frac{\tilde{\Phi}_{k-1}(u)}{J_{\widetilde{T}^k}(u)} \Big/ \frac{\tilde{\Phi}_{k-1}(v)}{J_{\widetilde{T}^k}(v)}\right| = \left|\log \frac{\Psi_k(u)}{\Psi_k(v)}\right| = \log \frac{\tilde{\Psi}_{k-1}(u)}{\tilde{\Psi}_{k-1}(v)} + \log \frac{J_{\widetilde{T}}\left(\widetilde{T}^{k-1}(v)\right)}{J_{\widetilde{T}}\left(\widetilde{T}^{k-1}(u)\right)},$$

which together with Sub-lemma 4.1.22 and Lemma 4.1.17 we get

$$\left|\log \frac{\tilde{\Phi}_{k-1}(u)}{J_{\widetilde{T}^k}(u)} \Big/ \frac{\tilde{\Phi}_{k-1}(v)}{J_{\widetilde{T}^k}(v)}\right| \leq C_5\beta^{\tilde{s}\left(\widetilde{T}^{k-1}(u),\widetilde{T}^{k-1}(v)\right)} + \bar{C}_{\widetilde{T}}\beta^{\tilde{s}\left(\widetilde{T}^k(u),\widetilde{T}^k(v)\right)} \leq 2C_5\beta^{\tilde{s}\left(\widetilde{T}^k(u),\widetilde{T}^k(v)\right)},$$

in particular, we have

$$\frac{\tilde{\Phi}_{k-1}(u)}{J_{\widetilde{T}^k}(u)} \Big/ \frac{\tilde{\Phi}_{k-1}(v)}{J_{\widetilde{T}^k}(v)} \leq e^{2C_5},$$

and this implies that

$$\max_{v\in\Omega} \frac{\tilde{\Phi}_{k-1}(v)}{J_{\widetilde{T}^k}(v)} \Big/ \min_{v\in\Omega} \frac{\tilde{\Phi}_{k-1}(v)}{J_{\widetilde{T}^k}(v)} \leq e^{2C_5}.$$

Consequently,

$$\tilde{\Phi}_{k-1}(v) \leq \frac{e^{2C_5}}{\varepsilon} J_{\widetilde{T}^k}(v)\varepsilon \min_{v\in\Omega} \frac{\tilde{\Phi}_{k-1}(v)}{J_{\widetilde{T}^k}(v)} = \frac{e^{2C_5}}{\varepsilon}(\tilde{\Phi}_{k-1}(v) - \tilde{\Phi}_k(v)).$$

Hence, we conclude that $\tilde{\Phi}_{k-1}(v) - \tilde{\Phi}_k(v) \geq \varepsilon_1 \tilde{\Phi}_{k-1}(v)$, where $\varepsilon_1 = \frac{\varepsilon}{e^{2C_5}}$. □

Remark 4.1.23. *It is important to mention that the constant $\varepsilon > 0$ in Sublemma 4.1.22 depends on δ, which depends on β. Hence the constant ε_1 in Lemma 4.1.21 depends on β.*

4.1.1.4 Densities in Real Time

We are now going to define the corresponding sequence of densities in real time iterations under $T \times T$, with the help of induced time densities. Let us introduce functions $\Phi_0 \geq \Phi_1 \geq \Phi_2 \geq \cdots$ on $\Delta \times \Delta$ as follows: for $v \in \Delta \times \Delta$

$$\Phi_n(v) = \tilde{\Phi}_i(v), \quad \text{if} \quad S_i(v) \leq n < S_{i+1}(v). \tag{4.1.33}$$

Since, by definition, $S_0 \leq i_1 \leq S_{i_1} < S_{i_1+1}$, this implies that for all $n \leq i_1$, $\Phi_n = \tilde{\Phi}_j$ for some $0 \leq j < i_1 + 1$, and $\tilde{\Phi}_j = \Phi$ for all $0 \leq j < i_1 + 1$, we have

$$\Phi_n = \Phi \text{ for all } n \leq i_1.$$

Note that for all $n \geq 1$ we can write

$$\Phi = \Phi_n + \sum_{k=1}^{n} (\Phi_{k-1} - \Phi_k). \tag{4.1.34}$$

For each $k \geq 1$, let $A_k = \cup_i A_{k,i}$, where

$$A_{k,i} = \{u \in \Delta \times \Delta : k = S_i(u)\}.$$

Note that $A_{k,i} \cap A_{k,j} = \emptyset$ for $i \neq j$ (because $S_i(u) \neq S_j(u)$, for $i \neq j$), and each $A_{k,i}$ is a union of elements of $\tilde{\xi}_i$.

Remark 4.1.24. *By definition, for any $\Omega \in \tilde{\xi}_i|_{A_{k,i}}$, we have $S_{i-1}|_\Omega < S_i|_\Omega = k$, and $S_{i-1}|_\Omega \leq k-1$. This implies that $\Phi_{k-1} - \Phi_k = \tilde{\Phi}_{i-1} - \tilde{\Phi}_i$ on $\Omega \in \tilde{\xi}_i|_{A_{k,i}}$, and $\Phi_k = \Phi_{k-1}$ on $\Delta \times \Delta \backslash A_k$.*

Our main result in this subsection is

Proposition 4.1.25. *There exists $C_6 > 0$, depending on $C''_{\varphi_1}, C''_{\varphi_2}$, such that for all $n \geq 1$,*

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq 2P\{S > n\} + C_6 \sum_{i=1}^{\infty} (1 - \varepsilon_1)^i P\{S_i \leq n < S_{i+1}\}.$$

Proof. From (4.1.6) and (4.1.34), for each $n \geq 1$ we have

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq I_1 + I_2, \tag{4.1.35}$$

where

$$I_1 = |(\pi_{1*} - \pi_{2*})(T \times T)_*^n (\Phi_n(m \times m))|$$

and

$$I_2 = \sum_{k=1}^{n} |(\pi_{1*} - \pi_{2*})[(T \times T)_*^n ((\Phi_{k-1} - \Phi_k)(m \times m))]| .$$

First we are going to compute I_1. We have

$$I_1 = |(\pi_{1*} - \pi_{2*})(T \times T)_*^n (\Phi_n(m \times m))| \leq 2 \int_{\Delta \times \Delta} \Phi_n d(m \times m). \tag{4.1.36}$$

Notice that $\Delta \times \Delta = \{S_{i_1} > n\} \cup \{S_{i_1} \leq n\}$ and we can further split the above union as

$$\{S_{i_1} \leq n\} = \bigcup_{i=i_1}^{\infty} \{S_i \leq n < S_{i+1}\}$$

and

$$\{S_{i_1} > n\} = \{S_1 > n\} \bigcup \left(\bigcup_{i=1}^{i_1-1} \{S_i \leq n < S_{i+1}\} \right),$$

thus we can write

$$\int \Phi_n d(m \times m) = \int_{\{S_{i_1} > n\}} \Phi_n d(m \times m) + \sum_{i=i_1}^{\infty} \int_{\{S_i \leq n < S_{i+1}\}} \Phi_n d(m \times m). \tag{4.1.37}$$

We have already observed that $\Phi_n|_{\{S_{i_1} > n\}} = \Phi$, thus we have

$$\int_{\{S_{i_1} > n\}} \Phi_n d(m \times m) = \int_{\{S_{i_1} > n\}} \Phi d(m \times m) = P\{S_{i_1} > n\}. \tag{4.1.38}$$

Moreover,

$$\begin{aligned} P\{S_{i_1} > n\} &= P\{S_1 > n\} + \sum_{i=1}^{i_1-1} P\{S_i \leq n < S_{i+1}\} \\ &= P\{S > n\} + (1-\varepsilon_1)^{-i_1} \sum_{i=1}^{i_1-1} (1-\varepsilon_1)^{i_1} P\{S_i \leq n < S_{i+1}\}. \end{aligned} \tag{4.1.39}$$

Note that we can find $K_2 > 1$ such that for any $1 \leq i \leq i_1 - 1$ we have

$$(1-\varepsilon_1)^{i_1} \leq K_2(1-\varepsilon_1)^i.$$

By using the above fact and (4.1.39) we can write

$$P\left\{S_{i_1}>n\right\} \leq P\{S>n\}+K_2\left(1-\varepsilon_1\right)^{-i_1} \sum_{i=1}^{i_1-1}\left(1-\varepsilon_1\right)^i P\left\{S_i \leq n<S_{i+1}\right\}. \quad (4.1.40)$$

It follows from (4.1.33) and Lemma 4.1.21 we can get for all $i \geq i_1$

$$\begin{aligned}\int_{\{S_i \leq n<S_{i+1}\}} \Phi_n d(m \times m) &= \int_{\{S_i \leq n<S_{i+1}\}} \tilde{\Phi}_i d(m \times m) \\ &= \left(1-\varepsilon_1\right)^{i-i_1} P\left\{S_i \leq n<S_{i+1}\right\}.\end{aligned} \quad (4.1.41)$$

By combining (4.1.36), (4.1.37), (4.1.38), (4.1.40) and (4.1.41), we obtained

$$\begin{aligned}I_1 \leq 2P\{S>n\} &+ 2K_2\left(1-\varepsilon_1\right)^{-i_1} \sum_{i=1}^{i_1-1}\left(1-\varepsilon_1\right)^i P\left\{S_i \leq n<S_{i+1}\right\} \\ &+ 2\left(1-\varepsilon_1\right)^{-i_1} \sum_{i=i_1}^{\infty}\left(1-\varepsilon_1\right)^i P\left\{S_i \leq n<S_{i+1}\right\}.\end{aligned}$$

Consequently,

$$I_1 \leq 2P\{S>n\}+2K_2\left(1-\varepsilon_1\right)^{-i_1} \sum_{i=1}^{\infty}\left(1-\varepsilon_1\right)^i P\left\{S_i \leq n<S_{i+1}\right\}. \quad (4.1.42)$$

Now we are going to compute I_2. We have

$$\begin{aligned}I_2 &= \sum_{k=1}^{n}\left|\left(\pi_{1*}-\pi_{2*}\right)\left[(T \times T)_*^n\left(\left(\Phi_{k-1}-\Phi_k\right)(m \times m)\right)\right]\right| \\ &\leq 2 \sum_{k=1}^{n} \int_{\Delta \times \Delta}\left(\Phi_{k-1}-\Phi_k\right) d(m \times m) \\ &= 2 \sum_{k=1}^{n} \int_{\Delta \times \Delta \backslash A_k}\left(\Phi_{k-1}-\Phi_k\right) d(m \times m)+2 \sum_{k=1}^{n} \int_{A_k}\left(\left(\Phi_{k-1}-\Phi_k\right) d(m \times m).\end{aligned} \quad (4.1.43)$$

From Remark 4.1.24, we have

$$2 \sum_{k=1}^{n} \int_{\Delta \times \Delta \backslash A_k}\left(\Phi_{k-1}-\Phi_k\right) d(m \times m)=0, \quad (4.1.44)$$

and writing $A_k = \bigcup_{i=1}^{\infty} A_{k,i} = \left(\bigcup_{i=1}^{i_1} A_{k,i}\right) \bigcup \left(\bigcup_{i=i_1+1}^{\infty} A_{k,i}\right)$, we have

$$\begin{aligned}\int_{A_k} \left(\tilde{\Phi}_{k-1} - \tilde{\Phi}_k\right) d(m \times m) = &\sum_{i=1}^{i_1} \int_{A_{k,i}} \left(\tilde{\Phi}_{i-1} - \tilde{\Phi}_i\right) d(m \times m) \\ &+ \sum_{i=i_1+1}^{\infty} \int_{A_{k,i}} \left(\tilde{\Phi}_{i-1} - \tilde{\Phi}_i\right) d(m \times m).\end{aligned} \tag{4.1.45}$$

Since by definition $\tilde{\Phi}_i = \Phi$, for all $0 \leq i \leq i_1$, the first sum in the above equality vanishes. On the other hand, by (4.1.22) and Lemma 4.1.21 we have

$$\begin{aligned}\sum_{i=i_1+1}^{\infty} \int_{A_{k,i}} \left(\tilde{\Phi}_{i-1} - \tilde{\Phi}_i\right) d(m \times m) &= \sum_{i=i_1+1}^{\infty} \int_{A_{k,i}} \varepsilon J_{\tilde{T}^i}(u) \min_{v \in A_{k,i}} \frac{\tilde{\Phi}_{i-1}(v)}{J_{\tilde{T}^i}(v)} d(m \times m) \\ &\leq \varepsilon \sum_{i=i_1+1}^{\infty} \int_{A_{k,i}} \frac{J_{\tilde{T}^i}(u)}{J_{\tilde{T}^i}(u)} \tilde{\Phi}_{i-1}(u) d(m \times m) \\ &= \varepsilon \sum_{i=i_1+1}^{\infty} \int_{A_{k,i}} \tilde{\Phi}_{i-1}(u) d(m \times m) \\ &\leq \varepsilon \sum_{i=i_1+1}^{\infty} \int_{A_{k,i}} (1-\varepsilon_1)^{i-i_1-1} \Phi(u) d(m \times m) \\ &= \varepsilon \sum_{i=i_1+1}^{\infty} (1-\varepsilon_1)^{i-i_1-1} P\{S_i = k\}.\end{aligned} \tag{4.1.46}$$

It follows from (4.1.43), (4.1.44), (4.1.45) and (4.1.46) that

$$I_2 \leq 2\varepsilon \sum_{k=1}^{n} \sum_{i=i_1+1}^{\infty} (1-\varepsilon_1)^{i-i_1-1} P\{S_i = k\} = 2\varepsilon \sum_{i=i_1+1}^{\infty} (1-\varepsilon_1)^{i-i_1-1} P\left(\bigcup_{k=1}^{n} \{S_i = k\}\right). \tag{4.1.47}$$

For each $1 \leq k \leq n$, we have $\{S_i = k\} \subset \{S_i \leq n < S_{i+1}\}$, and this implies that $\bigcup_{k=1}^{n} \{S_i = k\} \subset \{S_i \leq n < S_{i+1}\}$. Then from (4.1.47) we have

$$I_2 \leq 2\varepsilon (1-\varepsilon_1)^{-i_1-1} \sum_{i=i_1+1}^{\infty} (1-\varepsilon_1)^i P\{S_i \leq n < S_{i+1}\}. \tag{4.1.48}$$

It follows from (4.1.35), (4.1.42) and (4.1.48) that

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq 2P\{S > n\} + C_6 \sum_{i=1}^{\infty} (1-\varepsilon_1)^i \, P\left\{S_i \leq n < S_{i+1}\right\},$$

with $C_6 = 2(K_2 + \frac{\varepsilon}{1-\varepsilon_1})\,(1-\varepsilon_1)^{-i_1}$. Recall that i_1 (hence C_6) depends on C''_{φ_1}, C''_{φ_2}. $\square$

Corollary 4.1.26. *There exists $C_7 > 0$, depending on $C''_{\varphi_1}, C''_{\varphi_2}$, such that for all $n \geq 1$,*

$$|T_*^n\lambda_1 - T_*^n\lambda_2| \leq 2P\{S > n\} + C_7 \sum_{i=1}^{\infty} (1-\varepsilon_1)^i \,(i+1)(m \times m)\left\{S > \frac{n}{i+1}\right\}.$$

Proof. For each $i \geq 1$ we have

$$P\left\{S_i \leq n < S_{i+1}\right\} \leq \sum_{j=0}^{i} P\left\{S_{j+1} - S_j > \frac{n}{i+1}\right\}. \tag{4.1.49}$$

By Proposition 4.1.20,

$$P\left\{S_{j+1} - S_j > \frac{n}{i+1}\right\} \leq C_4(m \times m)\left\{S > \frac{n}{i+1}\right\}. \tag{4.1.50}$$

Combining (4.1.49) and (4.1.50), we get

$$P\left\{S_i \leq n < S_{i+1}\right\} \leq C_4(i+1)(m \times m)\left\{S > \frac{n}{i+1}\right\},$$

which, together with Proposition 4.1.25, gives us the required result. $\square$

4.1.2 Polynomial decay

We are going to prove item 1) of Proposition 4.1.4. Assume that there are $C > 0$ and $a > 1$ such that for all $n \geq 1$ we have $m\{R > n\} \leq Cn^{-a}$. From (4.1.7) there exists $\hat{C}$ such that

$$m\{\hat{R} > n\} = \sum_{\ell > n} m\{R > \ell\} \leq \hat{C}n^{-a+1}. \tag{4.1.51}$$

By Lemma 4.1.14, Lemma 4.1.15 and (4.1.51) we obtain, respectively:

(A_1) There exists $\varepsilon_0 > 0$, depending on $C'_{\varphi_1}, C'_{\varphi_2}$, such that for all $k \geq 1$ and $\Gamma \in \xi_k$ with $S|_\Gamma > \tau_{k-1}$, we have

$$P(S = \tau_k | \Gamma) \geq \varepsilon_0.$$

(A_2) For all $n, k \geq 0$ and for all $\Gamma \in \xi_k$, we have

$$P(\tau_{k+1} - \tau_k > n + n_0 | \Gamma) \leq C_2\hat{C}n^{-a+1}.$$

By Lemma A.2.2 there exists $\bar{C} > 0$ such that

$$P\{S > n\} \leq (1-\varepsilon_0)^{\left[\frac{n}{2n_0+1}\right]} + \bar{C}\sum_{j=1}^{\infty}(j+1)^{a+1}(1-\varepsilon_0)^{j-1}n^{-a+1},$$

where $\left[\frac{n}{2n_0+1}\right]$ is the smallest integer $\geq \frac{n}{2n_0+1}$. This implies that there exists $\bar{C}_1 > 0$

$$P\{S > n\} \leq \bar{C}_1 n^{-a+1}. \tag{4.1.52}$$

By similar arguments used to estimate $P\{S > n\}$, there exists $\bar{C}_2 > 0$ such that

$$(m \times m)\{S > \frac{n}{i+1}\} \leq \bar{C}_2\left(\frac{n}{i+1}\right)^{-a+1}. \tag{4.1.53}$$

By using the estimates (4.1.52) and (4.1.53) in Corollary 4.1.26, we get

$$|T_*^n\lambda_1 - T_*^n\lambda_2| \leq C'n^{-a+1} \text{ for some } C' > 0. \tag{4.1.54}$$

Considering $\lambda_1 = \lambda$ and $\lambda_2 = \nu$ in (4.1.54) we conclude

$$|T_*^n\lambda - \nu| \leq C'n^{-a+1}.$$

4.1.3 (Stretched) Exponential decay

We are now going to prove the remaining item 2) of Proposition 4.1.4. Assume that there are $C, c > 0$ and $0 < a \leq 1$ such that $m\{R > n\} \leq Ce^{-cn^a}$ for all $n \geq 1$. Then, there exists $\hat{C} > 0$ such that

$$m\{\hat{R} > n\} = \sum_{\ell > n} m\{R > \ell\} \leq \hat{C}e^{-cn^a}. \tag{4.1.55}$$

By Lemma 4.1.14, Lemma 4.1.15 and (4.1.55) we obtain, respectively:

(B_1) There exists $\varepsilon_0 > 0$, depending on $C'_{\varphi_1}, C'_{\varphi_2}$, such that for all $k \geq 1$ and $\Gamma \in \xi_k$ with $S|_\Gamma > \tau_{k-1}$, we have

$$P(S = \tau_k|\Gamma) \geq \varepsilon_0.$$

(B_2) For all $n, j \geq 0$ and for all $\Gamma \in \xi_j$, we have

$$P(\tau_{j+1} - \tau_j > n + n_0|\Gamma) \leq C_2\hat{C}e^{-cn^a}.$$

By Lemma A.2.3 we get

$$P\{S > n\} \leq (1-\varepsilon_0)^{[\varepsilon n^a]} + \bar{C}e^{-c_0 n^a}.$$

This implies that

$$P\{S > n\} \leq \bar{C}_3 e^{-c_0 n^a} \text{ for some } \bar{C}_3 > 0. \tag{4.1.56}$$

By similar arguments used to estimate $P\{S > n\}$, there exists $\bar{C}_4 > 0$ such that

$$(m \times m)\{S > \frac{n}{i+1}\} \leq \bar{C}_4 e^{-c_0 (\frac{n}{i+1})^a}. \tag{4.1.57}$$

Finally, by using (4.1.56) and (4.1.57) in Corollary 4.1.26, we have

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq C' e^{-c' n^a} \text{ for some } C', c' > 0. \tag{4.1.58}$$

Considering $\lambda_1 = \lambda$ and $\lambda_2 = \nu$ in (4.1.58) we conclude that

$$|T_*^n \lambda - \nu| \leq C' e^{-c' n^a}.$$

Remark 4.1.27. *Recall that ε_1 depends on β. This does not affect exponent 'a' in the polynomial case. On the other hand, this causes the exponent c' to depend on β in the (stretched) exponential case.*

4.2 Central Limit Theorem

In this section, we prove Corollary B. The idea is that we first establish the CLT for the tower map and then we conclude for the given dynamical system through a semi-conjugacy. We start by recalling a version of [52, Theorem 1.1].

Theorem 4.2.1. *Let $(X, \mathcal{F}, \upsilon)$ be a probability space, $T\colon X \to X$ a (non-invertible) measurable map and υ an ergodic T-invariant probability measure. Let $\varphi \in L^\infty(X, \upsilon)$ be such that $\int \varphi d\upsilon = 0$. Assume*

i) $\displaystyle\sum_{n=1}^{\infty} \left| \int (\varphi \circ T^n) \varphi d\upsilon \right| < \infty$

ii) $\displaystyle\sum_{n=1}^{\infty} \left(\hat{T}^{*n} \varphi \right)(x)$ *is absolutely convergent for υ-a.e. x,*

where $\hat{T}^*$ is the dual of the operator $\hat{T}\varphi = \varphi \circ T$. Then the Central Limit Theorem holds for φ if and only if φ is not coboundary.

The dual operator $\hat{T}^*$ above is the Perron-Frobenius operator with respect to T and υ, that is

$$(\hat{T}^*\varphi)(x) = \sum_{y \in T^{-1}(x)} \frac{\varphi(y)}{J_T(y)},$$

where J_T here is defined in terms of the measure υ.

Proposition 4.2.2. *Let $T : \Delta \to \Delta$ be the tower map of an aperiodic induced WGM expanding map f^R with a coprime block, $R \in L^1(m)$, and let ν be the unique mixing T-invariant probability measure. If $m\{R > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 2$, then the CLT is satisfied for all $\varphi \in \mathcal{F}_\beta(\Delta)$ if and only if φ is not coboudary.*

Proof. From *ii)* in Lemma 4.1.2 and Proposition 4.1.4 there is some $C' > 0$ such that

$$\mathrm{Cor}_\nu(\varphi, \varphi \circ T^n) \leq C' n^{-a+1}. \tag{4.2.1}$$

Let $\phi = \varphi - \int \varphi d\nu$, so that $\int \phi d\nu = 0$. We are going to show that ϕ satisfies conditions i) and ii) of the Theorem 4.2.1. It is straightforward to check that

$$\mathrm{Cor}_\nu(\varphi, \varphi \circ T^n) = \mathrm{Cor}_\nu(\phi, \phi \circ T^n) = \left| \int (\phi \circ T^n) \phi d\nu \right|.$$

It follows from (4.2.1) that condition i) holds. Since m and ν are equivalent measures, it suffices to verify condition ii) for m-a.e. $x \in \Delta$.

The operator $\hat{T}^*$ is defined in terms of the invariant measure, so for a measure $\lambda \ll m$ it sends $\frac{d\lambda}{d\nu}$ to $\frac{dT_*\lambda}{d\nu}$. We can write

$$(\hat{T}^{*n}\phi)(x) = \frac{1}{\rho(x)} (\hat{P}^n(\phi\rho))(x),$$

where $\hat{P}$ is the Perron-Frobenius operator with respect to m, that sends $\frac{d\lambda}{dm}$ to $\frac{dT_*\lambda}{dm}$ and $\rho = \frac{d\nu}{dm}$. We shall now write ϕ as the difference of the densities of two (positive) measures of similar regularity to ϕ. We let $\tilde{\phi} = b(\phi + a)$, for some large a, with $b > 0$ chosen such that $\int \tilde{\phi} \rho dm = 1$. We define the probability measures λ_1, λ_2 by

$$\frac{d\lambda_1}{dm} = (b\phi + \tilde{\phi})\rho, \quad \frac{d\lambda_2}{dm} = \tilde{\phi}\rho.$$

Similar to Lemma 4.1.5 we can check that $\frac{d\lambda_1}{dm}, \frac{d\lambda_2}{dm} \in \mathcal{F}_\beta^+(\Delta)$. Moreover,

$$b^{-1} \left(\frac{d\lambda_1}{dm} - \frac{d\lambda_2}{dm} \right) = \phi\rho.$$

Following Section 4.1.1.2, for these given measures λ_1 and λ_2 we have

$$T_*^n\lambda_1 - T_*^n\lambda_2 = \pi_{1*}(T\times T)_*^n\left([\Phi_n + \sum_{k=1}^{n}(\Phi_{k-1}-\Phi_k)](m\times m)\right) + \pi_{2*}(T\times T)_*^n\left([\Phi_n + \sum_{k=1}^{n}(\Phi_{k-1}-\Phi_k)](m\times m)\right).$$

Let ψ_n, ψ_n' be the densities with respect to m of the first and the second term above, respectively.

Since $\hat{P}$ is a linear operator, we see that

$$\begin{aligned}
|\hat{P}^n(\phi\rho)| &= \left|\hat{P}^n\left(b^{-1}\left(\frac{d\lambda_1}{dm} - \frac{d\lambda_2}{dm}\right)\right)\right| \\
&= b^{-1}\left|\frac{dT_*^n\lambda_1}{dm} - \frac{dT_*^n\lambda_2}{dm}\right| \\
&\leq b^{-1}\left(\psi_n + \psi_n'\right).
\end{aligned}$$

Then by using above estimates on I_1, I_2 and estimates in Section 4.1.2, we have

$$\int \psi_n dm = \int \psi_n' dm = \int (\Phi_n + \sum_{k=1}^{n}(\Phi_{k-1}-\Phi_k))d(m\times m) \leq Cn^{-a+1},$$

for some $C > 0$.

This implies that

$$\sum_{n=1}^{\infty}\int_{\Delta}|\hat{P}^n(\phi\rho)|dm \leq C'\sum_{n=1}^{\infty}n^{-a+1} < \infty,$$

for some $C' > 0$.

We have that $\sum_{n=1}^{\infty}|\hat{P}^n(\phi\rho)(x)|$ is convergent for m-a.e. x. This implies that

$$\sum_{n=1}^{\infty}|(\hat{T}^{*n}\phi)(x)| = \frac{1}{\rho(x)}\sum_{n=1}^{\infty}|\hat{P}^n(\phi\rho)(x)|$$

is convergent for m-a.e. x, and condition ii) holds. □

Now we are ready to conclude the proof of Corollary B.

Since $\varphi \in \mathcal{H}_\eta$, by Lemma 4.1.1, $\varphi\circ\pi \in \mathcal{F}_{\beta^\eta}(\Delta)$. Notice that φ is not coboundary if and only if $\varphi\circ\pi$ is not coboundary.

For every interval $J \subset \mathbb{R}$,

$$\begin{aligned}
&\pi_*\nu\left\{y \in M : \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left(\varphi(f^i(y)) - \int \varphi d\pi_*\nu\right) \in J\right\} \\
&\quad = \nu\left\{x \in \Delta : \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left(\varphi(f^i(\pi(x))) - \int \varphi d\pi_*\nu\right) \in J\right\} \qquad (4.2.2)\\
&\quad = \nu\left\{x \in \Delta : \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left(\varphi \circ \pi(T^i(x)) - \int \varphi \circ \pi d\nu\right) \in J\right\}.
\end{aligned}$$

The proof of Corollary B follows now from Proposition 4.2.2 since $\pi_*\nu = \mu$.

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Chapter 5

Statistical properties for non-Hölder observables

In this chapter, we prove Theorem D. Basically this result extends Theorem A by considering larger classes of observables. More precisely, in Theorem D we are going to obtain the decay of correlations for the same dynamical system considered in Theorem A, but for larger classes of observables with weaker regularity than Hölder. To prove Theorem D, we follow a similar approach as in the proof of Theorem A. We consider the tower map $T : \Delta \to \Delta$ of an aperiodic induced WGM map $f^R : \Delta_0 \to \Delta_0$, with a coprime block and $R \in L^1(m)$. So in this case, we already know that there exists mixing T-invariant probability measure ν. We first obtain the decay of correlations for the tower map for several classes of observables $(V1 - V4)$ on Δ (these classes defined in Section 5.1 just below), corresponding to observable classes $(R1 - R4)$ on M. Then, in similar fashion, we transfer the information obtained for the tower map to a given dynamical system $f : M \to M$. In order to obtain the decay of correlations for the tower map for classes of observables $(V1 - V4)$, we make an extension of the coupling arguments used in Chapter 4.

In this chapter, we will use some notations from Chapter 4, but we will consider their meanings in the context of the current chapter.

5.1 Decay of correlations for the tower map for larger classes of observables

In this section, our key result is Proposition 5.1.1, which establishes the decay of correlations for (T, ν) for larger classes of observables. As in Chapter 4, to obtain a decay of correlations for (f, μ), it suffices to obtain a decay of correlations for (T, ν). This means that the proof of Theorem D can be deduced from Proposition 5.1.1.

Let us define some classes of observables on the tower. Given a bounded function $\varphi : \Delta \to \mathbb{R}$, we define the variation for each $n \geq 0$ as

$$v_n(\varphi) = \sup\{|\varphi(x) - \varphi(y)| : s(x, y) \geq n\}.$$

Consider the following regularity classes:

- $(V1, \tau) = \{\varphi : \exists D > 0 \text{ s.t. } v_n(\varphi) \leq D\tau^n\}$, $\tau \in (0, 1)$.

- $(V2,\tau) = \{\varphi : \exists D > 0 \text{ s.t. } v_n(\varphi) \leq De^{-n^\tau}\}$, $\tau \in (0,1)$.
- $(V3,\tau) = \{\varphi : \exists D > 0 \text{ s.t. } v_n(\varphi) \leq De^{-(\log n)^\tau}\}$, $\tau > 1$.
- $(V4,\tau) = \{\varphi : \exists D > 0 \text{ s.t. } v_n(\varphi) \leq Dn^{-\tau}\}$, $\tau > 1$.

Consider the following spaces, which are important for the extended coupling argument:

$$\mathcal{I}(\Delta) = \{\varphi : \Delta \to \mathbb{R} : v_n(\varphi) \to 0, \quad \text{as} \quad n \to \infty\},$$

and

$$\mathcal{I}^+(\Delta) = \Big\{\varphi \in \mathcal{I}(\Delta) : \exists C'_\varphi > 0 \text{ s.t. } \varphi(x) > 0 \text{ and} \\ \left|\frac{\varphi(x)}{\varphi(y)} - 1\right| \leq C'_\varphi v_{s(x,y)}(\varphi) + C''\beta^{s(x,y)}, \forall x, y \in \omega \in \eta\Big\},$$

where $C'' > 0$ is a fixed constant to be specified in Lemma 5.1.3. The following result implies the decay of correlations for the tower map in terms of the return times.

Proposition 5.1.1. *Let $T : \Delta \to \Delta$ be the tower map of an aperiodic induced WGM map f^R, with a coprime block and $R \in L^1(m)$. If ν is the unique mixing T-invariant probability measure such that $\frac{d\nu}{dm} \in \mathcal{I}^+(\Delta)$, then*

1) if $m\{R > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 1$, given $\kappa > 0$, there is $0 < \zeta < 1$ such that, for any probability measure λ, with $\frac{d\lambda}{dm} \in \mathcal{I}^+(\Delta)$ and $v_n(\frac{d\lambda}{dm}) \leq Dn^{-\tau}$, for some $\tau > \frac{2}{\zeta}$, $D > 0$, and $v_0(\frac{d\lambda}{dm}) < \kappa$, we have

i) if $\tau = \frac{a+1}{\zeta}$, there is $D' > 0$ such that $|T^n_*\lambda - \nu| \leq D'\,(n^{1-a}\log n)$;

ii) otherwise, there is $D' > 0$ such that $|T^n_*\lambda - \nu| \leq D' \max\{n^{1-a}, n^{2-\zeta\tau}\}$;

2) if $m\{R > n\} \leq Ce^{-cn}$ for some $C, c > 0$, then

i) for any probability measure λ, with $\frac{d\lambda}{dm} \in \mathcal{I}^+(\Delta)$ and $v_n(\frac{d\lambda}{dm}) \leq D\tau^n$, for some $\tau \in (0,1)$, $D > 0$, there are $D', c' > 0$ such that

$$|T^n_*\lambda - \nu| \leq D'e^{-c'n};$$

ii) for any probability measure λ, with $\frac{d\lambda}{dm} \in \mathcal{I}^+(\Delta)$ and $v_n(\frac{d\lambda}{dm}) \leq De^{-n^\tau}$, for some $\tau \in (0,1)$, $D > 0$, there is $D' > 0$ such that

$$|T^n_*\lambda - \nu| \leq D'e^{-n^{\tau'}} \quad \textit{for every} \quad 0 < \tau' < \tau;$$

iii) for any probability measure λ, with $\frac{d\lambda}{dm} \in \mathcal{I}^+(\Delta)$ and $v_n(\frac{d\lambda}{dm}) \leq De^{-(\log(n))^\tau}$, for some $\tau > 1$, $D > 0$, there is $D' > 0$ such that

$$|T_*^n \lambda - \nu| \leq D' e^{-(\log n)^{\tau'}} \quad \textit{for every} \;\; 0 < \tau' < \tau;$$

iv) given $\kappa > 0$, there is $\zeta < 1$ such that, for any probability measure λ, with $\frac{d\lambda}{dm} \in \mathcal{I}^+(\Delta)$ and $v_n(\frac{d\lambda}{dm}) \leq D(n^{-\tau})$, for some $\tau > \frac{1}{\zeta}$, $D > 0$, and $v_0(\frac{d\lambda}{dm}) < \kappa$, there is $D' > 0$ such that

$$|T_*^n \lambda - \nu| \leq D' n^{1-\zeta\tau}.$$

Recall that for all $\varphi, \psi : M \to \mathbb{R}$,

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \mathrm{Cor}_\nu(\varphi \circ \pi, \psi \circ \pi \circ T^n).$$

This shows that in order to deduce the proof of Theorem D, for a given regularity of $\varphi : M \to \mathbb{R}$ in terms of $R_\varepsilon(\varphi)$, we need to have a suitable regularity for $\varphi \circ \pi : \Delta \to \mathbb{R}$. Particularly, we want that the observable classes $(R1 - R4)$ on M correspond to the classes $(V1 - V4)$ on Δ. For that we have the following lemma.

Lemma 5.1.2. *[53, Lemma 4] Let $T : \Delta \to \Delta$ be the tower map of an induced expanding WGM map $f^R : \Delta_0 \to \Delta_0$, then*

i) $\varphi \in (Ri, \tau) \Rightarrow \varphi \circ \pi \in (Vi, \tau')$, with $0 < \tau' < \tau$, for $i = 1, 2, 3$;

ii) $\varphi \in (R4, \tau) \Rightarrow \varphi \circ \pi \in (V4, \tau)$.

As discussed in Chapter 4, to obtain decay of correlation for the tower map it is enough to estimate $|T_*^n \lambda - \nu|$ with $\frac{d\lambda}{dm} = \varphi^* \frac{d\nu}{dm}$. First we would like to check the necessary regularity of $\frac{d\lambda}{dm} = \varphi^* \frac{d\nu}{dm}$, for a given φ in classes $(V1 - V4)$.

Lemma 5.1.3. *If $\varphi \in (Vj, \tau)$, $\tau \in (0, 1)$ for $j = 1, 2$, or $\tau > 1$ for $j = 3, 4$, then $\varphi^* \frac{d\nu}{dm} \in \mathcal{I}^+(\Delta)$.*

Proof. Set $\rho = \frac{d\nu}{dm}$. Since $0 < \rho \leq C_0$ and from Lemma 4.1.2 we have

$$\begin{aligned} |\varphi^*(x)\rho(x) - \varphi^*(y)\rho(y)| &\leq |\varphi^*(x)(\rho(x) - \rho(y))| + |\rho(y)(\varphi^*(x) - \varphi^*(y))| \\ &\leq 3|\rho(x) - \rho(y)| + C_0|\varphi(x) - \varphi(y)|. \end{aligned}$$

Thus $v_n(\varphi^*\rho) \leq 3C_\rho \beta^n + C_0 v_n(\varphi)$. This implies that $\varphi^*\rho \in \mathcal{I}(\Delta)$.

From Lemma 4.1.2, we also have

$$\begin{aligned}\left|\frac{\varphi^*(x)}{\varphi^*(y)}-1\right| &= \frac{1}{\varphi^*(y)}\left|\varphi^*(x)-\varphi^*(y)\right| \\ &\le 3|\varphi(x)-\varphi(y)| \\ &\le 3v_{s(x,y)}(\varphi).\end{aligned} \tag{5.1.1}$$

Since $\frac{1}{9} \le \frac{\varphi^*(x)}{\varphi^*(y)} \le 9$, for all $x, y \in \Delta$, then there exists $K_1' > 0$ such that

$$\left|\log \frac{\varphi^*(x)}{\varphi^*(y)}\right| \le K_1' \left|\frac{\varphi^*(x)}{\varphi^*(y)} - 1\right|. \tag{5.1.2}$$

It follows from (5.1.1) and (5.1.2) that for all $x, y \in \omega \in \eta$,

$$\begin{aligned}\left|\log \frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)}\right| &\le \left|\log \frac{\varphi^*(x)}{\varphi^*(y)}\right| + \left|\log \frac{\rho(x)}{\rho(y)}\right| \\ &\le 3K_1' v_{s(x,y)}(\varphi) + C\beta^{s(x,y)}.\end{aligned} \tag{5.1.3}$$

Since $0 < \frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)} \le 9e^C$, for all $x, y \in \omega \in \eta$, then we also have some uniform constant $K_2' > 0$ that for all $x, y \in \omega \in \eta$,

$$\left|\frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)} - 1\right| \le K_2' \left|\log \frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)}\right| \le K_2'(3K_1' v_{s(x,y)}(\varphi) + C\beta^{s(x,y)}).$$

Hence $\varphi^*\rho \in \mathcal{I}^+(\Delta)$, with $C'' = K_2' C$. □

We can extract the following remark about $\varphi^* \frac{d\nu}{dm}$ from the proof of Lemma 5.1.3, which is important because we make this kind of regularity assumption in the extended coupling argument.

Remark 5.1.4. *From the previous proof we can see that there exists $C''_{\varphi^*\rho} > 0$ such that for all $x, y \in \omega \in \eta$,*

$$\left|\log \frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)}\right| \le C''_{\varphi^*\rho} v_{s(x,y)}(\varphi) + C\beta^{s(x,y)}.$$

5.1.1 Extended coupling

We give the proof of Proposition 5.1.1 in these remaining sections. Extending coupling means that we deal simultaneously with both the coupling argument of Chapter 4 and the coupling argument of [53]. Note that most of the results of this subsection are the analogues of coupling argument of Chapter 4.

It is worth mentioning that we will use some of the results from [53]. Let λ_1 and λ_2 be probability measures on Δ with $\varphi_1 = \frac{d\lambda_1}{dm}, \varphi_2 = \frac{d\lambda_2}{dm} \in \mathcal{I}^+(\Delta)$.

We divide this subsection into two parts. In Part-I we estimate $P\{S > n\}$ and in Part-II we estimate $|T_*^n\lambda_1 - T_*^n\lambda_2|$ in terms of $P\{S > n\}$.

5.1.1.1 Part-I

In this part, we follow the coupling argument of Chapter 4 in order to estimate $P\{S > n\}$. We start with the following lemma, which is the analogous of Lemma 4.1.10.

Lemma 5.1.5. *There exists $D_1 > 0$, depending on C'_{φ_j} and $v_0(\varphi_j)$, for $j = 1, 2$, such that for all $n \geq 1$, $\omega \in \eta_n$ and $x, y \in T^n(\omega)$, we have*

$$\frac{dT_*^n(\lambda_j|\omega)}{dm}(x) \Big/ \frac{dT_*^n(\lambda_j|\omega)}{dm}(y) \leq D_1.$$

Moreover, the dependence of D_1 on C'_{φ_j} and $v_0(\varphi_j)$ can be removed if the number of visits of ω to Δ_0 up to n is sufficiently large.

Proof. Let $x_0, y_0 \in \omega$ be such that $T^n(x_0) = x$ and $T^n(y_0) = y$. We may write

$$\frac{dT_*^n(\lambda_j|\omega)}{dm}(x) \Big/ \frac{dT_*^n(\lambda_j|\omega)}{dm}(y) = \frac{J_{T^n}(y_0)}{J_{T^n}(x_0)} \frac{\varphi_j(x_0)}{\varphi_j(y_0)}. \tag{5.1.4}$$

By Lemma 4.1.8, there exists some $C'_T > 0$ such that

$$\frac{J_{T^n}(y_0)}{J_{T^n}(x_0)} \leq C'_T. \tag{5.1.5}$$

Considering i to be the number of visits of ω to Δ_0 until n, we have $s(x_0, y_0) \geq i$. Since $\varphi_j \in \mathcal{I}^+(\Delta)$ we may write

$$\frac{\varphi_j(x_0)}{\varphi_j(y_0)} = 1 + \left| \frac{\varphi_j(x_0)}{\varphi_j(y_0)} - 1 \right| \leq 1 + C'_{\varphi_j} v_i(\varphi_j) + C''\beta^i. \tag{5.1.6}$$

From (5.1.4), (5.1.5) and (5.1.6), we get

$$\frac{dT_*^n(\lambda_j|\omega)}{dm}(x) \Big/ \frac{dT_*^n(\lambda_j|\omega)}{dm}(y) \leq D_1,$$

where $D_1 = C'_T(1 + C'_{\varphi_j} v_i(\varphi_j) + C''\beta^i) \leq C'_T(1 + C'_{\varphi_j} v_0(\varphi_j) + C'')$. Note that the constant C'_T does not depend on φ_j. So the dependence of D_1 on C'_{φ_j} and $v_0(\varphi_j)$ can be removed if we take i sufficiently large. $\square$

Now we state two lemmas whose proofs follow the same steps of the proofs of Lemma 4.1.14 and Lemma 4.1.15. The only difference here is the dependence of the constants ε_0 and D_2. More precisely, the constants ε_0 of Lemma 5.1.6 and D_2 of Lemma 5.1.7 can be obtained in terms of constant D_1 of Lemma 5.1.5, which will imply the dependence of ε_0 and D_2 on C'_{φ_j} and $v_0(\varphi_j)$ for $j = 1, 2$.

Lemma 5.1.6. *There exists $\varepsilon_0 > 0$, depending on C'_{φ_j} and $v_0(\varphi_j)$ for $j = 1, 2$ such that for all $k \geq 1$ and $\Gamma \in \xi_k$ with $S|_\Gamma > \tau_{k-1}$, we have*

$$P(S = \tau_k|\Gamma) \geq \varepsilon_0.$$

Moreover, the dependence of ε_0 on C'_{φ_j} and $v_0(\varphi_j)$ for $j = 1, 2$ can be removed if we take k sufficiently large.

Lemma 5.1.7. *There exists D_2, depending on C'_{φ_j} and $v_0(\varphi_j)$ for $j = 1, 2$ such that for all $n, k \geq 0$ and for all $\Gamma \in \xi_k$, we have*

$$P(\tau_{k+1} - \tau_k > n + n_0|\Gamma) \leq D_2 m\{\hat{R} > n\}.$$

Moreover, the dependence of D_2 on C'_{φ_j} and $v_0(\varphi_j)$ for $j = 1, 2$ can be removed if we take k sufficiently large.

Finally we can obtain the estimate for $P\{S > n\}$. Recall that

$$m\{\hat{R} > n\} = \sum_{\ell > n} m\{R > n\},$$

which together with Lemma 5.1.6, Lemma 5.1.7, Lemma A.2.2 and Lemma A.2.3 provide the following result.

Lemma 5.1.8.

i) If $m\{R > n\} \leq Cn^{-a}$, for some $C > 0$ and $a > 1$, then $P\{S > n\} \leq \tilde{C}n^{-a+1}$, for some $\tilde{C} > 0$.

ii) If $m\{R > n\} \leq Ce^{-cn}$, for some $C, c > 0$, then $P\{S > n\} \leq \tilde{C}e^{-c_0 n}$, for some $\tilde{C}, c_0 > 0$.

5.1.1.2 Part-II

The key result of this part is Proposition 5.1.16. We will see simultaneous use of both coupling arguments of Chapter 4 and [53].

Let $\Phi(x,y)=\varphi_1(x)\varphi_2(y)$, and let C''_{φ_1} and C''_{φ_2} be constants such that, for $j=1,2$

$$\left|\log\frac{\varphi_j(x)}{\varphi_j(y)}\right| \le C''_{\varphi_j} v_{s(x,y)}(\varphi_j) \text{ for all } x,y\in\omega\in\eta. \tag{5.1.7}$$

The following auxiliary lemma describe explicitly C_Φ and $v_n(\Phi)$.

Lemma 5.1.9. *For all $u,v\in\Omega\in\tilde{\xi}$,*

$$\log\frac{\Phi(u)}{\Phi(v)} \le C_\Phi v_{\tilde{s}(u,v)}(\Phi),$$

where $C_\Phi = C''_{\varphi_1}+C''_{\varphi_2}$ and $v_{\tilde{s}(u,v)}(\Phi)=\max\{v_{\tilde{s}(u,v)}(\varphi_1), v_{\tilde{s}(u,v)}(\varphi_2)\}$.

Proof. Let $u=(x,x')$ and $v=(y,y')$, we may write

$$\begin{aligned}
\log\frac{\Phi(u)}{\Phi(v)} &= \log\frac{\varphi_1(x)}{\varphi_1(y)}+\log\frac{\varphi_2(x')}{\varphi_2(y')}\\
&\le C''_{\varphi_1}v_{s(x,y)}(\varphi_1)+C''_{\varphi_2}v_{s(x',y')}(\varphi_2), \quad \text{by (5.1.7)}\\
&\le C''_{\varphi_1}v_{\tilde{s}(u,v)}(\varphi_1)+C''_{\varphi_2}v_{\tilde{s}(u,v)}(\varphi_2), \quad \text{by Lemma 4.1.16}\\
&\le C_\Phi v_{\tilde{s}(u,v)}(\Phi).
\end{aligned}$$

□

Remark 5.1.10. *Notice that if*

$$\left|\log\frac{\varphi_1(x)}{\varphi_1(y)}\right| \le C''_{\varphi_1}v_{s(x,y)}(\varphi_1)+C''_{\varphi_2}v_{s(x,y)}(\varphi_2),$$

then Lemma 5.1.9 remains true with $C_\Phi = C''_{\varphi_1}+2C''_{\varphi_2}$.

The next lemma is the analogue of Lemma 4.1.19. We rewrite the proof in order to see the explicit dependence of constants.

Lemma 5.1.11. *There exists $D_3>0$, depending on C_Φ and $v_0(\Phi)$, such that for all $n\ge 1, \Omega\in\tilde{\xi}_n$ and $u,v\in\widetilde{T}^n(\Omega)$ we have*

$$\frac{d\widetilde{T}^n_*(P\mid\Omega)}{d(m\times m)}(u)\bigg/\frac{d\widetilde{T}^n_*(P\mid\Omega)}{d(m\times m)}(v)\le D_3.$$

Proof. Let $u_0,v_0\in\Omega$ be such that $\widetilde{T}^n(u_0)=u$ and $\widetilde{T}^n(v_0)=v$. Since $\Phi=dP/d(m\times m)$ we may write

$$\frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m \times m)}(u) \Bigg/ \frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m \times m)}(v) = \frac{J_{\widetilde{T}^n}(v_0)}{J_{\widetilde{T}^n}(u_0)} \cdot \frac{\Phi(u_0)}{\Phi(v_0)}. \tag{5.1.8}$$

Observing that for all $u_0, v_0 \in \Omega \in \tilde{\xi}_n$, we have $\tilde{s}(u_0, v_0) \geq n$, it follows from Lemma 4.1.17, that there is some $C'_{\widetilde{T}} > 0$ such that

$$\frac{J_{\widetilde{T}^n}(v_0)}{J_{\widetilde{T}^n}(u_0)} \leq C'_{\widetilde{T}}. \tag{5.1.9}$$

By Lemma 5.1.9, we have

$$\frac{\Phi(u_0)}{\Phi(v_0)} \leq e^{C_\Phi v_0(\Phi)}. \tag{5.1.10}$$

Using (5.1.8), (5.1.9) and (5.1.10), we get

$$\frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m \times m)}(u) \Bigg/ \frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m \times m)}(v) \leq C'_{\widetilde{T}} e^{C_\Phi v_0(\Phi)}.$$

Which gives the required conclusion with $D_3 = C'_{\widetilde{T}} e^{C_\Phi v_0(\Phi)}$, depending on C_Φ and $v_0(\Phi)$. □

Proposition 5.1.12. *There exists $D_4 > 0$, depending on C_Φ and $v_0(\Phi)$, such that for all $n, k \geq 0$, we have*

$$P\{S_{k+1} - S_k > n\} \leq D_4 (m \times m)\{S > n\}.$$

Remark 5.1.13. *The proof of Proposition 5.1.12 is similar to that of Proposition 4.1.20 and we omit it here. If we carry out the proof, we can see the use of Lemma 5.1.11 which implies that the constant D_4 of Proposition 5.1.12 depends on D_3, and D_3 (hence D_4) depends on C_Φ and $v_0(\Phi)$.*

The following lemma plays an important role in this setting, which is coming from the coupling argument of [53]. We remark that (5.1.11) and (5.1.12) of Lemma 5.1.14 are important for Lemma 5.1.15, and (5.1.13) is useful for Sections 5.1.2 and 5.1.3.

Lemma 5.1.14. *[53, Lemma 1] Given a sequence $v_i(\Phi)$, there exists a sequence $\varepsilon'_i \leq \frac{1}{2}$ such that*

$$v_i(\Phi) \prod_{j=1}^{i} \left(1 + \varepsilon'_j\right) \leq D_5, \tag{5.1.11}$$

$$\sum_{j=1}^{i} \left(\prod_{k=j}^{i} (1 + \varepsilon'_k) \right) \beta^{i-j+1} \leq D_5 \tag{5.1.12}$$

for some sufficiently large constant D_5 depending on $v_0(\Phi)$. Moreover, for any $\bar{D} > 1$ and $\bar{\delta} > 0$, with $\varepsilon_i = \bar{\delta}\varepsilon'_i$,

$$\prod_{j=1}^{i} \left(1 - \frac{\varepsilon_j}{\bar{D}}\right) \leq \hat{D} \max\{v_i(\Phi)^{\frac{\bar{\delta}}{\bar{D}}}, \theta^i\} \tag{5.1.13}$$

for some $\theta < 1$ and some $\hat{D} > 0$.

The following definition of sequence of densities $\tilde{\Phi}_0 \geq \tilde{\Phi}_1 \geq \tilde{\Phi}_2 \geq \cdots$, defined on $\Delta \times \Delta$, is slightly different from the one defined in Chapter 4. Consider a constant $\bar{\delta} > 0$, depending on C_Φ and $v_0(\Phi)$ to be chosen properly such that defining

$$\tilde{\Phi}_i(u) = \begin{cases} \Phi(u), & \text{if } i = 0 \\ \tilde{\Phi}_{i-1}(u) - \varepsilon_i J_{\tilde{T}^i}(u) \min_{v \in \Omega_i(u)} \frac{\tilde{\Phi}_{i-1}(v)}{J_{\tilde{T}^i}(v)}, & \text{if } i > 0. \end{cases} \tag{5.1.14}$$

for all $i \geq 1$ with $\varepsilon_i = \bar{\delta}\varepsilon'_i > 0$, the following holds (for details see [53, Lemma 2]):

Lemma 5.1.15. *There exists $D_6 > 1$, depending on C_Φ and $v_0(\Phi)$, such that for all $i \geq 1$ we have*

$$\tilde{\Phi}_i \leq (1 - \frac{\varepsilon_i}{D_6})\tilde{\Phi}_{i-1}.$$

We have the following key result of Part-II, which is an important step to prove Proposition 5.1.1.

Proposition 5.1.16. *There exists $D_7 > 0$, depending on C_Φ and $v_0(\Phi)$, such that for all $n \geq 1$,*

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq 2P\{S > n\} + D_7 \sum_{i=1}^{n} \prod_{j=1}^{i} \left(1 - \frac{\varepsilon_j}{D_6}\right) (i+1)(m \times m) \left\{S > \frac{n}{i+1}\right\}.$$

Proof. As in the proof of Proposition 4.1.25, for each $n \geq 1$ we have

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq I_1 + I_2, \tag{5.1.15}$$

where

$$I_1 = \left|(\pi_{1*} - \pi_{2*})\,(T\times T)_*^n\,(\Phi_n(m\times m))\right|$$

and

$$I_2 = \sum_{k=1}^{n} \left|(\pi_{1*} - \pi_{2*})\left[(T\times T)_*^n\,((\Phi_{k-1} - \Phi_k)\,(m\times m))\right]\right|.$$

In one hand, we have

$$\begin{aligned} I_1 &= \left|(\pi_{1*} - \pi_{2*})\,(T\times T)_*^n\,(\Phi_n(m\times m))\right| \\ &\le 2\int_{\Delta\times\Delta} \Phi_n d(m\times m) \\ &= 2\int_{\{S_1>n\}} \Phi_n d(m\times m) + 2\sum_{i=1}^{\infty}\int_{\{S_i\le n<S_{i+1}\}} \Phi_n d(m\times m). \end{aligned} \tag{5.1.16}$$

As $\Phi_n|_{\{S_1>n\}} = \Phi$, we have

$$\int_{\{S_1>n\}} \Phi_n d(m\times m) = \int_{\{S_1>n\}} \Phi d(m\times m) = P\left\{S_1 > n\right\}. \tag{5.1.17}$$

It follows from (4.1.33), (5.1.14) and Lemma 5.1.15 we can get for all $i \ge 1$,

$$\begin{aligned} \int_{\{S_i\le n<S_{i+1}\}} \Phi_n d(m\times m) &= \int_{\{S_i\le n<S_{i+1}\}} \tilde{\Phi}_i d(m\times m) \\ &= \prod_{j=1}^{i}\left(1-\frac{\varepsilon_j}{D_6}\right) P\left\{S_i \le n < S_{i+1}\right\}. \end{aligned} \tag{5.1.18}$$

By combining (5.1.16), (5.1.17) and (5.1.18), we obtain

$$I_1 \le 2P\{S>n\} + 2\sum_{i=1}^{\infty}\prod_{j=1}^{i}\left(1-\frac{\varepsilon_j}{D_6}\right) P\left\{S_i \le n < S_{i+1}\right\}. \tag{5.1.19}$$

On the other hand,

$$\begin{aligned} I_2 &\le 2\sum_{k=1}^{n}\int_{\Delta\times\Delta} (\Phi_{k-1} - \Phi_k)\,d(m\times m) \\ &= 2\sum_{k=1}^{n}\int_{\Delta\times\Delta\setminus A_k} (\Phi_{k-1} - \Phi_k)\,d(m\times m) + 2\sum_{k=1}^{n}\int_{A_k} (\Phi_{k-1} - \Phi_k)\,d(m\times m). \end{aligned} \tag{5.1.20}$$

From Remark 4.1.24, we have

$$2\sum_{k=1}^{n}\int_{\Delta\times\Delta\setminus A_k}(\Phi_{k-1}-\Phi_k)\,d(m\times m)=0, \tag{5.1.21}$$

and writing $A_k=\bigcup_{i=1}^{\infty}A_{k,i}$, we have

$$\int_{A_k}\left(\tilde{\Phi}_{k-1}-\tilde{\Phi}_k\right)d(m\times m)=\sum_{i=1}^{\infty}\int_{A_{k,i}}\left(\tilde{\Phi}_{i-1}-\tilde{\Phi}_i\right)d(m\times m). \tag{5.1.22}$$

By (5.1.14) and Lemma 5.1.15 we have

$$\begin{aligned}\sum_{i=1}^{\infty}\int_{A_{k,i}}\left(\tilde{\Phi}_{i-1}-\tilde{\Phi}_i\right)d(m\times m)&=\sum_{i=1}^{\infty}\int_{A_{k,i}}\varepsilon_i J_{\tilde{T}^i}(u)\min_{v\in A_{k,i}}\frac{\tilde{\Phi}_{i-1}(v)}{J_{\tilde{T}^i}(v)}d(m\times m)\\&\leq\sum_{i=1}^{\infty}\int_{A_{k,i}}\varepsilon_i\tilde{\Phi}_{i-1}(u)d(m\times m)\\&\leq\sum_{i=1}^{\infty}\int_{A_{k,i}}\varepsilon_i\prod_{j=1}^{i-1}\left(1-\frac{\varepsilon_j}{D_6}\right)\Phi(u)d(m\times m)\\&=\sum_{i=1}^{\infty}\varepsilon_i\prod_{j=1}^{i-1}\left(1-\frac{\varepsilon_j}{D_6}\right)P\{S_i=k\}.\end{aligned} \tag{5.1.23}$$

It follows from (5.1.20), (5.1.21), (5.1.22) and (5.1.23) that

$$I_2\leq 2\sum_{i=1}^{\infty}\varepsilon_i\prod_{j=1}^{i-1}\left(1-\frac{\varepsilon_j}{D_6}\right)P\left(\bigcup_{k=1}^{n}\{S_i=k\}\right). \tag{5.1.24}$$

For each $1\leq k\leq n$, we have $\{S_i=k\}\subset\{S_i\leq n<S_{i+1}\}$, and this implies that

$$\bigcup_{k=1}^{n}\{S_i=k\}\subset\{S_i\leq n<S_{i+1}\}.$$

From (5.1.24) we have

$$I_2\leq\frac{2\bar{\delta}D_6}{2D_6-\bar{\delta}}\sum_{i=1}^{\infty}\prod_{j=1}^{i}\left(1-\frac{\varepsilon_j}{D_6}\right)P\{S_i\leq n<S_{i+1}\}. \tag{5.1.25}$$

It follows from (5.1.15), (5.1.19) and (5.1.25), that

$$|T_*^n\lambda_1-T_*^n\lambda_2|\leq 2P\{S>n\}+2\left(1+\frac{\bar{\delta}D_6}{2D_6-\bar{\delta}}\right)\sum_{i=1}^{\infty}\prod_{j=1}^{i}\left(1-\frac{\varepsilon_j}{D_6}\right)P\left\{S_i\leq n<S_{i+1}\right\}. \tag{5.1.26}$$

For each $i \geq 1$ we have

$$P\{S_i \leq n < S_{i+1}\} \leq \sum_{j=0}^{i} P\left\{S_{j+1} - S_j > \frac{n}{i+1}\right\}, \tag{5.1.27}$$

and from Proposition 5.1.12,

$$P\left\{S_{j+1} - S_j > \frac{n}{i+1}\right\} \leq D_4(m \times m)\left\{S > \frac{n}{i+1}\right\}. \tag{5.1.28}$$

Combining (5.1.27) and (5.1.28), we get

$$P\{S_i \leq n < S_{i+1}\} \leq D_4(i+1)(m \times m)\left\{S > \frac{n}{i+1}\right\},$$

which, together with (5.1.26), implies

$$|T_*^n\lambda_1 - T_*^n\lambda_2| \leq 2P\{S > n\} + D_7 \sum_{i=1}^{n} \prod_{j=1}^{i} \left(1 - \frac{\varepsilon_j}{D_6}\right)(i+1)(m \times m)\left\{S > \frac{n}{i+1}\right\},$$

with $D_7 = 2D_4\left(1 + \frac{\bar{\delta}D_6}{2D_6 - \bar{\delta}}\right)$. We recall that $\bar{\delta}$, D_4 and D_6 (hence D_7) depend on C_Φ and $v_0(\Phi)$.

□

In the rest of this section, we set $\zeta = \frac{\bar{\delta}}{D_6}$, which can be seen to depend on C_Φ and $v_0(\Phi)$.

5.1.2 Polynomial return times

In this subsection, we combine several estimates, from which we will conclude the first item of Proposition 5.1.1. If $m\{R > n\} \leq Cn^{-a}$, for some $C > 0$ and $a > 1$, then by Lemma 5.1.8,

$$P\{S > n\} \leq \tilde{C}n^{-a+1}. \tag{5.1.29}$$

By similar arguments used to estimate $P\{S > n\}$, we have

$$(m \times m)\left\{S > \frac{n}{i+1}\right\} \leq \tilde{C}_1\left(\frac{n}{i+1}\right)^{-a+1} \quad \text{for some } \tilde{C}_1 > 0. \tag{5.1.30}$$

Class $(V4, \tau)$: Assume that $v_i(\Phi) \leq Di^{-\tau}$, for some $\tau > \frac{2}{\zeta}$, and $D > 0$.

By Lemma 5.1.14 we have

$$\prod_{j=1}^{i}\left(1-\frac{\varepsilon_j}{D_6}\right) \leq \tilde{D} i^{-\zeta\tau}, \text{ for some } \tilde{D} > 0. \tag{5.1.31}$$

By using (5.1.29), (5.1.30) and (5.1.31) in Proposition 5.1.16, we get

$$\begin{aligned}|T_*^n\lambda_1 - T_*^n\lambda_2| &\leq 2\tilde{C}n^{-a+1} + \tilde{C}_1\tilde{D}D_7\sum_{i=1}^{\infty} i^{-\zeta\tau}(i+1)\left(\frac{n}{i+1}\right)^{-a+1} \\ &\leq 2\tilde{C}n^{-a+1} + K_3'\tilde{C}_1\tilde{D}D_7 n^{-a+1}\sum_{i=1}^{\infty} i^{-\zeta\tau} i^a, \text{ for some } K_3' > 0.\end{aligned}$$

First we want to estimate the second term. For that we consider the following cases:

- $\tau = \frac{a+1}{\zeta}$. The sum is

  $$\sum_{i=1}^{n} i^{-\zeta\tau+a} = \sum_{i=1}^{n} i^{-1} \leq 1 + \int_1^n x^{-1}dx = 1 + \log(n) \leq \bar{D}\log n \text{ for some } \bar{D} > 0.$$

  So the whole term is $\leq D'(n^{1-a}\log n)$, for some $D' > 0$.

- $\tau > \frac{a+1}{\zeta}$. Here, $a - \zeta\tau < -1$, so the sum is bounded from above independently of n, and the whole term is $\leq D'n^{1-a}$, for some $D' > 0$.

- $\frac{2}{\zeta} < \tau < \frac{a+1}{\zeta}$. The sum is of order $n^{a+1-\zeta\tau}$, and so the whole term is $\leq D'n^{2-\zeta\tau}$, for some $D' > 0$.

Consequently,

i) if $\tau = \frac{a+1}{\zeta}$, then $|T_*^n\lambda_1 - T_*^n\lambda_2| \leq D'\left(n^{1-a}\log n\right)$ for some $D' > 0$;

ii) otherwise, $|T_*^n\lambda_1 - T_*^n\lambda_2| \leq D'\max\{n^{1-a}, n^{2-\zeta\tau}\}$ for some $D' > 0$.

5.1.3 Exponential return times

Second item of Proposition 5.1.1 will be a consequence of this subsection.

If $m\{R > n\} \leq Ce^{-cn}$, for some $C, c > 0$, then by Lemma 5.1.8 we have

$$P\{S > n\} \leq \tilde{C}e^{-c_0 n}. \tag{5.1.32}$$

By similar arguments used to estimate $P\{S > n\}$, we have

$$(m \times m)\left\{S > \frac{n}{i+1}\right\} \leq \tilde{C}_2 e^{\frac{-c_0 n}{i+1}}, \text{ for some } \tilde{C}_2 > 0. \tag{5.1.33}$$

Class $(V1, \tau)$: Assume that $v_i(\Phi) \leq D\tau^i$, for some $0 < \tau < 1$ and $D > 0$. By Lemma 5.1.14, we have

$$\prod_{j=1}^{i}\left(1 - \frac{\varepsilon_j}{D_6}\right) \leq \tilde{D}\theta_1^i \tag{5.1.34}$$

for some $0 < \theta_1 < 1$ and $\tilde{D} > 0$. By using (5.1.32), (5.1.33) and (5.1.34) in Proposition 5.1.16, we get

$$|T_*^n\lambda_1 - T_*^n\lambda_2| \leq 2\tilde{C}e^{-c_0 n} + \tilde{C}_2\tilde{D}D_7\sum_{i=1}^{n}\theta_1^i(i+1)\left(e^{-c_0(\frac{n}{i+1})}\right) \leq D'e^{-c'n},$$

for some $D', c' > 0$. By similar arguments as used in the above class, we have the following estimates for other classes.

Class $(V2, \tau)$: Assume that $v_i(\Phi) \leq De^{-i^\tau}$, for some $\tau \in (0,1)$ and $D > 0$. We have

$$|T_*^n\lambda_1 - T_*^n\lambda_2| \leq 2\tilde{C}e^{-c_0 n} + \tilde{C}_2\tilde{D}D_7\sum_{i=1}^{n}e^{-\zeta i^\tau}(i+1)\left(e^{-c_0(\frac{n}{i+1})}\right) \leq D'e^{-n^{\tau'}},$$

for every $0 < \tau' < \tau$ and some $D' > 0$.

Class $(V3, \tau)$: Assume that $v_i(\Phi) \leq De^{-(\log i)^\tau}$, for some $\tau > 1$, and $D > 0$. We have

$$|T_*^n\lambda_1 - T_*^n\lambda_2| \leq 2\tilde{C}e^{-c_0 n} + \tilde{C}_2\tilde{D}D_7\sum_{i=1}^{n}e^{-\zeta(\log i)^\tau}(i+1)\left(e^{-c_0(\frac{n}{i+1})}\right) \leq D'e^{-(\log(n))^{\tau'}},$$

for every $0 < \tau' < \tau$ and some $D' > 0$.

Class $(V4, \tau)$: Assume that $v_i(\Phi) \leq Di^{-\tau}$, for some $\tau > \frac{1}{\zeta}$ and $D > 0$. We have

$$|T_*^n\lambda_1 - T_*^n\lambda_2| \leq 2\tilde{C}e^{-c_0 n} + \tilde{C}_2\tilde{D}D_7\sum_{i=1}^{n}i^{-\zeta\tau}(i+1)\left(e^{-c_0(\frac{n}{i+1})}\right) \leq D'n^{1-\tau\zeta},$$

for some $D' > 0$.

Let us conclude the proof of Proposition 5.1.1. We notice that from the proof of Lemma 5.1.3, we have $v_n(\varphi^*\rho) \leq 3C_\rho\beta^n + C_0 v_n(\varphi)$. This implies that $v_n(\Phi) \leq 3C_\rho\beta^n + C_0 v_n(\varphi)$.

Now Taking $\lambda_1 = \lambda$, where λ is the probability measure on Δ such that $\frac{d\lambda}{dm} = \varphi^*\rho$, and $\lambda_2 = \nu$ in Sections 5.1.2 and 5.1.3, we get the required estimates for $|T_*^n\lambda - \nu|$. Let us check the dependence of the constants. From (5.1.3), $C_\Phi = 3K_1' + 2C$. This means that the constant C_Φ does not depend on φ. Therefore the constant $\zeta = \frac{\bar{\delta}}{D_4}$ will only depend on $v_0(\Phi)$, which can be bounded with $v_0(\varphi)$, as $v_0(\Phi) \leq 3C_\rho + C_0 v_0(\varphi)$.

5.2 Central Limit Theorem for non-Hölder observables

In this section we prove Corollary E via the following proposition, which provides the CLT for the tower map.

Proposition 5.2.1. *Let $T : \Delta \to \Delta$ be the tower map of an aperiodic induced WGM expanding map f^R, with a coprime block, $R \in L^1(m)$, and let ν be the unique mixing T-invariant probability measure. If $m\{R > n\} \leq Cn^{-a}$, for some $a > 2$ and $C > 0$, then the CLT is satisfied for all $\varphi \in (V4, \tau)$, with τ sufficiently large, if and only if φ is not coboudary.*

Proof. The proof follows in exactly the same manner as the one of Proposition 4.2.2. □

Since $\varphi \in (R4, \tau)$ for some $\tau > 1$, then by Lemma 5.1.2, $\varphi \circ \pi \in (V4, \tau)$. Notice that φ is not coboundary if and only if $\varphi \circ \pi$ is not coboundary. The proof of Corollary E follows now from (4.2.2) and Proposition 5.2.1.

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Chapter 6

Conclusions and topics for future research

In this final chapter, we provide a summary of the work presented throughout the thesis and suggest possible future projects.

6.1 Summary

The goal of this thesis was to investigate statistical properties of dynamical systems having induced WGM map. Particularly, we have managed to investigate the decay of correlations, CLT and large deviations for dynamical systems having induced WGM map. We obtained these statistical properties here using an inducing scheme. This means that we considered dynamical systems $f : M \to M$ having an induced WGM map $f^R : \Delta_0 \to \Delta_0$, and we defined a tower map $T : \Delta \to \Delta$ of the induced WGM map f^R. We first obtained some statistical properties for the tower map, and then these properties went back to the given dynamical system through a semi-conjugacy $\pi : \Delta \to M$.

The first goal of our work was to ensure the existence of suitable mixing measures for these dynamical systems. In Chapter 3 we managed to achieve mixing measures for these dynamical systems under some extra assumptions, namely the aperiodicity and the existence of a coprime block. Specifically, we obtained a mixing f-invariant probability measure μ, a mixing f^R-invariant probability measure ν_0 and a mixing T-invariant probability measure ν.

Our next goal was to describe the statistical properties of (f, μ). In Chapter 2, we provided formal statements of these properties and their proofs were presented in Chapter 4 and Chapter 5. In fact, in Chapter 4 we provided the proofs of statistical properties of (f, μ) for Hölder observables. As mentioned earlier, we first obtained some statistical properties such as the decay of correlations and CLT for (T, ν), and then these properties went back to (f, μ). In this context, we generalized L.-S. Young's coupling argument in the absence of full branch property in Section 4.1.1, which leads the decay of correlations and CLT for (T, ν). As a results, we obtained the similar estimates for the decay of correlations and CLT to those in [71] under weaker hypotheses. Moreover, we shared some remarks concerning the lack of full branch property in Section 4.1.1. In Chapter 5, we extended the results of Chapter 4, by considering the regularity classes $(V1 - V4)$ defined on Δ, which provided the statistical properties of (f, μ) for the larger classes of observables $(R1 - R4)$ on M.

6.2 Conclusion

I would like to conclude this thesis with the following remarks.

In this work we achieved results on the decay of correlations, CLT and large deviations for dynamical system $f : M \to M$, having an aperiodic induced WGM map $f^R : \Delta_0 \to \Delta_0$ with a coprime block and integrable return times.

Similarly, we can achieve other results for the given dynamical system $f : M \to M$, having an aperiodic induced WGM map $f^R : \Delta_0 \to \Delta_0$ with a coprime block and integrable return times, that are known for dynamical systems having an induced GM map (full branch) using a coupling argument. This is feasible because we have already developed in this thesis a coupling argument in the absence of the full branch property.

We keenly observed that constructing an induced WGM map $f^R : \Delta_0 \to \Delta_0$ for $f : M \to M$ is often a difficult task. It may be even more harder to construct an aperiodic induced WGM map f^R with a coprime block. As we have seen, the aperiodicity and coprime block are only used to get a unique mixing f-invariant probability measure μ equivalent to the reference measure m, such that $\mu(\Delta_0) > 0$. In order to apply the results of this thesis to a particular dynamical systems $f : M \to M$, then either construct an aperiodic induced WGM map $f^R : \Delta_0 \to \Delta_0$ with coprime block and integrable return time, or simply construct an induced WGM $f^R : \Delta_0 \to \Delta_0$ and ensure that f has a unique mixing invariant probability measure μ, equivalent to the reference measure m with $\mu(\Delta_0) > 0$. In the latter case, there is no need to verify the aperiodicity and coprime block conditions.

6.3 Topics for future projects

There are many possible future projects that can be investigated for the dynamical system studied in this thesis, namely the dynamical system having an induced WGM map.

One of future project is to obtain similar results as in [70], but under weak conditions on dynamical system. More precisely, in [70] the existence of a hyperbolic product structure known as the *Young structure* for a given dynamical system was assumed, from which exponential decay of correlations and CLT were obtained. The Young structure consists of a reference set Λ with a hyperbolic product structure that satisfies several properties. One of the property included was the existence of a Markov partition $\Lambda_1, \Lambda_1, \ldots$ of Λ in such a way that the quotient map becomes a GM map. It should be possible to relax some conditions on the Markov partition considered in the Young structure that forces the quotient map to satisfy the full branch property. If the quotient map becomes a WGM map, then by applying this thesis work should be possible to obtain similar results as in [70].

It would also be interesting to investigate other properties, such as almost sure invariance principle, entropy formula, and concentration inequalities of dynamical systems that have an induced WGM map. These results have already been discussed for dynamical systems with an induced GM map, using techniques other than the coupling argument. For instance, in [22, 30] it was considered an operator renewal theory as a method in order to estimate concentration inequality for dynamical system having induced GM map. So we might need to develop some technique in the absence of a full branch property other than coupling. We may refer to [48, 56] for almost sure invariance principle and to [11, 12] for entropy formula.

It should be interesting to consider random dynamical system where the fibre maps are WGM maps as considered in this thesis. In particular, we should obtain statistical properties, including quenched versions in more general settings than [5, 15, 44, 51, 65].

A natural question that may arise is "whether we can always induce GM map with the same return time decay once we have an induced WGM map with some return time decay?". In some cases, as for instance in [71, §6], [40] and [34], the induced WGM map is used to get a GM map, but to my knowledge none of them cover the general case.

All those topics will be subject of future work.

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Appendix A

Basics of Ergodic Theory and useful results

In this appendix, we recall some basic definitions in ergodic theory and we list some results that are used in this thesis. We also review an example of WGM map in which the density of invariant measure does not have positive lower bound. Moreover, for the sake of completeness, we provide the construction of an induced WGM map that we used in Section 2.4 and proofs of ergodicity and exactness of measure for WGM map.

A.1 Basic definitions

The following basics definitions of Ergodic Theory can be found in [69], and material related to partitions comes from [3, Chapter 2].

Definition A.1.1 (Measure space and measurable sets)**.** *Let M be a set with $\sigma-$ algebra $\mathcal{B}$ and a measure m. We call $(M, \mathcal{B})$ a measurable space, $(M, \mathcal{B}, m)$ a measure space and the elements of $\mathcal{B}$ measurable sets. If $m(M) < \infty$ then we say that m is finite measure and if $m(M) = 1$ then we call m a probability measure.*

From now on we consider a measure space $(M, \mathcal{B}, m)$.

Definition A.1.2 (Full measure set)**.** *We say that a measurable set $B \subset M$ has full m measure if $m(M \backslash B) = 0$.*

Definition A.1.3 (m mod 0 partition)**.** *A family $\mathcal{P}$ of measurable subsets of M is an (m mod 0) partition of M if there exists a full m measure set B such that the elements in the family $\{\omega \cap B : \omega \in \mathcal{P}\}$ are pairwise disjoint and their union is equal to B.*

Definition A.1.4 (Refinement of a partition)**.** *An (m mod 0) partition $\mathcal{P}_2$ is said to be a refinement of an (m mod 0) partition $\mathcal{P}_1$, if each element of $\mathcal{P}_1$ is an (m mod 0) union of elements in $\mathcal{P}_2$. In this case we write $\mathcal{P}_1 \prec \mathcal{P}_2$. We call the sequence $(\bigvee_{i=1}^{n} \mathcal{P}_i)_n$ increasing if $\bigvee_{i=1}^{n} \mathcal{P}_i \prec \bigvee_{i=1}^{n+1} \mathcal{P}_i$.*

Definition A.1.5 (Generating sequences)**.** *Let $(\mathcal{P}_n)_n$ be a sequence of an (m mod 0) partitions of M. We say that $(\mathcal{P}_n)_n$ generates $\mathcal{B}$ (m mod 0) if the σ-algebra generated by $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ coincides with $\mathcal{B}$ (m mod 0).*

Consider now a measurable map $f : M \to M$ and an (m mod 0) partition $\mathcal{P}$ of M.

Definition A.1.6 (Generating partition). *We say that $\mathcal{P}$ is generating partition if the increasing sequence of partitions $\mathcal{P}^n = \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P})$ generates $\mathcal{B}$ (m mod 0).*

Lemma A.1.7. *[3, Corollary 2.3] Let m be a measure such that $f_*m \ll m$. If $\mathcal{P}$ is a generating partition , then for all $\delta > 0$ and $A \in \mathcal{B}$ with $m(A) > 0$, there are $n \geq 1$ and $\omega \in \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{P})$ such that $m(\omega \setminus A) < \delta m(\omega)$.*

Definition A.1.8 (Jacobian). *Assume that there is countable (m mod 0) partition $\mathcal{P}$ of M into invertibility domains of f i.e, f is a bijection from each $\omega \in \mathcal{P}$ to $f(\omega)$, with a measurable inverse. We say that a measurable function $J_f : M \to [0, \infty)$ is a Jacobian of f (with respect to m) if*

$$m(f(A)) = \int_A J_f dm,$$

for every measurable set $A \subset \omega \in \mathcal{P}$.

Lemma A.1.9. *[3, Lemma 2.11] Let $f : M \to M$ be a measurable map with an (m mod 0) partition $\mathcal{P}$ into invertiblity domains and $J_f > 0$. For all $n \geq 1$, f^n has a Jacobian and*

i) $J_{f^n} = \prod_{j=0}^{n-1} J_f \circ f^j$;

ii) if $\lambda \ll m$ and $\varphi = d\lambda/dm$, then for all $\omega \in \bigvee_{j=0}^{n-1} f^{-j}(\mathcal{P})$

$$\tfrac{df_*^n(\lambda|\omega)}{dm} \circ f^n|_\omega = \tfrac{\varphi}{J_{f^n}}|_\omega.$$

Definition A.1.10 (Invariant measure). *A measure m is f-invariant if for all $A \in \mathcal{B}$ we have $m(f^{-1}(A)) = m(A)$.*

Definition A.1.11 (Ergodic measure). *A measure m is ergodic if for every measurable and an invariant set A (i.e, $A \in \mathcal{B}$ and $f^{-1}(A) = A$), $m(A) = 0$ or $m(A) = 1$.*

Definition A.1.12 (Mixing measure). *An f-invariant measure m is mixing if*

$$\lim_{n \to \infty} m(f^{-n}(A) \cap B) = m(A)m(B),$$

for all $A, B \in \mathcal{B}$.

Definition A.1.13 (Exact measure). *An f-invariant measure m is exact if*

$$A \in \bigcap_{n \geq 0} f^{-n}(\mathcal{B}) \Longrightarrow m(A) = 0 \text{ or } m(M \setminus A) = 0.$$

Note that exact $\Longrightarrow$ mixing $\Longrightarrow$ ergodic.

Theorem A.1.14 (Birkhoff). *Let $f : M \to M$ be a measurable map and let m be an ergodic f-invariant probability measure. Then, for any function $\varphi : M \to \mathbb{R}$ in $L^1(M, m)$, and for m almost every x we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \int \varphi dm.$$

A.2 Some useful results

We now state some results that are used to get invariant measure for WGM map, to estimate $P\{S > n\}$ in the coupling arguments and to obtain the large deviations results.

Let $\Delta_0 \subset M$, with $m(\Delta_0) > 0$. Recall $\mathcal{F}_\beta(\Delta_0)$ and $\|.\|_\beta$ from Section 3.1.

Lemma A.2.1. *[3, Proposition 3.8] If $(\varphi_n)_n$ is a sequence in $\mathcal{F}_\beta(\Delta_0)$ such that $\|\varphi_n\|_\beta \leq C$, for all $n \in \mathbb{N}$, then $(\varphi_n)_n$ has a subsequence converging pointwise and in $L^1(\Delta_0, m)$ to a function $\varphi : \Delta_0 \to \mathbb{R}$ with $\|\varphi\|_\beta \leq C$.*

We say that a constant $C > 0$ has *bounded dependence* on D if for every $K_1 > 0$, there is $K_2 > 0$ such that $C \leq K_2$ whenever $D \leq K_1$.

Lemma A.2.2. *[3, Proposition 3.46] Let S and $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ be measurable functions on a measurable space M and $\xi_1 \prec \xi_2 \prec \cdots$ be measurable partitions such that $\tau_0, \ldots, \tau_j$ and $\{S > \tau_{j-1}\}$ are ξ_j-measurable, for all $j \geq 1$. Assume that P is a probability measure on M for which*

(A_1) there is $\varepsilon_0 > 0$ such that, for all $j \geq 1$ and $\Omega \in \xi_j$ with $S|_\Omega > \tau_{j-1}$,

$$P\left(S = \tau_j \mid \Omega\right) \geq \varepsilon_0;$$

(A_2) there are $C, a > 0$ and $n_0 \geq 0$ such that, for all $j \geq 0$ and $\Omega \in \xi_j$,

$$P\left(\tau_{j+1} - \tau_j > n + n_0 \mid \Omega\right) \leq C n^{-a}.$$

Then, there exists $\bar{C} > 0$ such that

$$P\{S > n\} \leq (1-\varepsilon_0)^{\left[\frac{n}{2n_0+1}\right]} + \bar{C} \sum_{j=1}^{\infty} (j+1)^{a+1} (1-\varepsilon_0)^{j-1} n^{-a}.$$

Moreover, if C has bounded dependence on some D, then $\bar{C}$ also has bounded dependence on D.

Lemma A.2.3. *[3, Proposition 3.48] Let S, k and $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ be measurable functions on a measurable space M such that $S(x) = \tau_k(x)$ and $\xi_1 \prec \xi_2 \prec \cdots$ be measurable partitions such that $\tau_0, \ldots, \tau_j$ and $\{S > \tau_{j-1}\}$ are ξ_j-measurable, for all $j \geq 1$. Assume that P is a probability measure on M for which*

(B_1) *there is $\varepsilon_0 > 0$ such that, for all $j \geq 1$ and $\Omega \in \xi_j$ with $S|_\Omega > \tau_{j-1}$,*

$$P(S = \tau_j \mid \Omega) \geq \varepsilon_0;$$

(B_1) *there are C, $0 < a \leq 1$ and $n_0 \geq 0$ such that, for all $j \geq 0$ and $\Omega \in \xi_j$,*

$$P(\tau_{j+1} - \tau_j > n + n_0 \mid \Omega) \leq Ce^{-cn^a}.$$

Then, there exists $\bar{C} > 0$, $\varepsilon > 0$ and $c_0 > 0$ such that

$$P\{S > n\} \leq (1 - \varepsilon_0)^{[\varepsilon n^a]} + \bar{C}e^{-c_0 n^a}.$$

Moreover, ε only depend on a, c and if C has bounded dependence on some D, then $\bar{C}$ also has bounded dependence on D.

Proposition A.2.4. *Let $f : M \to M$ be a nonsingular map with respect to an ergodic f invariant probability measure μ. Let $\mathcal{H} \subset L^\infty(M, \mu)$ be a Banach space and $\varphi \in \mathcal{H}$.*

1. *([55, Theorem 1.2]) Let $\beta > 0$ and suppose that for all $\psi \in L^\infty(M, \mu)$, we have $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' n^{-\beta}$. Then, for every $\varepsilon > 0$, there exists $C = C(\varphi, \varepsilon) > 0$ such that $LD_\mu(\varphi, \varepsilon, n) \leq Cn^{-\beta}$.*

2. *([8, Theorem D]) Let $c, a > 0$ and suppose that for all $\psi \in L^\infty(M, \mu)$, we have $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq C' e^{-cn^a}$. Then, for every $\varepsilon > 0$, there exists $C = C(\varphi, \varepsilon) > 0$ and $c' = c'(c, \varphi, \varepsilon) > 0$ such that $LD_\mu(\varphi, \varepsilon, n) \leq Ce^{-c' n^{a/(a+2)}}$.*

A map $f : M \to M$ being nonsingular with respect to μ in the above lemma means that $f_* m \ll \mu$ whenever $m \ll \mu$.

A.3 A WGM map lacking positive lower bound for the density

We now present an example of an aperiodic WGM map in which the density of the invariant measure does not have a positive lower bound. This was given in [64], but we modify it to the present formulation.

Example A.3.1. *Let $S \subset \mathbb{N}$ be a countable set and $\Sigma = S^{\mathbb{N}}$ the set of all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in S$ for every n. Consider the shift map $\sigma : \Sigma \to \Sigma$, given by*

$$\sigma((x_n)_n) = (x_{n+1})_n.$$

Consider the family of transition probabilities $(P(i, .))_{i\in S}$ on S is defined by

$$P(i, \{j\}) = P_{i,j} = \begin{cases} 1 - \frac{1}{i^2} & \text{if } j = 1, i > 1; \\ \frac{1}{i^2} & \text{if } j = i + 1; \\ 0 & \text{else} \end{cases}$$

Moreover, we define the finite measure p on S by $p(\{n\}) = p_n = \frac{1}{n^2}$. Let us take

$$\Sigma_P = \{(x_n)_n \in \Sigma : P_{x_n,x_{n+1}} > 0 \text{ for every } n\}.$$

Let m be the Markov measure given by

$$m([a_0, ..., a_n]) = p_{a_0} P_{a_k,a_{k+1}} ... P_{a_{n-1},a_n}.$$

Consider the partition $\mathcal{P}_0 = \{[a] \cap \Sigma_P : a \in S\}$ of Σ_P. We can easily see that (Σ_P, m, σ) is an aperiodic WGM map with respect to partition $\mathcal{P}_0$. In this case the the measure m is exact, but not σ-invariant. By Theorem 3.1.6, there exists an exact unique σ-invariant probability measure $\nu_0 \ll m$. Let us see explicitly form of the measure ν_0.

Consider the Ruelle-Perron-Frobenius operator, $\mathcal{L}_\varphi : C^0(\Sigma_P) \to C^0(\Sigma_P)$ (where $C^0(\Sigma_P)$ is the space of continuous functions) is given by

$$\mathcal{L}_\varphi g(y) = \sum_{x\in\sigma^{-1}(y)} e^{\varphi(x)} g(x),$$

where, $\varphi : \Sigma_P \to \mathbb{R}$ is a Hölder continuous function. Since the dual space of the $C^0(\Sigma_P)$ identified with the space of measures $\mathcal{M}(\Sigma_P)$ on Σ_P, by Riesz-Markov Theorem. Then, the dual of the above operator, $\mathcal{L}^*_\varphi : \mathcal{M}(\Sigma_P) \to \mathcal{M}(\Sigma_P)$, is defined as

$$\int g d(\mathcal{L}^*_\varphi \lambda) = \int (\mathcal{L}_\varphi g) d\lambda \text{ for every } g \in C^0(\Sigma_P) \text{ and } \lambda \in \mathcal{M}(\Sigma_P).$$

Take now $\varphi(x) = \log\left(\frac{p_{x_0} P_{x_0 x_1}}{P_{x_1}}\right)$ and $h(x) = \left(\frac{x_0}{(x_0-1)!}\right)^2$ defined on Σ_P. We can easily prove that $\mathcal{L}^*_\varphi m = m$ and $\mathcal{L}_\varphi h = h$. Now consider the measure $\nu_0 = hm$, such that for each measurable set $A \subset \Sigma_P$,

$$\nu_0(A) = \int_A h dm.$$

The measure ν_0 is σ-invariant and is equivalent to the reference measure m. As m is an exact measure, this implies that ν_0 is an exact invariant measure. Clearly, the density $h = \frac{d\nu_0}{dm}$ does not have positive lower bound.

A.4 Construction of an induced WGM

In this section we recall the induced map constructed in [41], that is considered in Section 2.4.

Let $M = S^1 \times [0,1]$ and let m denote the Lebesgue measure on M. We consider the map $f : M \to M$ defined by

$$f(\theta, x) = \left(g(\theta), f_{\alpha(\theta)}(x)\right),$$

where $g(\theta) = 4\theta \pmod 1$,

$$f_{\alpha(\theta)}(x) = \begin{cases} x\left(1 + 2^{\alpha(\theta)} x^{\alpha(\theta)}\right) & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} < x \leqslant 1, \end{cases}$$

and $\alpha : S^1 \to (0,1)$ is a C^1 map that has minimum $\alpha_{\min}$ and maximum $\alpha_{\max}$, with $\alpha_{\min} < \alpha_{\max}$.

Set $\Delta_0 = S^1 \times \left(\frac{1}{2}, 1\right] \subset M$. Now we want to build an induced map $f^R : \Delta_0 \to \Delta_0$ for f. For that we need to construct partition $\mathcal{P}_0$ of Δ_0 and define the return time function $R : \Delta_0 \to \mathbb{N}$.

For all $\theta \in S^1$, we define $X_0(\theta) = 1, X_1(\theta) = \frac{1}{2}$, and for $n \geqslant 2$,

$$X_n(\theta) = f_{\alpha(\theta)}^{-1}(X_{n-1}(g(\theta))) \in [0, \frac{1}{2}].$$

Consider $Y_n(\theta) = f_{\alpha(\theta)}^{-1}(X_{n-1}(g(\theta))) \in [\frac{1}{2}, 1]$ and $J_n(\theta) = [Y_{n+1}(\theta), Y_n(\theta)]$ for all $\theta \in S^1$. Consider $q > 0$ large enough to be chosen properly and

$$A_{s,n} = \left[s/4^{q+n}, (s+1)/4^{q+n}\right] \times J_n,$$

for $n \in \mathbb{N}$ and $0 \leqslant s \leqslant 4^{q+n} - 1$.

If I is an interval of S^1, then $I \times [Y_{n+1}, Y_n]$ stands for $\{(\theta, x) : \theta \in I, x \in [Y_{n+1}(\theta), Y_n(\theta)]\}$.

Consider $\mathcal{P}_0 = \{A_{s,n}\}$, and $R|_{A_{s,n}} = n$. Set

$$K_i = [i/4^q, (i+1)/4^q] \times [\frac{1}{2}, 1] \text{ for } 0 \leq i \leq 4^q - 1.$$

Clearly f^R maps each $A_{s,n}$ bijectively onto some K_i. Consequently, induced map $f^R : \Delta_0 \to \Delta_0$ satisfy the Markov condition $W1$) with respect to the partition $\mathcal{P}_0$, and it has the Long branch property $W5$). The next lemma provide the expansion with respect to the following metric.

For (θ_1, x_1) and $(\theta_2, x_2) \in S^1 \times [0,1]$, set

$$d'\left(\left(\theta_1, x_1\right), \left(\theta_2, x_2\right)\right) = a\left|x_1 - x_2\right| + \left|\theta_1 - \theta_2\right|$$

for some appropriate $a > 0$.

Lemma A.4.1. *[41, Proposition 2.6] On each $A_{s,n}$, the map f^R is expanding with respect to the distance d'.*

In particular, Lemma A.4.1 implies the Separability condition $W2$).

Lemma A.4.2. *[41, Proposition 2.9] There exists a constant C such that, for every $A_{s,n}$, for every (θ_1, x_1) and $(\theta_2, x_2) \in A_{s,n}$,*

$$\left|\frac{\det Df^R\left(\theta_1, x_1\right)}{\det Df^R\left(\theta_2, x_2\right)} - 1\right| \leqslant Cd\left(f^R\left(\theta_1, x_1\right), f^R\left(\theta_2, x_2\right)\right).$$

Lemma A.4.2 provide the distortion control, and this implies the Gibbs property $W4$). Thus we conclude that $f^R : \Delta_0 \to \Delta_0$ is an induced WGM map with respect to the countable partition $\mathcal{P}_0 = \{A_{s,n}\}$.

The following lemma is useful to prove $R \in L^1(m)$.

Lemma A.4.3. *[41, Proposition 2.2] There exists $C > 0$ such that for all $n \in \mathbb{N}$, for all $\theta \in S^1$,*

$$\frac{1}{Cn^{1/\alpha_{\min}}} \leqslant X_n(\theta) \leqslant \frac{C}{n^{1/\alpha_{\max}}}.$$

Therefore, we have

$$m(R > n) = m\left(S^1 \times [\frac{1}{2}, Y_{n+1}]\right) = \frac{1}{2} m\left(S^1 \times [0, X_n]\right) \leq \frac{1}{2}\frac{C}{n^{1/\alpha_{\max}}}.$$

Since $\alpha_{\max} < 1$, this implies that $\sum_{n \geq 0} m(R > n) < \infty$ and hence $R \in L^1(m)$. Moreover, $m(R > n) \leq Cn^{-a}$ with $a = 1/\alpha_{\max}$ and some $C > 0$.

Now we want to check that f^R is aperiodic and has a coprime block. Since $\left(f^R\right)^2 (A_{s,n}) = \Delta_0$ for each $A_{s,n} \in \mathcal{P}_0$, this implies that f^R is an aperiodic induced WGM.

Pick now $A_{0,1}, A_{0,2}, A_{0,3}, ..., A_{0,N} \in \mathcal{P}_0$ for $N \geq 2$. We have $R(A_{0,i}) = i$ and this implies that $\gcd\{R(A_{0,i})\}_i = 1$. By construction of partition elements $A_{0,i}$, we have

$$f^R(A_{0,i}) = [0/4^q, 1/4^q] \times [\frac{1}{2}, 1] \supseteq A_{0,i} \text{ for all } i.$$

Hence we conclude that f^R is an aperiodic induced WGM with $R \in L^1(m)$ and has a coprime block.

A.5 Ergodicity and exactness of the invariant measure for a WGM map

In this section, for the sake of the reader's convenience, we present the proofs of ergodicity and exactness of the measure ν_0 of Theorem 3.1.6. These were originally proved in [1, Theorem 2.8 and Theorem 3.2] respectively. In fact [1, Theorem 2.8 and Theorem 3.2] require a weaker condition than $W4$) of Definition2.1.1.

Proposition A.5.1. *If $F : \Delta_0 \to \Delta_0$ is an irreducible WGM map, then the F-invariant probability measure $\nu_0 \ll m$ is ergodic. Moreover, if F is aperiodic then ν_0 is exact.*

We first recall some definitions that will be useful for the proof of Proposition A.5.1. A set $W \subset \Delta_0$ is called a *wandering set* for F if the sets $\{F^{-k}(W)\}_{k=0}^{\infty}$ are disjoint. Let $W(F)$ denote the collection of measurable wandering sets. The map F is called *dissipative* if $D(F) = \Delta_0$ (m mod 0), where the *dissipative part* $D(F)$ is the union of elements $W(F)$.

The set $C(F) = \Delta_0 \backslash D(F)$ is called the *conservative part* of F. If $C(F) = \Delta_0$ (m mod 0), then F is called *conservative.*

Now we present some results that will be used in the proof of Proposition A.5.1. First we would like to translate the bounded distortion formula given in [1] to the setting considered here, i.e. in terms of the WGM map $F : \Delta_0 \to \Delta_0$.

From i) of Lemma 3.1.1, there exists some $K \geq 1$ such that for all $n \geq 1$ and $x, y \in \alpha \in \mathcal{P}_0^n$,

$$\frac{1}{K} \leq \frac{J_{F^n}(x)}{J_{F^n}(y)} \leq K. \tag{A.5.1}$$

Also we have for any $x \in \alpha$,

$$m(F^n(\alpha)) = \int_\alpha J_{F^n}(y)dm(y) = J_{F^n}(x) \int_\alpha \frac{J_{F^n}(y)}{J_{F^n}(x)} dm(y). \tag{A.5.2}$$

It follows from (A.5.1) and (A.5.2) that for any $x \in \alpha$,

$$\frac{1}{K} m(\alpha) J_{F^n}(x) \leq m(F^n(\alpha)) \leq K m(\alpha) J_{F^n}(x). \tag{A.5.3}$$

Take now $\beta \in \tilde{\mathcal{P}}_0 = \bigcup_{n \in \mathbb{N}} \mathcal{P}_0^n$ such that $\alpha' = \alpha \cap F^{-n}(\beta) \neq \emptyset$.

Similar to (A.5.3) we have

$$\frac{1}{K} m(\alpha') J_{F^n}(z) \leq m(F^n(\alpha')) \leq K m(\alpha') J_{F^n}(z),$$

which together with (A.5.3) and the fact that $F^n(\alpha') = \beta$, implies

$$\frac{1}{K^3} \leq \frac{m(F^n(\alpha)) m(\alpha')}{m(\beta) m(\alpha)} \leq K^3$$

or, equivalently,

$$\frac{1}{K^3} \leq \frac{m(F^{-n}(\beta)|\alpha)}{m(\beta|F^n(\alpha))} \leq K^3. \tag{A.5.4}$$

The inequality (A.5.4) was called the *bounded distortion formula* in [1].

Lemma A.5.2. *[1, Lemma 2.4] Let $F : \Delta_0 \to \Delta_0$ be a WGM map, and let Δ_c be the conservative part of F. If $A \in \tilde{\mathcal{P}}_0$, then*

$$\sum_{i=1}^{\infty} m(F^{-i}(A)) = \infty \implies A \subset \Delta_c \ (m \ mod \ 0)$$

and

$$\sum_{i=1}^{\infty} m(F^{-i}(A)) < \infty \implies A \subset \Delta_0 \backslash \Delta_c \ (m \ mod \ 0) \ .$$

In particular, the conservative part Δ_c is the union of the sets in $\tilde{\mathcal{P}}_0$.

Lemma A.5.3. *[1, Lemma 2.7] Let $F : \Delta_0 \to \Delta_0$ be an irreducible WGM map, If $\alpha \in \tilde{\mathcal{P}}_0$, then*

$$\forall A \in \mathcal{A}, \ m(A) > 0, \ \exists \ n \in \mathbb{N} \ s.t. \ m(\alpha \cap F^{-n}(A)) > 0.$$

Lemma A.5.4. *[1, Theorem 2.8] An irreducible WGM map $F : \Delta_0 \to \Delta_0$ is conservative.*

Proof. Since ν_0 is an F-invariant probability measure, this implies that there is no wandering subset of Δ_0 of positive ν_0 measure. So, the conservative part Δ_c of F has positive ν_0 measure. Since $\nu_0 \ll m$, then $m(\Delta_c) > 0$, and by Lemma A.5.2, Δ_c is the union of sets in $\tilde{\mathcal{P}}_0$.

Assume that $F : \Delta_0 \to \Delta_0$ is not conservative. Then there exists a wandering set $W \in \mathcal{A}$ such that $m(W) > 0$. By Lemma A.5.3, we have $m(\Delta_c \cap F^{-n}(W)) > 0$ for some $n \in \mathbb{N}$. The set $W_n = \Delta_c \cap F^{-n}(W)$ is also a wandering set for F. Since $W_n \subset \Delta_c \subset \Delta_0$, we have

$$\bigcup_{k=1}^{\infty} F^{-k}(W_n) \subset \Delta_0 \backslash W_n. \tag{A.5.5}$$

Since Δ_c is the conservative part of F and $W_n \subset \Delta_c$, then we have

$$\bigcup_{k=1}^{\infty} F^{-k}(W_n) \subset \Delta_c,$$

and this contradicts (A.5.5). Hence F is conservative. □

We will use the martingale convergence theorem (see e.g. [33]) in the proof of Proposition A.5.1, which ensures that, for $\varphi \in L^1(\Delta_0, m)$, $\mathbb{E}(\varphi|\mathcal{A}_n)$ converges almost surely and in $L^1(\Delta_0, m)$ to a function $\varphi : \Delta_0 \to \mathbb{R}$, where $\mathcal{A}_n$ is the σ-algebra generated by $\bigcup_{i=1}^{n} \mathcal{P}_0^i$. In particular, for almost every $x \in \Delta_0$,

$$\mathbb{E}(\mathbb{1}_A|\mathcal{A}_n)(x) \to \mathbb{1}_A(x) \quad \text{as} \quad n \to \infty,$$

for any measurable set $A \subset \Delta_0$.

The proof of Proposition A.5.1 is divided into Lemma A.5.5 and Lemma A.5.6.

Lemma A.5.5. *[1, Theorem 2.8] If $F : \Delta_0 \to \Delta_0$ is an irreducible WGM map, then the F-invariant probability measure $\nu_0 \ll m$ is ergodic.*

Proof. Take $A \in \mathcal{A}$ such that $F^{-1}(A) = A$ and $\nu_0(A) > 0$. Since $\nu_0 \ll m$, it is sufficient to prove that m is ergodic. For $n \in \mathbb{N}$, we can find $\alpha \in \mathcal{P}_0^n$ such that $\alpha \cap F^{-n}(A) \neq \emptyset$. By (A.5.4) we have

$$\frac{m(A|\alpha)}{m(A|F^n(\alpha))} = \frac{m(F^{-n}(A)|\alpha)}{m(A|F^n(\alpha))} \in [K^{-3}, K^3].$$

Since $\alpha \in \mathcal{P}_0^n$, then there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \mathcal{P}_0$ such that $\alpha = \bigcap_{i=1}^{n} F^{-i+1}(\alpha_i)$. We denote $\alpha = [\alpha_1, \alpha_2, ..., \alpha_n]$.

By the martingale convergence theorem, for m almost every $x \in \Delta_0$,

$$m(A|[\alpha_1(x), \alpha_2(x), ..., \alpha_n(x)]) \text{ conververges to } \mathbb{1}_A(x) \text{ as } n \to \infty,$$

where, for $n \geq 1$, $\alpha_n(x) \in \mathcal{P}_0$ is such that $F^{n-1}(x) \in \alpha_n$. By Lemma A.5.4, F is conservative, then for m almost every $x \in \alpha$, $F^t(x) \in \alpha$ for infinitely many t, and this means that $[\alpha_1(x), \alpha_2(x), ..., \alpha_n(x)] = [\alpha_{1+t}(x), \alpha_{2+t}(x), ..., \alpha_{n+t}(x)]$. Thus

$$\lim_{t\to\infty} \frac{m(A|[\alpha_{1+t}(x), \alpha_{2+t}(x), ..., \alpha_{n+t}(x)]}{m(A|F^n(\alpha))} = \frac{\mathbb{1}_A(x)}{m(A|F^n(\alpha))} \in [K^{-3}, K^3].$$

Therefore,

$$A = \bigcup_{\alpha \in \tilde{\mathcal{P}}_0, m(\alpha\cap A)>0} \alpha \ (m \bmod 0)\ .$$

Since $m(A) > 0$, then there exist at least one $\alpha \in \tilde{\mathcal{P}}_0$ such that $\alpha \subset A$ (m mod 0).

By irreducibility, for any $\beta \in \tilde{\mathcal{P}}_0$, there exist $k \geq 0$ such that $m(\alpha \cap F^{-k}(\beta)) > 0$, this implies that $\beta \subset A$ (m mod 0). Hence $A = \Delta_0$ (m mod 0), and m is ergodic. □

Lemma A.5.6. *[1, Theorem 3.2] If $F : \Delta_0 \to \Delta_0$ is an aperiodic WGM map, then the F-invariant probability measure $\nu_0 \ll m$ is exact.*

Proof. Take $A \in \bigcap_{n=1}^{\infty} F^{-n}(\mathcal{A})$ such that $\nu_0(A) > 0$, then

$$\forall n \geq 0\ \exists\ A_n \in \mathcal{A}\ s.t.\ A = F^{-n}(A_n).$$

It follows that $A_n = F^n(A)$, and

$$F^{-k}(A_{n+k}) = F^{-k}(F^{n+k}(A)) = F^n(A) = A_n$$

Since $\nu_0 \ll m$, it is sufficient to prove that m is exact. As $m(A) > 0$, then for every $k \geq 1$ there exists $\beta = [\beta_1, \beta_2, ..., \beta_k] \in \mathcal{P}_0^k$ (means that $\beta = \bigcap_{i=1}^{k} F^{-i+1}(\beta_i)$ for some $\beta_1, \beta_2, ..., \beta_k \in \mathcal{P}_0$) such that $m(A \cap \beta) > 0$.

Since by Lemma A.5.4 and Lemma A.5.5 (F, m) is conservative and ergodic, we can define for almost every $x \in \Delta_0$,

$$\tau(x) = \min\{n \in \mathbb{N} : F^n(x) \in \beta\}.$$

Define $F^* : \Delta_0 \to \beta$ by

$$F^*(x) = F^{\tau(x)}.$$

Note that $F^*|_\beta = F_\beta$, where $F_\beta : \beta \to \beta$ is an induced map defined as $F_\beta(x) = F^{q(x)}(x)$ with $q(x) = \min\{n \in \mathbb{N} : F^n(x) \in \beta\}$. Since (F, m) is conservative and ergodic, this implies that F_β is also conservative and ergodic. Hence the map (F^*, m) is conservative and ergodic on β.

For any integer $t \geq 0$, let

$$\tau_t(x) = \sum_{j=0}^{t-1} \tau((F^*)^j(x)).$$

Clearly, $(F^*)^t(x) = F^{\tau_t(x)}x$ and $\beta = [\beta_1, \beta_2, ..., \beta_k] = [\beta_{1+\tau_t(x)}, \beta_{2+\tau_t(x)}, ..., \beta_{k+\tau_t(x)}]$.

Now for every $x \in \Delta_0$ and $t \geq 1$, set

$$\beta' = [\beta_1(x), \beta_2(x), ..., \beta_{k+\tau_t(x)}(x)].$$

Since $m(A \cap \beta') > 0$, then $m(F^{k+\tau_t(x)}(A) \cap F^{k+\tau_t(x)}(\beta')) > 0$, this implies that $\beta' \cap F^{-(k+\tau_t(x))}(A_{k+\tau_t(x)}) \neq \emptyset$. By using (A.5.4) we have

$$\frac{m(A|\beta')}{m(A_{k+\tau_t(x)}|F^k(\beta))} = \frac{m(F^{-(k+\tau_t(x))}(A_{k+\tau_t(x)})|\beta')}{m(A_{k+\tau_t(x)}|F^{k+\tau_t(x)}(\beta'))} \in [K^{-3}, K^3]. \tag{A.5.6}$$

By martingale convergence theorem, for almost every $x \in A$,

$$m(A|[\beta_1(x), \beta_2(x), ..., \beta_{k+\tau_t(x)}(x)]) \to \mathbb{1}_A(x) \text{ as } t \to \infty.$$

We must have

$$\liminf_t m(A_{k+\tau_t(x)}|F^k(\beta)) > 0,$$

and, in particular,

$$\liminf_t m(A_{k+\tau_t(x)}) > 0.$$

Since $(F^*|_\beta, m)$ is conservative and ergodic on β, $(F^*)^{-1}(\beta) = \Delta_0$, and $m(A \cap \beta) > 0$, it follows that

$$\bigcup_{n=0}^{\infty} (F^*)^{-n}(A) = \Delta_0 \ (m \bmod 0) \ .$$

Assume $x \in \Delta_0$. We can find $\ell \in \mathbb{N}$ such that $(F^*)^\ell(x) \in A$, and

$$\liminf_t m(A_{k+\tau_t((F^*)^\ell(x))}) > 0.$$

Then

$$\exists\, \varepsilon > 0 \text{ s.t. } m(A_{k+\tau_t((F^*)^\ell(x))}) \geq \varepsilon \quad \forall \ \text{ t large} \ . \tag{A.5.7}$$

Recall that

$$A_{k+\tau_{t+\ell}(x)} = A_{k+\tau_t((F^*)^\ell(x))+\tau_\ell(x)} = F^{\tau_\ell(x)}(A_{k+\tau_t((F^*)^\ell(x))}). \tag{A.5.8}$$

We claim that

$$\exists\, \varepsilon_1 > 0 \text{ s.t. } B \in \mathcal{A},\ m(B) \geq \varepsilon \implies m(F^{\tau_\ell(x)}(B)) \geq \varepsilon_1. \tag{A.5.9}$$

First note that there exists

$$P_\varepsilon \subseteq \mathcal{P}_0^{\tau_\ell(x)-1} \text{ s.t. } |P_\varepsilon| < \infty \text{ and } m\left(\bigcup_{\omega \in P_\varepsilon} \omega\right) \geq 1 - \frac{\varepsilon}{2}.$$

Since from the definition of P_ε, if $m(B) \geq \varepsilon$ there exists $\omega \in P_\varepsilon$ such that $m(B \cap \omega) \geq \frac{\varepsilon}{2|P_\varepsilon|}$. As $F^{\tau_\ell(x)}$ is nonsingular on each $\omega \in P_\varepsilon$ and $m(B \cap \omega) \geq \frac{\varepsilon}{2|P_\varepsilon|}$, then there exists ε_1 such that $m(F^{\tau_\ell(x)}(B)) \geq \varepsilon_1$, so the claim is established.

From (A.5.7), (A.5.8) and (A.5.9), we get

$$\liminf_t m(A_{k+\tau_t(x)}) > 0 \ m \text{ almost every on } \Delta_0. \tag{A.5.10}$$

Now we want to prove that

$$\limsup_t m(A_{k+\tau_t(x)} \cap F^k(\beta)) > 0 \ m \text{ almost every on } \Delta_0.$$

From (A.5.10), for each $x \in \Delta_0$ with $m(A_{k+\tau_t(x)}) \geq \varepsilon > 0$ for some $\varepsilon > 0$ and large t. By the aperiodicity of F,

$$\exists\, r_0 \in \mathbb{N} \text{ s.t. } m(F^r(\beta_k)) \geq 1 - \frac{\varepsilon}{4},\ \forall\, r \geq r_0.$$

By the ergodicity of $F^*|_\beta$,

$$\exists\, r_1 \geq r_0,\ s \geq 1, \text{ s.t. } \sum_{t=1}^{\infty} \mathbb{1}_{\{\tau_s = r_1\}} \circ (F^*)^t = \infty \ m \text{ almost every on } \Delta_0.$$

Similarly, as before,

$$\exists\, P_{r_1,\varepsilon} \subset \mathcal{P}_0^{r_1-1} \cap F(\beta_k) \text{ s.t. } |P_{r_1,\varepsilon}| < \infty \text{ and } m\left(\bigcup_{\omega \in P_{r_1,\varepsilon}} F^{r_1}(\omega)\right) \geq 1 - \frac{\varepsilon}{2}.$$

Therefore,

$$\exists\, \omega \in P_{r_1,\varepsilon} \text{ s.t. } m(A_{k+\tau_t(x)} \cap F^{r_1}(\omega)) \geq \frac{\varepsilon}{2|P_{r_1,\varepsilon}|}, \text{ for large t.}$$

As before, there exists $\varepsilon_1 > 0$ such that

$$\text{if } B \in \mathcal{A} \text{ and } \omega \in P_{r_1,\varepsilon} \text{ are s.t. } m(B \cap F^{r_1}(\omega)) \geq \frac{\varepsilon}{2|P_{r_1,\varepsilon}|}, \text{ then } m(F^{-r_1}(B) \cap \omega) \geq \varepsilon_1,$$

and hence $m(F^{-r_1}(B) \cap F(\beta_k)) \geq \varepsilon_1$. This shows that

$$m(A_{k+\tau_t(x)-r_1} \cap F(\beta_k)) = m(F^{-r_1}(A_{k+\tau_t(x)}) \cap F(\beta_k)) \geq \varepsilon_1, \text{ for large t.}$$

Now, by choice of r_1, we have

$$\begin{aligned}\varepsilon_1 &\leq m(A_{k+\tau_{i+s}(x)-r_1} \cap F(\beta_k)) \\ &= m(A_{k+\tau_i(x)+\tau_s((F^*)^i x)-r_1} \cap F(\beta_k)) = m(A_{k+\tau_i(x)} \cap F(\beta_k)),\end{aligned}$$

for infinitely many i, and thus

$$\limsup_t m(A_{k+\tau_t(x)} \cap F^k(\beta)) \geq \varepsilon_1 \ m \text{ almost every on } \Delta_0. \tag{A.5.11}$$

From (A.5.6), (A.5.10) and (A.5.11), we deduce that

$$m(A|[\beta_1(x), \beta_2(x), ..., \beta_{k+\tau_t(x)}(x)]) \in [M'K^{-3}, M''K^3], \text{for some } M', M'' > 0,$$

for large t. Again by the martingale convergence theorem, we conclude that $A = \Delta_0$ (m mod 0). This shows that m is exact. $\square$