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GYROSTAT DYNAMICS ON A CIRCULAR ORBIT

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ABSTRACT

The attitude control of a modern satellite is a crucial condition for its operation. In this work, is investigated the dynamics of an asymmetrical inertial distribution gyrostat satellite, subjected to gravitational torque, moving along a circular orbit in a central Newtonian gravitational field.

To solve the problem it is proposed a symbolic-numerical method for determining all equilibrium orientations of an asymmetrical gyrostat satellite in the orbital coordinate system with a given gyrostatic torque and given principal central moments of inertia. The conditions of equilibria are obtained depending on four dimensionless parameters of the system.

The evolution of the domains in the study of equilibria and stability is carried out in great detail. All bifurcation values of parameters at which there is a change of numbers of equilibrium orientations are determined with great accuracy. For each equilibrium orientation the sufficient conditions of stability are obtained as a result of analysis of the generalized integral of energy used as a Lyapunov's function.

It is shown that the number of equilibria of a gyrostat satellite in the general case must be between 8 and 24, and that the number of stable equilibria changes from 4 to 2.

CONTENTS

1 INTRODUCTION

Attitude control of space bodies has been one of the most interesting and challenging problems in aerospace systems from the first half of the twentieth century to present time. A modern spacecraft is a multi-body system, composed of structure, hardware, antennas, mirrors, propulsion systems, etc., connected together by flexible rods or other kinds of attaching links, and do not feature simple geometric shapes or inertial distributions like the ones used in the early satellite designs.

Modern satellites are designed to use different kinds of torque to achieve and keep stabilization, for example, one side of the satellite pointing towards Earth if it has an antenna or camera. There are several ways to induce external torque to improve the equilibria and stability. The passive methods can use a gravitational field, magnetic field, aerodynamic drag, solar radiation pressure or the properties of rotating bodies, and the active methods which uses flywheels or thrusters for example. The passive methods are recognized to have an extended operating live than the active ones, due to the last ones carry out limited quantity of propellant or also due to the reliability of the spinning rotors.

It has long been recognized that introducing a source of angular momentum (for example, a constant speed symmetric rotor) into such a satellite can significantly improve its attitude stability properties. Special cases have been studied where the angular momentum is not aligned with a principal axis, and in such a configuration this causes another side of the satellite to face towards Earth when the satellite is in equilibrium relative to gravitational torques. This approach requires selecting the angular momentum vector or a set of angular momentum vectors, then finding some or all of the associated satellite equilibrium orientations and determining whether they are stable or unstable.

It was shown in by *Sarychev et al* that a panoramic view of all the possibilities involved can be obtained by adopting a different approach, and rise what is the set of satellite orientations which can be made into equilibrium orientations by proper choice of an internal angular momentum vector within the satellite, namely if this choice lie down in the special cases considered in [11], [12], [13] and [14]. Since the studied approach identifies all the possible equilibria, it is of considerable interest to perform an exhaustive study of stability to give the most complete point of view of the general case.

Regarding the subject in study, is carried out a detailed equilibria and stability of equilibria analysis for such gyrostat satellites families. The parameters involved are the gyrostat inertias and the parameters of angular momentum, including internal and orbital. The boundaries in which the equilibria parameters change are identified using the theory of bifurcation[1]. Regarding the stability analysis the Lyapunov's[2] stability scheme (using the Hamiltonian[3] as a Lyapunov's function) is the best way to identify this regions due to its simplicity.

There are two ways to focus on the gyrostat general case. The first way, also called the direct problem, which the principal moments of inertia and the projections of the

[1]Theory of Bifurcation is a mathematical approach widely used in the study of dynamical systems, in which small and smooth changes in the parameters values of a system cause a qualitative change in its behavior.
[2]Alexander Lyapunov (1857-1918) was a notable Russian mathematician and physics responsible for the development of the stability theory of dynamical systems and also several other important contributions to mathematics, physics and probabilities theory.
[3]Hamiltonian mechanics was developed by William Rowan Hamilton (1805-1865) as a reformulation of classical mechanics, which later was used in the formulation of quantum mechanics.

gyrostatic motion vector are given and there is the need to determine all nine directions cosines (or the three Euler-angles[4]) to find all the equilibrium positions of the gyrostat satellite. The second way or the inverse problem principal moments of inertia and three direction cosines are given, and there is the need to find the remaining six direction cosines that satisfies the equilibria conditions.

The inverse problem has already been solved in the general case by various methods (See Bibliography [1] to [4]). The direct problem only has been solved for special conditions were the total angular momentum of the rotors coincides with one of the axes or lies in one of the coordinate plane of the frame $Oxyz$ (See Bibliography [9] to [15]). The goal is then to achieve the unknown conditions of the direct method, it is expected to encounter equations with high level of complexity which will be responsible for countless hours of high level computation. If simple results are achieved, an enormous leap into the understanding of the gyrostat general case will bring to the satellite construction industry new an optimized solutions which will give birth to new and more optimized spacecraft's.

[4] [4] The Euler angles are three angles introduced by Leonhard Euler (1707 – 1783) to describe the orientation of a rigid body in a rotation scheme.

2 LITERATURE REVIEW

Since the beginning of the space exploration and the understanding of gyrostatic systems, scientists have tried to solve the general case of equilibria and stability of equilibria. But due to the complexity of the calculations involved the general case direct method has never been deeply explored. With the advance of the numerical computation the task has become easier, as the calculations for specific situations became more accessible, and applied to the construction of satellites and/or other space systems. Nevertheless, the full understanding of the general case boundaries, implications and consequences was not investigated due to the huge task and multiple cases to take into account. Scientific investigations were focused first into special cases, where it is much simpler to develop an analytical approach and achieve a set of rules for those specific cases.

Much of the important authors regarding gyrostat dynamics have been reviewed. The most relevant author for the accomplishment of this particular study was *Sarychev et al*. The study of all the particular cases [12] to [15] has led to the accomplishment of a more deep analysis into the general case of gyrostat satellite dynamics in a circular orbit subjected to a gravitational torque.

It has been shown by *Sarychev* in 1965 [20] and *Likins & Roberson* in 1966 [21] that a satellite in a central Newtonian force field in a circular orbit has no more than 24 equilibrium orientations with 4 of them stable. To the case in study was added inside the body a static and balanced rotating rotors with constant speed relativity to the body, this has led to new equilibria orientations which are very interesting for practical satellite applications, such solutions can be used in the current space technologies to design satellite active or semi-passive control systems.

Several authors have studied special cases for multiple equilibria of rotating rigid bodies and gyrostats using both analytical and numerical methods. *Sarychev et al* in all the studied cases from [12] to [15] has achieved the analytical equations for both equilibria and stability of the equilibria, namely in [12], which *Sarychev et al* studied the dynamics of a gyrostat satellite with a single nonzero component of vector of gyrostatic moment; the researchers used a set of parameterizations that led them to confirm that for a circular orbit a gyrostat satellite may have no more than 24 equilibrium positions, and that the equilibria in this particular case can be described as function of just a couple of parameterized factors. The study of the evolution of the regions of the necessary conditions of stability was described by a numerical-analytical method in the plane of the considered parameters. This work also described successfully the bifurcation values corresponding to the qualitative variation of the shape of the stability regions.

It was shown by *Sarychev and Mirer* in [13] the relative equilibria of a gyrostat satellite with internal angular moment along the principal axis. Again, the rather complicated system of equations was successfully reduced using a set of parameterized factors, it was shown that nine domains exist in the plane quadrant of the parameterized values, and that a fixed number of equilibria solutions exist in every possible domain. The conditions for stability of the equilibrium positions were also given for all the presented group of solutions and estimated for several inertial configurations.

In [14], *Sarychev et al* studied the dynamics of a gyrostat satellite with the vector of gyrostatic moment in the principal plane of inertia. Here all positions of equilibrium are determined, and the conditions of their existence are analyzed. Also determined were the bifurcation values of the parameterized dimensionless factors, at which the number of equilibrium position changes. Moreover, and as result of the analysis of the generalized integral of energy, the sufficient conditions of stability for each equilibrium orientation were derived. The evolution of the regions where the sufficient conditions of stability are valid were investigated under the variation of the systems parameterized dimensionless parameters.

In [15] *Sarychev and Gutnik* studied the relative equilibria for the general case of a gyrostat satellite. The approach used is very similar to the one in this work, for the calculation of equilibria the authors use Euler angles instead of dimensionless parameters as used on the current study. It was a first approach to the study of the gyrostat general problem, the authors introduced a method to parameterize the different coefficients in order to make the analysis easier. However, no results on equilibria, stability of equilibria and bifurcation are given.

3 GENERAL GYROSTAT CONCEPTS

3.1 DIRECTION COSINES

To describe a motion of a spacecraft with respect to the orbital reference system there is the need to introduce a proper coordinates system. The coordinate need to be carefully selected, as can be seen later, if proper chosen, the coordinate system will simplify the gyrostat both equilibria and stability calculations.

Euler observed in [24] that the generalized displacement of a rigid body with one fixed point is a rotation about an axis through that point. In the present context, Euler's angles theorem requires that the displacement of one point with respect to other be a rotation about some axis through their common origin, like figure 3.1 shows.
Subsequently can be said that Euler angles can represent a sequence of three elemental rotations, i.e. rotations about the axes of the coordinate system.

In mathematical terms, Euler angles are represented by rotation matrices, which are used to perform a rotation in the three dimensional space. Rotational matrices also provide a mean of numerically representing an arbitrary rotation of the axes about the origin without appealing to the angular specification.

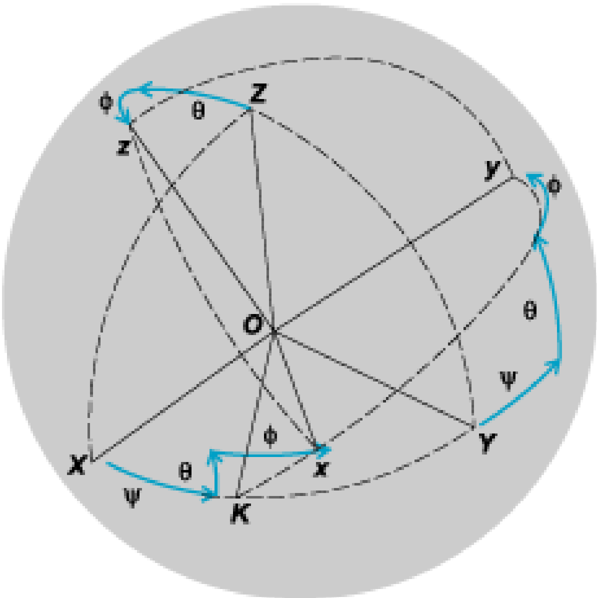

Figure 3.1 Representation of the Euler angles. Adapted from McGraw-Hill Concise Encyclopaedia of Physics.

Euler angles [24] can be achieved by several different set of rotations, the first rotation can occur in any of the three axes, covering three choices. Because two consecutive rotations cannot take place on the same axis, two alternatives are possible for the second rotation and two choices are again possible for the last rotation. With this we have 3x2x2 possibilities. The 12 Euler angles configurations corresponds to the sequences labeled as 1-2-1, 1-2-3, 1-3-1, 1-3-2, 2-1-2, 2-1-3, 2-3-1, 2-3-2, 3-1-2, 3-1-3, 3-2-1 and 3-2-3.

Peter C. Hughes in Spacecraft Attitude Dynamics [23] made a further analysis to the Euler rotation possibilities, is shown that can be divided into two different groups, there are 6 sets of symmetric rotations or classic Euler angles, 1-2-1, 1-3-1, 2-3-2, 2-1-2, 3-1-3 and 3-2-

3, and 6 sets of asymmetric rotations or also called Cardan or Tait-Bryan angles[5], 1-2-3, 1-3-2, 2-3-1, 2-1-3, 3-1-2 and 3-2-1.

After setting the proper rotation scheme is highly desirable to create the direction cosine matrix. This matrix represents the cosines of the angles between a certain vector and the three coordinate axes.

Further on the study of stability, is necessary to use the direction cosines into the energy scheme in the form of an expanded Taylor series[6]. This means that the directions cosines will then represent small displacements from the equilibrium position.

As seen previously, there are plenty of sequences to represent any kind of system, the option of which set of rotations belongs exclusively to the designer itself.

The system base change matrix corresponds to:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = [A] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{3.1}$$

Because [A] is the cosine direction matrix, system (3.1) becomes:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{3.2}$$

The angular orientation of the body-fixed base is thought to be the result of three successive rotations. Taking figure 3.1 as reference, the first rotations is carried out about the axis Oz through an angle $\boldsymbol{\psi}$, the second rotation is through the axis Ox and gives origin to the angle ϑ, and finally the third rotation is around the axis Oz and gives origin to the angle $\boldsymbol{\varphi}$.

Applying the matrix notation scheme, the first rotation is represented as:

$$A_\psi = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.3}$$

The second rotation is represented as:

$$A_\vartheta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{bmatrix} \tag{3.4}$$

And the third and final rotation is represented as:

$$A_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

[5] Named after Peter Guthrie Tait (1831-1901) and George H. Bryan (1864-1928) which differ from Euler angles by the replacement by non-homologous planes (perpendicular when angles are zero).

[6] Taylor series is a represents a function as an infinite sum of terms which are calculated from the values of the function's derivatives at a single point. This mathematical concept was discovered by the Scottish mathematician James Gregory (1638-1675).

Multiplying out the matrix's (3.3) by (3.4) by (3.5) gives the direction cosines (3.6).

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = A_{\psi} A_{\vartheta} A_{\varphi} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leftrightarrow$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\varphi - \sin\psi\cos\vartheta\sin\varphi & -\cos\psi\sin\varphi - \sin\psi\cos\vartheta\cos\varphi & \sin\psi\sin\vartheta \\ \sin\psi\cos\varphi + \cos\psi\cos\vartheta\sin\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\vartheta\cos\varphi & -\cos\psi\sin\vartheta \\ \sin\vartheta\sin\varphi & \sin\vartheta\cos\varphi & \cos\vartheta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (3.6)$$

Where the angle of precession can be defined as the change of orientation of the pivotal axis and analytically represented by ψ, the angle of nutation is defined as the nodding motion in or inclination in the axis of rotation and analytically represented by ϑ, and the angle of rotation can be defined as the rotation around its axis and analytically represented by φ.

Therefore the direction cosine expressions, also might be referred as the result of a 3-1-3 rotation is described as, and confirmed in [17] to [19]:

$$\begin{cases} a_{11} = \cos\psi\cos\varphi - \sin\psi\cos\vartheta\sin\varphi \\ a_{12} = -\cos\psi\sin\varphi - \sin\psi\cos\vartheta\cos\varphi \\ a_{13} = \sin\psi\sin\vartheta \\ a_{21} = \sin\psi\cos\varphi + \cos\psi\cos\vartheta\sin\varphi \\ a_{22} = -\sin\psi\sin\varphi + \cos\psi\cos\vartheta\cos\varphi \\ a_{23} = -\cos\psi\sin\vartheta \\ a_{31} = \sin\vartheta\sin\varphi \\ a_{32} = \sin\vartheta\cos\varphi \\ a_{33} = \cos\vartheta \end{cases} \quad (3.7)$$

It is often necessary to calculate the rotation angles, as can be seen in chapter 7, to obtain the stability results, the values of the angles are needed. Therefore, is necessary to achieve a simplified mathematical expression for (3.7) which covers all the angle expressions. It is the derived from (3.7) the expressions which permit us to achieve all the angles expressions.

$$\begin{cases} \cos\vartheta = a_{33} \\ \cos\psi = -a_{23} / \sin\vartheta \\ \cos\varphi = a_{32} / \sin\vartheta \\ \sin\vartheta = \sigma\sqrt{1-\cos^2\vartheta}, (\sigma = \pm 1) \\ \sin\psi = a_{13} / \sin\vartheta \\ \sin\varphi = a_{31} / \sin\vartheta \end{cases}$$

If $(\psi, \vartheta, \varphi)$ are the angles associated with $\sigma = +1$, then the angles associated with $\sigma = -1$ are $(\psi + \pi, -\vartheta, \varphi + \pi)$. Both triplets as will be seen further ahead will produce one and the same final result.

The choice of this kind of representation and specific rotation angles is because the stationary rotation of a spacecraft corresponds to a fixed position of its axis with respect to the orbital reference system defined by angles $\psi = \psi_0 = const.$ and $\vartheta = \vartheta_0 = const.$ and also from the uniform rotation of the spacecraft about this axis.

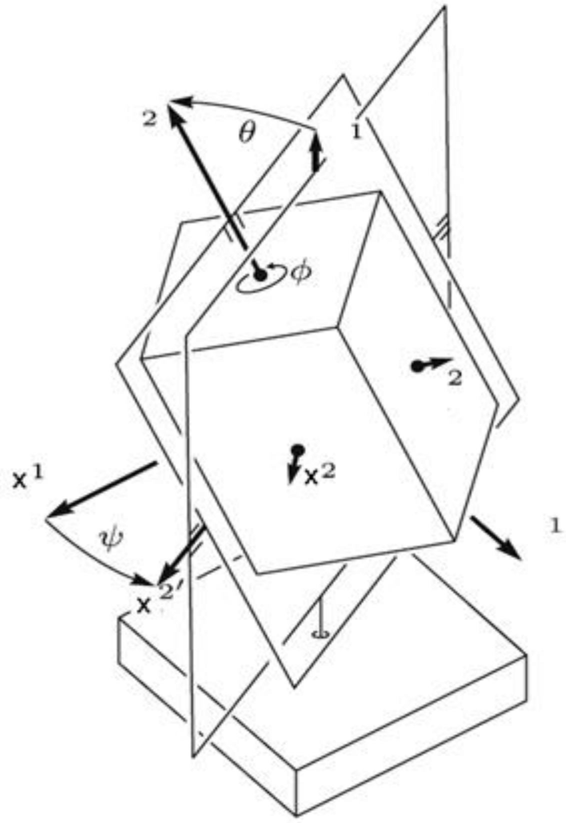

Figure 3.2: Representation of the Euler angles in a two-gimbal suspension.
* The upper coefficients represent rotation sequence.

Adapted from McGraw-Hill Concise Encyclopedia of Physics.

The matrix (3.6) and system of equations (3.7) are orthogonal matrixes, so they obey to the following properties:

$$[A]^{-1} = [A]^T$$

$$\det([A]) = 1$$

$$[A]^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Because $[A][A]^{-1} = [I]$ it follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.8}$$

Performing the calculations of system (3.8):

$$\begin{bmatrix} a_{11}^2+a_{12}^2+a_{13}^2 & a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} & a_{11}a_{31}+a_{12}a_{32}+a_{13}a_{33} \\ a_{21}a_{11}+a_{22}a_{12}+a_{23}a_{13} & a_{21}^2+a_{22}^2+a_{23}^2 & a_{21}a_{31}+a_{22}a_{32}+a_{23}a_{33} \\ a_{31}a_{11}+a_{32}a_{12}+a_{33}a_{13} & a_{31}a_{21}+a_{32}a_{22}+a_{33}a_{23} & a_{31}^2+a_{32}^2+a_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.9)$$

From system of equations (3.9), the following expressions are achieved:

$$\begin{cases} a_{11}^2+a_{12}^2+a_{13}^2=1 \\ a_{21}^2+a_{22}^2+a_{23}^2=1 \\ a_{31}^2+a_{32}^2+a_{33}^2=1 \\ a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23}=0 \\ a_{11}a_{31}+a_{12}a_{32}+a_{13}a_{33}=0 \\ a_{21}a_{31}+a_{22}a_{32}+a_{23}a_{33}=0 \end{cases} \quad (3.10)$$

Now the remaining orthogonal conditions can be found by:

$$[A]^{-1}=\frac{\mathrm{Adj}[A]}{\det[A]}$$

Where:

$$\det[A]=1$$

$$[A]^{-1}=[A]^{T}$$

Taking into account the algebra properties the remaining orthogonal conditions can be calculated having into account the following system:

$$Adj[A]=\begin{bmatrix} a_{22}a_{33}-a_{23}a_{32} & a_{13}a_{32}-a_{12}a_{33} & a_{12}a_{23}-a_{13}a_{22} \\ a_{23}a_{31}-a_{21}a_{33} & a_{11}a_{33}-a_{13}a_{31} & a_{13}a_{21}-a_{11}a_{23} \\ a_{21}a_{32}-a_{22}a_{31} & a_{12}a_{31}-a_{11}a_{32} & a_{11}a_{22}-a_{12}a_{21} \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Which lead to the following equations:

$$\begin{cases} a_{11}=a_{22}a_{33}-a_{23}a_{32} \\ a_{12}=a_{23}a_{31}-a_{21}a_{33} \\ a_{13}=a_{21}a_{32}-a_{22}a_{31} \\ a_{21}=a_{13}a_{32}-a_{12}a_{33} \\ a_{22}=a_{11}a_{33}-a_{13}a_{31} \\ a_{23}=a_{12}a_{31}-a_{11}a_{32} \\ a_{31}=a_{12}a_{23}-a_{13}a_{22} \\ a_{32}=a_{13}a_{21}-a_{11}a_{23} \\ a_{33}=a_{11}a_{22}-a_{12}a_{21} \end{cases} \quad (3.11)$$

Because a satellite fixed reference frame is undergoing a rotation, the elements of the rotation matrix will be in function of time. The functional relationships between the direction cosines and their rates are detailed by Jens Wittenburg in Dynamics of Multibody Systems [26] and are known as the kinematic equations of Poisson[7], which can be applied to the studied system in the following form:

$$\begin{cases} \dot{a}_{11} = a_{12}r - a_{13}q + \omega_3 a_{21} - \omega_2 a_{31} \\ \dot{a}_{12} = a_{13}p - a_{11}r + \omega_3 a_{22} - \omega_2 a_{32} \\ \dot{a}_{13} = a_{11}q - a_{12}p + \omega_3 a_{23} - \omega_2 a_{33} \\ \dot{a}_{21} = a_{22}r - a_{23}q + \omega_1 a_{31} - \omega_3 a_{11} \\ \dot{a}_{22} = a_{23}p - a_{21}r + \omega_1 a_{32} - \omega_3 a_{12} \\ \dot{a}_{23} = a_{21}q - a_{22}p + \omega_1 a_{33} - \omega_3 a_{13} \\ \dot{a}_{31} = a_{32}r - a_{33}q + \omega_2 a_{11} - \omega_1 a_{21} \\ \dot{a}_{32} = a_{33}p - a_{31}r + \omega_2 a_{12} - \omega_1 a_{22} \\ \dot{a}_{33} = a_{31}q - a_{32}p + \omega_2 a_{13} - \omega_1 a_{23} \end{cases} \tag{3.12}$$

Were *p*, *q* and r are the projections of the angular velocity into x, y and z axis and ω_1, ω_2 and ω_3 are the projections of the orbital angular velocity.

Because the satellite is small when compared with the size of its orbit can be considered that the angular velocity in relation to x and z axis be $\omega_1 = \omega_3 = 0$, and the angular velocity in the plane *y* be $\omega_2 = \omega$. Because the study is conducted in a circular orbit then $\omega_2 = \omega_0$. Taking this considerations, the system (3.12) take the following form:

$$\begin{cases} \dot{a}_{11} = a_{12}r - a_{13}q - \omega a_{31} \\ \dot{a}_{12} = a_{13}p - a_{11}r - \omega a_{32} \\ \dot{a}_{13} = a_{11}q - a_{12}p - \omega a_{33} \\ \dot{a}_{21} = a_{22}r - a_{23}q \\ \dot{a}_{22} = a_{23}p - a_{21}r \\ \dot{a}_{23} = a_{21}q - a_{22}p \\ \dot{a}_{31} = a_{32}r - a_{33}q + \omega a_{11} \\ \dot{a}_{32} = a_{33}p - a_{31}r + \omega a_{12} \\ \dot{a}_{33} = a_{31}q - a_{32}p + \omega a_{13} \end{cases} \tag{3.13}$$

The system of equations (3.13) is the kinematic equations of Poisson for the presented system.
The Poisson kinematic equations are used for solving the problem of determining the body position for a certain given angular velocity. Poisson kinematic equations possesses an advantage when compared with Euler kinematic equations, this ones are linear and present no singularities. The drawback is that the former equations have a higher dimensionality when compared with the Euler equations[8].

[7] Developed by the French mathematician Siméon Denis Poisson (1781-1840) and gives the functional relationships between the orientation angles and their rates of change.

[8] Developed by the Swiss mathematician Leonhard Euler (1707-1783) which in classical mechanics describe the rotation equations of a rigid body, using a rotating reference frame with its axes fixed to the body and parallel to the body's principal axes of inertia.

3.2 INERTIAL INEQUALITY TRIANGLE

Taking into consideration the general definition, the inertia matrix for a general body is symmetric and the terms along the diagonal are called inertia moments. The inertia matrix must respect the constraints among its elements, it is possible to define 3 triangular inequalities and 6 inertial distributions that relates the inertia moments, which represent constraints on the admissible values of inertia moments. These constraints are the sum of two inertia moments, as the following example can demonstrate.

$$I_{XX} = A = \iiint \left(y^2 + z^2\right) dm$$
$$I_{YY} = B = \iiint \left(x^2 + z^2\right) dm$$
$$I_{ZZ} = C = \iiint \left(x^2 + y^2\right) dm$$

Where $I_{XX}, I_{YY}, I_{ZZ}, A, B$ and C are the inertia moments of a given general shape.
Taking for example the calculus of I_{XX} :

$$I_{XX} = A = \iiint \left(y^2 + z^2\right) dm \; + \iiint \left(x^2 + z^2\right) dm \; - \iiint \left(x^2 + y^2\right) dm \geq 0 \;\Leftrightarrow$$

$$\Leftrightarrow \iiint \left(y^2 + z^2 + x^2 + z^2 - x^2 - y^2\right) dm \quad \geq 0 \Leftrightarrow$$
$$\Leftrightarrow 2\iiint z^2 \; dm \geq 0$$

The remaining inequalities are obtained by a simple rotation of the axis x, y and z, and doing so, the inertia moments configurations can be described by the following inequalities.

$$A + B \geq C$$
$$A + C \geq B$$
$$B + C \geq A$$

Taking into account the previous demonstrated three triangle inequality rules, it is possible to achieve the six inertial distributions. Each inertia distribution can be applied to actual design properties. The Inertia distributions which rise from the triangle inequality are:

$$A > B > C$$
$$A > C > B$$
$$B > A > C$$
$$B > C > A$$
$$C > A > B$$
$$C > B > A$$

It has been shown by *Sarychev et al* in [1] and [2] that exists all the inertial configurations above mentioned. The choice of such configuration is because this inertial distribution has a gravitic gradient possible stability scheme and also by the designer choice.

4 GENERAL GYROSTAT EQUATIONS OF MOTION

Let us consider the problem of rotational motion of a gyrostat satellite representing a solid body (also called platform) with rotors inside, which are balanced both statically and dynamically. We assume the angular velocity of rotation of these rotors to be constant relative to the satellite mainframe, while the center of mass of the satellite moves along a circular orbit in a homogeneous gravitational field.

We introduce two right-handed Cartesian reference frames with an origin at the satellite's center of mass O. $OXYZ$ is the orbital reference frame whose axis OZ is directed along the radius vector connecting the centers of mass of the satellite and the Earth; the OX axis is directed along the linear velocity vector of the center of mass O. $Oxyz$ is the satellite-fixed reference frame; where Ox, Oy and Oz are the satellite's principal axes of inertia.

Let us define orientation of the $Oxyz$ reference frame relative to the orbital reference frame by the angle $\boldsymbol{\psi}$ (angle of precession), ϑ (angle of nutation), $\boldsymbol{\varphi}$ (angle of rotation), (See figure 3.1 and figure 4.1).

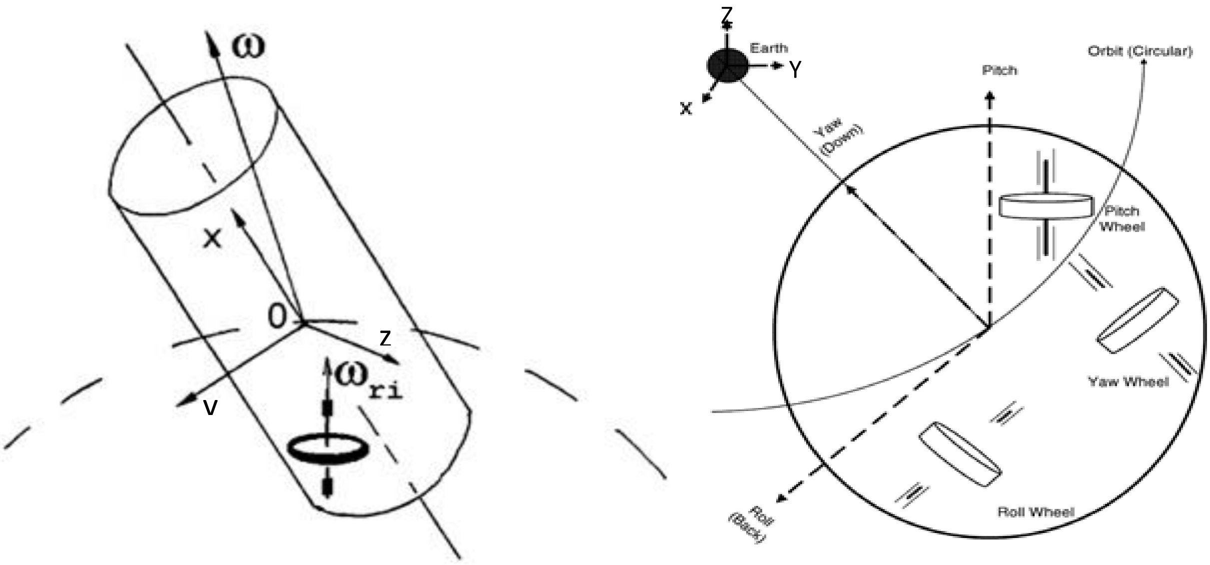

Figure 4.1: Spacecraft and Orbital Coordinates Representation.

Adapted from McGraw-Hill Concise Encyclopedia of Physics.

The installation of rotors (may also be called flywheels) will change the equilibria scheme of a solid body to positions more favorable for practical applications. The use of the gravitational-gradient torque effect in conjunction with flywheels will improve significantly the accuracy and orientation of such space application.
The total angular velocity of a gyrostat satellite $\vec{\omega}$ is described as the sum of the angular velocity components, and is defined by:

$$\vec{\omega} = p\vec{x} + q\vec{y} + r\vec{z} = \vec{\omega}_s + \vec{\omega}_0 \tag{4.1}$$

Were p, q and r are the projections of the angular velocity into x, y and z axis, $\vec{\omega}_s$ the internal gyrostat angular velocity and $\vec{\omega}_0$ the orbital angular velocity.

And by definition the gyrostat angular velocity mentioned in [23] and [26]:

$$\overrightarrow{\omega_S} = (\dot{\psi}a_{21} + \dot{\varphi})\vec{x} + (\dot{\psi}a_{22} + \dot{\vartheta}\sin\varphi)\vec{y} + (\dot{\psi}a_{23} + \dot{\vartheta}\cos\varphi)\vec{z} \tag{4.2}$$

And the orbital angular velocity is:

$$\overrightarrow{\omega_0} = \omega_0\vec{Y} \tag{4.3}$$

Writing $\overrightarrow{\omega_0}$ in matricidal format, (4.3) is presented as:

$$\begin{bmatrix}\omega_{0x}\\ \omega_{0y}\\ \omega_{0z}\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}0\\ \omega_0\\ 0\end{bmatrix} \tag{4.4}$$

Or:

$$\begin{bmatrix}\omega_{0x}\\ \omega_{0y}\\ \omega_{0z}\end{bmatrix} = \begin{bmatrix}\omega_0 a_{21}\\ \omega_0 a_{22}\\ \omega_0 a_{23}\end{bmatrix} \tag{4.5}$$

Re-writing back equation (4.3):

$$\overrightarrow{\omega_0} = (\omega_0 a_{21})\vec{x} + (\omega_0 a_{22})\vec{y} + (\omega_0 a_{23})\vec{z} \tag{4.6}$$

Replacing equations (4.2) and (4.6) into equation (4.1), can be obtained the several components of absolute angular velocity (rotational - around itself and translation – around a central attracting body) projections of the gyrostat satellite in the axis Ox, Oy and Oz:

$$\begin{aligned} p &= \dot{\psi}a_{31} + \dot{\vartheta}\cos\varphi + \omega_0 a_{21} = \overline{p} + \omega_0 a_{21}, \\ q &= \dot{\psi}a_{32} - \dot{\vartheta}\sin\varphi + \omega_0 a_{22} = \overline{q} + \omega_0 a_{22}, \\ r &= \dot{\psi}a_{33} + \dot{\varphi} + \omega_0 a_{23} = \overline{r} + \omega_0 a_{23}. \end{aligned} \tag{4.7}$$

Where the dot designates the derivative as function of time t.

$$\begin{cases} \overline{h}_1 = \sum_{k=1}^{n} J_k \hat{\psi}_k \dot{\varphi}_k \\ \overline{h}_2 = \sum_{k=1}^{n} J_k \hat{\theta}_k \dot{\varphi}_k \\ \overline{h}_3 = \sum_{k=1}^{n} J_k \hat{\varphi}_k \dot{\varphi}_k \end{cases} \tag{4.8}$$

Being $\overline{h}_1, \overline{h}_2$ and $\overline{h}_3$ the projection of the gyrostatic motion vector, J_k is the axial moment of inertia of *k*-th rotor; $\alpha_k, \beta_k, \gamma_k$ are the constant direction cosines of the symmetry axis of the *k*-th rotor in the coordinate system $Ox_1x_2x_3$ and $\dot{\varphi}_k$ is the constant angular velocity of the *k*-th rotor relative to the gyrostat.

4.1 KINETIC ENERGY

The kinetic energy of a body and/or system can be simply described as the energy which possesses due to its motion.
According to *Koenig's theorem*[9], the kinetic energy of a rigid body is equivalent to the sum of the kinetic energy of a mass point located at the center of mass and the kinetic energy relative to the center of mass.

The kinetic energy of a gyro-wheel with moment of inertia I_S, rotating with angular velocity ω_S is described as $\frac{1}{2} I_S \omega_S^2$. So as long as ω_S remains constant, the kinetic energy due to rotation is constant, whether there is precession or not. If a mass m is applied to one end of the rotation-axis of the gyroscope the rotation-axle will dip through an angle $\boldsymbol{\varphi}$ and a torque will be developed about the torque-axis.
It is known that a torque produce kinetic energy to a rotating body only when the torque produces angular velocity about the torque-axis, in this case the torque is responsible for producing the dip $\boldsymbol{\varphi}$; this physical model is the basic concept behind the restoring torque for stabilization of gyrostat platforms.

Indicating the coordinates of the satellite's center of mass in inertial frame as x, y and z, the mass of the satellite as m and using expression of rotational kinetic energy about the center of mass, the kinetic energy of satellite is described as:

$$T = \frac{1}{2}\left(Ap^2 + Bq^2 + Cr^2\right) + \frac{1}{2}m\left(\dot{\varphi}_{kx}^2 + \dot{\varphi}_{ky}^2 + \dot{\varphi}_{kz}^2\right)$$

Then the kinetic energy of a gyrostat satellite is defined by [15] as:

$$T = \frac{1}{2}\left(Ap^2 + Bq^2 + Cr^2\right) + \bar{h}_1 p + \bar{h}_2 q + \bar{h}_3 r \qquad (4.9)$$

Where the first term of equation (4.9) is due to the rotation around itself and translation around the central attracting body, and the second term is due to the rotation of the rotors inside the gyrostat platform.

Because the studied gyrostat is confined to a circular orbit, the projections of the gyrostatic motion vector can be manipulated in function of the orbital velocity as follows:

$$\begin{cases} h_1 = \dfrac{\bar{h}_1}{\omega_0} \\ h_2 = \dfrac{\bar{h}_2}{\omega_0} \\ h_3 = \dfrac{\bar{h}_3}{\omega_0} \end{cases}$$

[9] Developed by Johann Samuel König (1712-1757) the theorem expresses the kinetic energy of a system of particles in terms of the velocities of the individual particles. Specifically, it states that the kinetic energy of a system of particles is the sum of the kinetic energy associated to the movement of the center of mass and the kinetic energy associated to the movement of the particles relative to the center of mass.

Taking into account the system of equations (4.7), equation (4.9) can be simplified as follows:

$$T=\frac{1}{2}\left[A(\bar{p}+\omega_0 a_{21})^2+B(\bar{q}+\omega_0 a_{22})^2+C(\bar{r}+\omega_0 a_{23})^2\right]+\omega_0\left[h_1(\bar{p}+\omega_0 a_{21})+h_2(\bar{q}+\omega_0 a_{22})\right.$$
$$\left.+h_3(\bar{r}+\omega_0 a_{23})\right]\Leftrightarrow$$

$$T=\frac{1}{2}\left[A\left(\bar{p}^2+\omega_0^2 a_{21}^2+2\bar{p}\omega_0 a_{21}\right)+B\left(\bar{q}^2+\omega_0^2 a_{22}^2+2\bar{q}\omega_0 a_{22}\right)+C\left(\bar{r}^2+\omega_0^2 a_{23}^2+2\bar{r}\omega_0 a_{23}\right)\right]+$$
$$+\omega_0\left[h_1(\bar{p}+\omega_0 a_{21})+h_2(\bar{q}+\omega_0 a_{22})+h_3(\bar{r}+\omega_0 a_{23})\right]\Leftrightarrow$$

$$T=\frac{1}{2}\left(A\bar{p}^2+B\bar{q}^2+C\bar{r}^2\right)+\frac{1}{2}\omega_0^2\left(Aa_{21}^2+Ba_{22}^2+Ca_{23}^2\right)+\omega_0\left(A\bar{p}a_{21}+B\bar{q}a_{22}+C\bar{r}a_{23}\right)+$$
$$+\omega_0\left(h_1\bar{p}+h_2\bar{q}+h_3\bar{r}\right)+\omega_0^2\left(h_1a_{21}+h_2a_{22}+h_3a_{23}\right) \quad (4.10)$$

Equation (4.10) represents the total kinetic energy of the gyrostat satellite along its orbit around a central gravitating body, and is described in function of the principal's moments of inertia, the projections of the gyrostatic motion vector, the projections of the angular velocity and the direction cosines.

4.2 FORCE FUNCTION

The force function can be defined as the moment created around the center of gravity of the gyrostat due to the gravitational effects created on the gyrostat, and has the same magnitude that the potential energy but differs from it in sign, which can be quantified with (4.11).

$$U = \int_B dU \Leftrightarrow U = -GM \sum_{i=1}^{n} \int_P \int_R \frac{dm_p dm_R}{R_P} \tag{4.11}$$

With U being the Force Function, and considering a uniform dense spherical body where G is the universal gravitational constant 6.67259x10[24] Nm[2]/kg[2], M the mass of the central attracting body, m_p and m_R the mass of the platform and the mass of the rotors respectively and R_P the radius from the satellite center of mass to the central graviting body center of mass.
Because the gravitational forces acting upon the platform and the rotors are approximately the same, we can treat it as a unique body with mass m.

$$U = -GM \int_B \frac{dm}{R_P} \Leftrightarrow U = -k \int_B \frac{dm}{R_P} \Leftrightarrow U = -k \sum_i \frac{m_i}{R_P} \Leftrightarrow U = -k \sum_i \frac{m_i}{\sqrt{R_0^2 + 2R_0 r_i' + r_i'^2}} \Leftrightarrow$$

$$U = -km_i \frac{1}{\sqrt{R_0^2 + 2R_0 r_i' + r_i'^2}} \Leftrightarrow U = -km_i \frac{1}{R_0 \sqrt{1+\varepsilon}}$$

With $\varepsilon = \frac{2R_0 r_i' + r_i'^2}{R_0^2}$ and $k = GM$

So the equation $U = -k \sum_i \frac{m_i}{R_0} \Leftrightarrow U = -\frac{kM}{R_{CM}}$ represents the gravitational force function of a particle with the rigid body mass located at its centre of mass, and the term:

$$U = -k \sum_i \frac{m_i}{R_0} \left[-\frac{1}{2} \frac{2R_0 r_i' + r_i'^2}{R_0^2} \right] = -\frac{k}{R_0^3} R_0 \sum_i m_i r_i' + \frac{k}{2R_0^3} \sum_i m_i r_i'^2$$

The first sum on the right hand side gives the dipole term, which vanishes because the origin of the body fixed frame system is at the centre of mass of the rigid body.
The second term on the right hand side is from an order of R^{-3} and for consistency, it must be added it to the contribution from ε^2 term in the Taylor series expansion, as shown below.

$$U = -\frac{k}{2R_0^5} \sum_i m_i \frac{3}{8} (2R_0 r_i')^2 \Leftrightarrow U = -\frac{3}{2} \frac{k}{R_0^3} \sum_i m_i (\hat{r}_0 r_i')^2$$

To simplify the above equation, we need to find the centripetal forces which the gyrostat is subjected to, for that the Centripetal Force F_C and Gravity Forced F_G are equalled.

$$F_C = F_G \Leftrightarrow ma_c = G\frac{mM}{R^2} \Leftrightarrow m\frac{v^2}{R} = G\frac{mM}{R^2} \Leftrightarrow \frac{(\omega R)^2}{R} = G\frac{M}{R^2} \Leftrightarrow \omega^2 R = G\frac{M}{R^2} \Leftrightarrow$$

$$\Leftrightarrow \omega^2 = G\frac{M}{R^3}$$

Being $k = GM$ we can make $\omega^2 = \frac{k}{R^3}$ and so the equation $U = -\frac{3}{2}\frac{k}{R_0^3}\sum_i m_i(\hat{r}_0 r_i')^2$ can be simplified to:

$$U = -\frac{3}{2}\omega_0^2 \sum_i m_i(\hat{r}_0 r_i')^2$$

Applying our terminology it came:

$$U = -\frac{3}{2}\omega_0^2 \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}^T \Leftrightarrow$$

$$U = -\frac{3}{2}\omega_0^2\left(Aa_{31}^2 + Ba_{32}^2 + Ca_{33}^2\right) \tag{4.12}$$

Equation (4.12) represents the force function of the gyrostat satellite along its orbit around a central gravitating body in function of the orbital velocity, principal moments of inertia and direction cosines.

4.3 INTEGRAL OF ENERGY

The integral of energy or also called the Hamiltonian, is equal to the total energy of the system, kinetic plus potential (in our case, but can involve other forces such as magnetic or solar pressure for example).
The Hamiltonian mechanics is much closed linked with the Lagrangian Mechanics[10], the fact that the Hamiltonian is an important physical quantity, whereas the physical meaning of the Lagrangian is more "obscure", is one of the appealing features of the Hamiltonian approach. Both the Lagrangian and Hamiltonian have the dimensions of energy and both approaches can be called energy methods. They are characterized by the use of scalar quantities rather than the vectors encountered in the direct use of Newton's second law[11]. This has both the theoretical advantage of leading to very general formulations of mechanics and the practical benefit of avoiding some vector manipulations when changing between coordinate systems (in fact, Lagrangian and Hamiltonian methods were developed before modern vector notation was developed).
In the problem of attitude motion of a gyrostat satellite in a circular orbit, the Hamiltonian of our system is described by H and mathematically described as:

$$H = T_2 - T_0 - U = const. \tag{4.13}$$

The existence of T_0 is due to the fact that the orbital coordinate system $OXYZ$ is not inertial.

From (4.13) it follows that:

$$\begin{aligned} T_2 &= \frac{1}{2}\left(A\bar{p}^2 + B\bar{q}^2 + C\bar{r}^2\right) \\ T_0 &= \frac{1}{2}\omega_0^2\left(Aa_{21}^2 + Ba_{22}^2 + Ca_{23}^2\right) + \omega_0^2\left(h_1a_{21} + h_2a_{22} + h_3a_{23}\right) \end{aligned} \tag{4.14}$$

Now substituting (4.12) and (4.14) into equation (4.13) is obtained the integral of energy for the studied system.

$$H = \frac{1}{2}(A\bar{p}^2 + B\bar{q}^2 + C\bar{r}^2) - \frac{1}{2}\omega_0^2\left(Aa_{21}^2 + Ba_{22}^2 + Ca_{23}^2\right) - \omega_0^2\left(h_1a_{21} + h_2a_{22} + h_3a_{23}\right) +$$
$$+\frac{3}{2}\omega_0^2\left(Aa_{31}^2 + Ba_{32}^2 + Ca_{33}^2\right)$$

Having into account (3.10), the integral of energy becomes:

$$\begin{cases} a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{cases} \Leftrightarrow \begin{cases} a_{22}^2 = 1 - a_{21}^2 - a_{23}^2 \\ a_{33}^2 = 1 - a_{31}^2 - a_{31}^2 \end{cases}$$

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[10] Developed by the Italian-French mathematician Joseph-Louis Lagrange (1736-1813) the Lagrangian mechanics is a re-formulation of classical mechanics using the principle of stationary action (also called the principle of least action). Lagrangian mechanics applies to systems whether or not they conserve energy or momentum, and it provides conditions under which energy, momentum or both are conserved.
[11] The vector sum of the forces *F* on an object is equal to the mass *m* of that object multiplied by the acceleration vector *a* of the object.

$$H = \frac{1}{2}\left(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2\right) - \frac{1}{2}\omega_0^2\left[Aa_{21}^2 + B\left(1 - a_{21}^2 - a_{23}^2\right) + Ca_{23}^2\right] - \omega_0^2\left(h_1 a_{21} + h_2 a_{22} + h_3 a_{23}\right) +$$
$$+\frac{3}{2}\omega_0^2\left[Aa_{31}^2 + Ba_{32}^2 + C\left(1 - a_{31}^2 - a_{32}^2\right)\right] \Leftrightarrow$$
$$H = \frac{1}{2}\left(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2\right) - \frac{1}{2}\omega_0^2\left[a_{21}^2(A - B) + a_{23}^2(C - B) + B\right] - \omega_0^2\left(h_1 a_{21} + h_2 a_{22} + h_3 a_{23}\right) +$$
$$+\frac{3}{2}\omega_0^2\left[a_{31}^2(A - C) + a_{32}^2(B - C) + C\right] \Leftrightarrow$$

$$H = \frac{1}{2}\left(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2\right) - \frac{1}{2}\omega_0^2\left[a_{21}^2(A - B) + a_{23}^2(C - B)\right] - \omega_0^2\left(h_1 a_{21} + h_2 a_{22} + h_3 a_{23}\right) +$$
$$+\frac{3}{2}\omega_0^2\left[a_{31}^2(A - C) + a_{32}^2(B - C)\right] + \frac{1}{2}\omega_0^2 B + \frac{3}{2}\omega_0^2 C$$

Because of in our system we defined that $B > A > C$, the above integral of energy becomes:

$$\frac{1}{2}\left(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2\right) + \frac{1}{2}\omega_0^2\left[a_{21}^2(B - A) + a_{23}^2(B - C)\right] - \omega_0^2\left(h_1 a_{21} + h_2 a_{22} + h_3 a_{23}\right) +$$
$$+\frac{3}{2}\omega_0^2\left[a_{31}^2(A - C) + a_{32}^2(B - C)\right] = H + \frac{1}{2}\omega_0^2 B + \frac{3}{2}\omega_0^2 C$$

$$\frac{1}{2}\left(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2\right) + \frac{1}{2}\omega_0^2\left[a_{21}^2(B - A) + a_{23}^2(B - C)\right] - \omega_0^2\left(h_1 a_{21} + h_2 a_{22} + h_3 a_{23}\right) +$$
$$+\frac{3}{2}\omega_0^2\left[a_{31}^2(A - C) + a_{32}^2(B - C)\right] = H + \frac{1}{2}\omega_0^2 B + \frac{3}{2}\omega_0^2 C$$

$$\frac{1}{2}\left(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2\right) + \frac{1}{2}\omega_0^2\left[a_{21}^2(B - A) + a_{23}^2(B - C)\right] - \omega_0^2\left(h_1 a_{21} + h_2 a_{22} + h_3 a_{23}\right) +$$
$$+\frac{3}{2}\omega_0^2\left[a_{31}^2(A - C) + a_{32}^2(B - C)\right] = \overline{H} \tag{4.15}$$

Were $\overline{H} = H + \frac{1}{2}\omega_0^2 B + \frac{3}{2}\omega_0^2 C$

It can be easily seen that the integral of energy $\overline{H}$ (4.15) is constant during our entire path because all the parameters are constant in time, and because so, can be used as a Lyapunov's function in the basic Lyapunov's stability theorem.

5 GENERAL GYROSTAT EQUILIBRIUM ORIENTATIONS

The study of equilibria can be defined as the identification of the conditions under which the considered system is either at rest or in uniform motion. Each such equilibria position suggests a potential attitude stabilization scheme, especially if the equilibrium is stable. The term *general gyrostat* appears several times during this investigation, and it means that is an asymmetrical inertial distribution gyrostat satellite subjected to a gravitational torque.
To find the equilibrium orientations with respect to the orbital reference frame it is needed to find the solutions where the system is either at rest or in uniform motion, to find such results, the conditions must be developed from Lagrange general equation, or more precisely called the Lagrangian, which can be roughly called as a method of least energy of a system. Regarding the system in study, the Lagrangian is in function of Kinetic Energy, Force Function and system coordinates.

$$\frac{d}{dt}\left(\frac{\partial T}{d\dot{n}}\right)-\frac{\partial T}{\partial n}=\frac{\partial U}{\partial n}$$

Where n is the generalized coordinate. Now expanding the Lagrange equation into the studied coordinate scheme:

$$\begin{cases}\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial p}\right)-r\dfrac{\partial T}{\partial q}+q\dfrac{\partial T}{\partial r}=a_{33}\dfrac{\partial U}{\partial a_{32}}-a_{32}\dfrac{\partial U}{\partial a_{33}}\\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial q}\right)-p\dfrac{\partial T}{\partial r}+r\dfrac{\partial T}{\partial p}=a_{31}\dfrac{\partial U}{\partial a_{33}}-a_{33}\dfrac{\partial U}{\partial a_{31}}\\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial r}\right)-q\dfrac{\partial T}{\partial p}+p\dfrac{\partial T}{\partial q}=a_{32}\dfrac{\partial U}{\partial a_{31}}-a_{31}\dfrac{\partial U}{\partial a_{32}}\end{cases} \tag{5.1}$$

In the above set of equations T and U are the equations defined in (4.10) and (4.12).

The differential equations of (5.1) are:

$$\frac{\partial T}{\partial p}=Ap+\omega_0\bar{h}_1,\quad \frac{\partial T}{\partial q}=Bq+\omega_0\bar{h}_2,\quad \frac{\partial T}{\partial r}=Cr+\omega_0\bar{h}_3$$
$$\frac{\partial U}{\partial a_{31}}=-3\omega_0^2Aa_{31},\quad \frac{\partial U}{\partial a_{32}}=-3\omega_0^2Ba_{32},\quad \frac{\partial U}{\partial a_{33}}=-3\omega_0^2Ca_{33}$$

Replacing the derivations above calculated into the set of equations (5.1):

$$\begin{cases}\dfrac{d}{dt}(Ap)-rBq+qCr+3\omega_0^2a_{32}a_{33}(B-C)-\bar{h}_2r+\bar{h}_3q=0\\ \dfrac{d}{dt}(Bq)-pCr+rAp-3\omega_0^2a_{31}a_{33}(A-C)-\bar{h}_3p+\bar{h}_1r=0\\ \dfrac{d}{dt}(Cr)-qAp+pBq+3\omega_0^2a_{31}a_{32}(A-C)-3\omega_0^2a_{31}a_{32}(B-C)-\bar{h}_1q+\bar{h}_2p=0\end{cases} \Leftrightarrow$$

$$\begin{cases} A\dot{p} - rBq + qCr + 3\omega_0^2 a_{32}a_{33}(B-C) - \bar{h}_2 r + \bar{h}_3 q = 0 \\ B\dot{q} - pCr + rAp - 3\omega_0^2 a_{31}a_{33}(A-C) - \bar{h}_3 p + \bar{h}_1 r = 0 \\ C\dot{r} - qAp + pBq + 3\omega_0^2 a_{31}a_{32}(B-A) - \bar{h}_1 q + \bar{h}_2 p = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} A\dot{p} + (C-B)qr - 3\omega_0^2 a_{32}a_{33}(C-B) - (\bar{h}_2 r - \bar{h}_3 q) = 0 \\ B\dot{q} + (A-C)rp - 3\omega_0^2 a_{31}a_{33}(A-C) - (\bar{h}_3 p - \bar{h}_1 r) = 0 \\ C\dot{r} + (B-A)pq - 3\omega_0^2 a_{31}a_{32}(B-A) - (\bar{h}_1 q - \bar{h}_2 p) = 0 \end{cases} \quad (5.2)$$

The set of equations (5.2) represents the gyrostat angular velocity projected into the principal axis of inertia Ox, Oy and Oz.
Having into account $\psi = \psi_0 = const.$, $\vartheta = \vartheta_0 = const.$, $\varphi = \varphi_0 = const.$, $\dot{\psi} = \ddot{\psi} = \dot{\vartheta} = \ddot{\vartheta} = \dot{\varphi} = \ddot{\varphi} = 0$, $p = \omega_0 a_{21}$, $q = \omega_0 a_{22}$ and $r = \omega_0 a_{23}$ we get:

$$\begin{cases} -(B-C)\omega_0^2 a_{22}a_{23} + 3\omega_0^2(B-C)a_{32}a_{33} - \bar{h}_2\omega_0 a_{23} + \bar{h}_3\omega_0 a_{22} = 0 \\ -(C-A)\omega_0^2 a_{21}a_{23} + 3\omega_0^2(C-A)a_{31}a_{33} - \bar{h}_3\omega_0 a_{21} + \bar{h}_1\omega_0 a_{23} = 0 \\ -(A-B)\omega_0^2 a_{21}a_{22} + 3\omega_0^2(A-B)a_{31}a_{32} - \bar{h}_1\omega_0 a_{22} + \bar{h}_2\omega_0 a_{21} = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} (B-C)\omega_0^2(a_{22}a_{23} - 3a_{32}a_{33}) + \bar{h}_2\omega_0 a_{23} - \bar{h}_3\omega_0 a_{22} = 0 \\ (C-A)\omega_0^2(a_{21}a_{23} - 3a_{31}a_{33}) + \bar{h}_3\omega_0 a_{21} - \bar{h}_1\omega_0 a_{23} = 0 \\ (A-B)\omega_0^2(a_{21}a_{22} - 3a_{31}a_{32}) + \bar{h}_1\omega_0 a_{22} - \bar{h}_2\omega_0 a_{21} = 0 \end{cases}$$

Simplifying the above set of equations with $h_i = \dfrac{\bar{h}_i}{\omega_0}$ (i=1,2,3) it is obtained the stationary solutions of the integral of energy (4.15), which describes the equilibrium positions of the spacecraft with respect to the orbital reference system. Replacing then the $P = Q = R = 0$ and the constant angles $\psi = \psi_0$, $\vartheta = \vartheta_0$, $\varphi = \varphi_0$ in (4.15), the dynamical equations of the spacecraft in scalar form are presented as follows:

$$\begin{aligned} &(C-B)(a_{22}a_{23} - 3a_{32}a_{33}) - h_2 a_{23} + h_3 a_{22} = P = 0 \\ &(A-C)(a_{23}a_{21} - 3a_{33}a_{31}) - h_3 a_{21} + h_1 a_{23} = Q = 0 \\ &(B-A)(a_{21}a_{22} - 3a_{31}a_{32}) - h_1 a_{22} + h_2 a_{21} = R = 0 \end{aligned} \quad (5.3)$$

The direction cosines presented in (3.7) can now be introduced into the studied system, with this, can now be defined the spacecraft angular positions with respect to the orbital reference system as follows:

	x **P=0**	**y** **Q=0**	**z** **R=0**
X	a_{11}	a_{12}	a_{13}
Y	a_{21}	a_{22}	a_{23}
Z	a_{31}	a_{32}	a_{33}

(5.4)

The projection into the orbital coordinate frame between X and x is a_{11}, between X and y is a_{12}, and so on.

Projecting equations (5.3) into the axis's of the orbital coordinate system:

$$OX : Pa_{11} + Qa_{12} + Ra_{13} = 0$$
$$OY : Pa_{21} + Qa_{22} + Ra_{23} = 0$$
$$OZ : Pa_{31} + Qa_{32} + Ra_{33} = 0$$

The above system is equivalent to the system (5.3) and can be transformed:

$$\begin{cases} 4(Aa_{21}a_{31} + Ba_{22}a_{32} + Ca_{23}a_{33}) + (h_1a_{31} + h_2a_{32} + h_3a_{33}) = 0 \\ Aa_{11}a_{31} + Ba_{12}a_{32} + Ca_{13}a_{33} = 0 \\ Aa_{11}a_{21} + Ba_{12}a_{22} + Ca_{13}a_{23} + (h_1a_{11} + h_2a_{12} + h_3a_{13}) = 0 \end{cases} \qquad (5.5)$$

It can be seen that equations (5.5) depend on 6 dimensional parameters (h_1, h_2, h_3, A, B and C). The goal is now to transform system of equations (5.5) into dimensionless parameters to reduce the number of parameters in order to simplify the calculations.

Introducing (3.10) into (5.5):

$$\begin{cases} 4(Aa_{21}a_{31} + B(-a_{21}a_{31} - a_{23}a_{33}) + Ca_{23}a_{33}) + (h_1a_{31} + h_2a_{32} + h_3a_{33}) = 0 \\ Aa_{11}a_{31} + B(-a_{11}a_{31} - a_{13}a_{33}) + Ca_{13}a_{33} = 0 \\ Aa_{11}a_{21} + B(-a_{11}a_{21} - a_{13}a_{23}) + Ca_{13}a_{23} + (h_1a_{11} + h_2a_{12} + h_3a_{13}) = 0 \end{cases} \qquad (5.6)$$

And simplifying by means of $\nu = \dfrac{(B-A)}{(B-C)}$ and $H_i = \dfrac{h_i}{(B-C)}$, system (5.6) is transformed into:

$$\begin{cases} -4(\nu a_{21}a_{31} + a_{23}a_{33}) + (H_1a_{31} + H_2a_{32} + H_3a_{33}) = 0 \\ \nu a_{11}a_{31} + a_{13}a_{33} = 0 \\ [\nu a_{11}a_{21} + a_{13}a_{23}] - (H_1a_{11} + H_2a_{12} + H_3a_{13}) = 0 \end{cases} \qquad (5.7)$$

The system of equations (5.6) and (5.7) corresponds to the studied system represented in the referential $OXYZ$. It can be seen now in (5.7) that the gyrostat satellite equilibria in the orbital coordinate system depends only of 4 dimensional parameters H_1, H_2, H_3 and ν.

Now, is convenient to make a small remark, remembering chapter 3.2 – Inertial inequality triangle, we have made a choice that our inertial parameters are $B > A > C$, having this into account, it can be seen that for the parameter $\nu \approx 0$ when $B \approx A$ and $\nu \approx 1$ when $A \approx C$, following this can be assumed that $0 < \nu < 1$.

Taking (5.4) into account, the system (5.6) and (5.7) is considered a system of three equations with unknowns ψ_0, ϑ_0 and φ_0. In order to solve it, is necessary to add the six orthogonally conditions, which then is used to study the equilibrium orientations of the gyrostat satellite system.

From equation 2 on (5.5), equation 1 and 5 in (3.10) the following system of equations is achieved:

$$\begin{cases} a_{11} = \dfrac{a_{32}a_{33}(C-B)}{\left[a_{32}^2a_{33}^2(C-B)^2 + a_{31}^2a_{33}^2(A-C)^2 + a_{31}^2a_{32}^2(B-A)^2\right]^{1/2}} \\ a_{12} = \dfrac{a_{31}a_{33}(A-C)}{\left[a_{32}^2a_{33}^2(C-B)^2 + a_{31}^2a_{33}^2(A-C)^2 + a_{31}^2a_{32}^2(B-A)^2\right]^{1/2}} \\ a_{13} = \dfrac{a_{31}a_{32}(B-A)}{\left[a_{32}^2a_{33}^2(C-B)^2 + a_{31}^2a_{33}^2(A-C)^2 + a_{31}^2a_{32}^2(B-A)^2\right]^{1/2}} \end{cases} \tag{5.8}$$

Introducing $Z = \left[a_{32}^2a_{33}^2(C-B)^2 + a_{31}^2a_{33}^2(A-C)^2 + a_{31}^2a_{32}^2(B-A)^2\right]^{1/2}$:

$$\begin{cases} a_{11} = \dfrac{a_{32}a_{33}(C-B)}{Z} \\ a_{12} = \dfrac{a_{31}a_{33}(A-C)}{Z} \\ a_{13} = \dfrac{a_{31}a_{32}(B-A)}{Z} \end{cases} \tag{5.9}$$

Going back to the determinant solutions (5.4) (which corresponds to equations 3.11), and introducing $I_3 = Aa_{31}^2 + Ba_{32}^2 + Ca_{33}^2$ to facilitate the developing of the calculations, the remaining direction cosines take the following form:

$$\begin{cases} a_{21} = a_{31}\dfrac{(I_3 - A)}{Z} \\ a_{22} = a_{32}\dfrac{(I_3 - B)}{Z} \\ a_{23} = a_{33}\dfrac{(I_3 - C)}{Z} \end{cases} \tag{5.10}$$

Now with the complete set of equations, is introduced $Aa_{21}a_{31} + Ba_{22}a_{32} + Ca_{23}a_{33} = Z$ and $F = h_1a_{31} + h_2a_{32} + h_3a_{33}$, by the first equation of system (5.7) we have: $Z = \dfrac{F}{4}$, doing so, the set of direction cosines can be presented as:

$$\begin{cases} a_{11} = 4\dfrac{a_{32}a_{33}(C-B)}{F} \\ a_{12} = 4\dfrac{a_{31}a_{33}(A-C)}{F} \\ a_{13} = 4\dfrac{a_{31}a_{32}(B-A)}{F} \\ a_{21} = 4a_{31}\dfrac{(I_3-A)}{F} \\ a_{22} = 4a_{32}\dfrac{(I_3-B)}{F} \\ a_{23} = 4a_{33}\dfrac{(I_3-C)}{F} \end{cases} \tag{5.11}$$

As it was shown in *Sarychev and Gutnik* (1984) [15], a system with the second equation of (5.7) and the first, second, fourth, fifth and sixth equations in (3.10) can be solved for $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ if $A \neq B \neq C$, using dimensionless parameters H_1, H_2, H_3, υ.

Let us notice also that solutions of system (5.11) exist only when a_{31}, a_{32} and a_{33} with none two of them could vanish simultaneously, otherwise we get some special cases[12]. These special cases have been solved when the vector of gyrostatic momentum is located along the satellite's principal central axis of inertia Ox_2 and when $h_1 = 0, h_2 \neq 0, h_3 = 0$ in *Sarychev and Mirer* (2001) [13] and *Sarychev et al.* (2005) [12], *Longman et al.* (1981) [22], and when the vector of gyrostatic moment is located in the satellite's principal central plane of inertia Ox_1x_3 of the frame $Ox_1x_2x_3$ and $h_1 \neq 0, h_2 = 0, h_3 \neq 0$, *Sarychev et al.* (2008) [14].

Replacing equations (5.11) in the first and third equations of (5.5) and adding the third equation of (3.10) it gives origin to the following three equations:

$$\begin{cases} 16\left[a_{32}^2a_{33}^2(B-C)^2 + a_{31}^2a_{33}^2(C-A)^2 + a_{31}^2a_{32}^2(A-B)^2\right] = (h_1a_{31} + h_2a_{32} + h_3a_{33})^2 \\ 4(B-C)(C-A)(A-B)a_{31}a_{32}a_{33} + \\ +\left[h_1(B-C)a_{32}a_{33} + h_2(C-A)a_{31}a_{33} + h_3(A-B)a_{31}a_{32}\right](h_1a_{31} + h_2a_{32} + h_3a_{33}) = 0 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{cases} \tag{5.12}$$

To determine the remaining direction cosines a_{31}, a_{32} and a_{33}, if system (5.12) is solved, then relations (5.11) allow us to find the other six director cosines.

12 See bibliography [12], [13] and [14].

Following the simplification process, multiplying the right side of the first equation by $a_{31}^2 + a_{32}^2 + a_{33}^2 = 1$.

$$\begin{cases} 16\left[a_{32}^2 a_{33}^2 (B-C)^2 + a_{31}^2 a_{33}^2 (C-A)^2 + a_{31}^2 a_{32}^2 (A-B)^2\right] = (h_1 a_{31} + h_2 a_{32} + h_3 a_{33})^2 \left(a_{31}^2 + a_{32}^2 + a_{33}^2\right) \\ 4(B-C)(C-A)(A-B) a_{31} a_{32} a_{33} + \\ + \left[h_1 (B-C) a_{32} a_{33} + h_2 (C-A) a_{31} a_{33} + h_3 (A-B) a_{31} a_{32}\right](h_1 a_{31} + h_2 a_{32} + h_3 a_{33}) = 0 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{cases}$$

And introducing the dimensionless parameters:

$x = \frac{a_{31}}{a_{33}},\ y = \frac{a_{32}}{a_{33}},\ \nu = \frac{B-A}{B-C},\ H_i = \frac{h_i}{B-C},\ (i = 1, 2, 3)$, and divide the first equation of the above set of equations by $(B-C)^2 a_{33}^4$, and the second equation by $(B-C)^3 a_{33}^3$.

With this, we get the following system:

$$\begin{cases} 16\left[y^2 + x^2(\nu-1)^2 + \nu^2 x^2 y^2\right] = (H_1 x + H_2 y + H_3)^2 \left(x^2 + y^2 + 1\right) \\ 4\nu(1-\nu)xy + \left[H_1 y + H_2(\nu-1)x - H_3 \nu xy\right](H_1 x + H_2 y + H_3) = 0 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{cases} \tag{5.13}$$

It can be seen that:

$$\begin{cases} x = \dfrac{a_{31}}{a_{33}} \\ y = \dfrac{a_{32}}{a_{33}} \end{cases} \Leftrightarrow \begin{cases} a_{31} = x a_{33} \\ a_{32} = y a_{33} \end{cases} \tag{5.14}$$

Reminding that $F = h_1 a_{31} + h_2 a_{32} + h_3 a_{33}$, $I_3 = A a_{31}^2 + B a_{32}^2 + C a_{33}^2$, $\nu = \frac{B-A}{B-C}$ and $H_i = \frac{h_i}{B-C}, (i = 1, 2, 3)$, equations (5.11) can be manipulated as shown below.

$$\begin{cases} a_{11} = \dfrac{-4a_{32}a_{33}}{H_1a_{31}+H_2a_{32}+H_3a_{33}} \\ a_{12} = \dfrac{4a_{31}a_{33}(1-\nu)}{H_1a_{31}+H_2a_{32}+H_3a_{33}} \\ a_{13} = \dfrac{4\nu a_{31}a_{32}}{H_1a_{31}+H_2a_{32}+H_3a_{33}} \\ a_{21} = \dfrac{4a_{31}\left[\nu a_{32}^2 - a_{33}^2(1-\nu)\right]}{H_1a_{31}+H_2a_{32}+H_3a_{33}} \\ a_{22} = \dfrac{-4a_{32}\left[\nu a_{31}^2 + a_{33}^2\right]}{H_1a_{31}+H_2a_{32}+H_3a_{33}} \\ a_{23} = \dfrac{4a_{33}\left[a_{31}^2(1-\nu)+a_{32}^2\right]}{H_1a_{31}+H_2a_{32}+H_3a_{33}} \end{cases} \tag{5.15}$$

Replacing equations (5.14) into the third equation of system (5.13):

$$\begin{cases} 16\left[y^2 + x^2(\nu-1)^2 + \nu^2x^2y^2\right] = (H_1x+H_2y+H_3)^2(x^2+y^2+1) \\ 4\nu(1-\nu)xy + \left[H_1y + H_2(\nu-1)x - H_3\nu xy\right](H_1x+H_2y+H_3) = 0 \\ x^2a_{33}^2 + y^2a_{33}^2 + a_{33}^2 = 1 \end{cases} \Leftrightarrow$$

$$\begin{cases} 16\left[y^2 + x^2(\nu-1)^2 + \nu^2x^2y^2\right] = (H_1x+H_2y+H_3)^2(x^2+y^2+1) \\ 4\nu(1-\nu)xy + \left[H_1y + H_2(\nu-1)x - H_3\nu xy\right](H_1x+H_2y+H_3) = 0 \\ a_{33}^2 = \dfrac{1}{x^2+y^2+1} \end{cases} \tag{5.16}$$

Converging our attention only in the first two equations of system of equations (5.16), and expand them in function of x and y, (5.16) can be presented as follows:

$$\begin{cases} y^2\left[H_2(H_1 - H_3\nu x)\right] + y\left\{H_1H_3 + \left[(1-\nu)(4\nu - H_2^2) + H_1^2 - H_3^2\nu\right]x\right\} + \left[-(1-\nu)H_2x(H_1x+H_3)\right] = 0 \\ y^4\left\{H_2^2\right\} + y^3\left\{2H_2(H_1x+H_3)\right\} + y^2\left\{(H_2^2+H_3^2-16) + 2H_1H_3x + (H_1^2+H_2^2-16\nu^2)x^2\right\} + \\ + y\left\{2H_2(H_1x+H_3)(1+x^2)\right\} + \left\{(H_1x+H_3)^2(1+x^2) - 16(1-\nu)^2x^2\right\} = 0 \end{cases} \tag{5.17}$$

Equations (5.17) will be the re-ordered to easily obtain the y coefficients as:

$$a_0 y^2 + a_1 y + a_2 = 0,$$
$$b_0 y^4 + b_1 y^3 + b_2 y^2 + b_3 y + b_4 = 0, \quad (5.18)$$

The first equation of (5.18) corresponds to the second equation of (5.16) and the second equation of (5.18) corresponds to the first equation of (5.16).

Where the coefficients are:

$$\begin{aligned}
a_0 &= H_2(H_1 - \nu H_3 x),\\
a_1 &= H_1H_3 + [4\nu(1-\nu) + H_1^2 - (1-\nu)H_2^2 - \nu H_3^2]x - \nu H_1 H_3 x^2,\\
a_2 &= -(1-\nu)H_2(H_1x + H_3)x;\\
b_0 &= H_2^2,\\
b_1 &= 2H_2(H_1x + H_3),\\
b_2 &= (H_2^2 + H_3^2 - 16) + 2H_1H_3x + (H_1^2 + H_2^2 - 16\nu^2)x^2,\\
b_3 &= 2H_2(H_1x + H_3)(1 + x^2),\\
b_4 &= (H_1x + H_3)^2(1 + x^2) - 16(1-\nu)^2x^2.
\end{aligned} \quad (5.19)$$

The objective now is to apply the Sylvester matrix theorem[13] together with the resultant determinant theory to system of equations (5.18), this is done in order to eliminate the variable y from the system of equations.

$$R(x) = \begin{vmatrix} a_0 & a_1 & a_2 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & b_4 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 \end{vmatrix} = 0 \quad (5.20)$$

The resultant of (5.20) in is described in the following form:

$$p_0x^{12} + p_1x^{11} + p_2x^{10} + p_3x^9 + p_4x^8 + p_5x^7 + p_6x^6 + p_7x^5 + p_8x^4 + p_9x^3 + p_{10}x^2 + p_{11}x + p_{12} = 0 \quad (5.21)$$

The known studies of the gyrostat general case applying the direct method end up in the achievement of equation (5.21), see *Sarychev and Gutnik* in [15]. From now on all results should be carefully interpreted.

[13] Named after James Joseph Sylvester (1814-1897) the theorem provides the necessary and sufficient conditions to solve multi-polynomial by constructing an single variant expression such that all the roots of the original multi-polynomial equation are represented in the roots of the single variant expression.

When calculating the resultant of (5.20), and simplified by a common factor of $16H_2^2$, the following coefficients arise:

$$p_0 = -H_1^4 H_3^4 \nu^6$$
$$p_1 = 2H_1^3 H_3^3 \nu^5 \left[2H_1^2 - H_2^2(\nu-1) - 2\nu\left(H_3^2 + 2\nu - 2\right)\right]$$

$$p_2 = -H_1^2 H_3^2 \nu^4 \left\{6H_1^4 + H_2^4(\nu-1)^2 - H_2^2(\nu-1)\left[16\left(\nu^3-\nu^2\right)+(\nu-1)+H_3^2(1-7\nu)\right]+\right.$$
$$+H_1^2\left[\left(-25\nu^2+26\nu-1\right)+H_3^2\left(\nu^2-16\nu+1\right)+H_2^2\left(\nu^2-8\nu+7\right)\right]+2\nu^2\left[3H_3^4+8(\nu-1)^2-\right.$$
$$\left.\left.-4H_3^2\left(2\nu^2-7\nu+5\right)\right]\right\}$$

$$p_3 = 2H_1 H_3 \nu^3 \left\{2H_1^6 + H_1^4\left[\left(-13\nu^2+14\nu-1\right)+2H_3^2\left(\nu^2-6\nu+1\right)+H_2^2\left(\nu^2-5\nu+4\right)\right]+\right.$$
$$+H_3^2\left[-H_2^4(\nu-1)^2(2\nu-1)+H_2^2(\nu-1)\nu\left[H_3^2(1-4\nu)+\left(16\nu^3-16\nu^2+\nu-1\right)\right]+2\nu^3\left[-H_3^4+\right.\right.$$
$$\left.\left.+8(\nu-1)^2(4\nu-5)+2H_3^2\left(7-11\nu+4\nu^2\right)\right]\right]-H_1^2\left[H_2^4(\nu-2)(\nu-1)^2+H_2^2(\nu-1)\left[\left(16\nu^3-16\nu^2\right.\right.\right.$$
$$\left.\nu-1\right)+H_3^2\left(3\nu^2-13\nu+3\right)\Big]+2\nu\left[-2(\nu-1)^2(5\nu-1)+H_3^4\left(\nu^2-6\nu+1\right)+H_3^2\left(18\nu^3-53\nu^2+\right.\right.$$
$$\left.\left.\left.\left.+38\nu-3\right)\right]\right]\right\}$$

$$p_4 = -\nu^2\left\{H_1^8 + H_1^6\left[-1+10\nu-9\nu^2+H_2^2\left(3-4\nu+\nu^2\right)+2H_3^2\left(3-8\nu+3\nu^2\right)\right]+H_3^2\left[H_2^6(\nu-1)^4+\right.\right.$$
$$+\left\{H_3^2-16(\nu-1)^2\right\}\nu^4\left(-4+H_3^2+4\nu\right)^2+H_2^4(\nu-1)^2\nu\left\{-8(\nu-1)^2(2\nu+1)+H_3^2(3\nu-2)\right\}+H_2^2(\nu-1)$$
$$\left.\nu^2\left\{H_3^4(3\nu-1)+16(\nu-1)^3(1+8\nu)+H_3^2\left(17-49\nu+64\nu^2-32\nu^3\right)\right\}\right]+H_1^2\left[H_2^6(\nu-1)^4+H_2^4(\nu-1)^2\right.$$
$$\left\{-(\nu-1)^2(1+8\nu)+2H_3^2\left(4-9\nu+4\nu^2\right)\right\}+H_2^2(\nu-1)\nu\left\{8(\nu-1)^3(1+2\nu)+H_3^4\left(14-33\nu+13\nu^2\right)+\right.$$
$$\left.H_3^2\left(-4+38\nu-98\nu^2+64\nu^3\right)\right\}+2\nu^2\left\{-8(\nu-1)^4+H_3^6\left(3-8\nu+3\nu^2\right)+4H_3^2(\nu-1)^2\left(7-63\nu+52\nu^2\right)+\right.$$
$$\left.\left.H_3^4\left(-23+134\nu-187\nu^2+76\nu^3\right)\right\}\right]-H_1^4\left[H_2^4(\nu-1)^2(2\nu-3)+H_2^2(\nu-1)\left\{-2+4\nu-19\nu^2+17\nu^3+\right.\right.$$
$$\left.H_3^2\left(13-33\nu+14\nu^2\right)\right\}+\nu\left\{-8(\nu-1)^2(3\nu-1)+H_3^4\left(16-37\nu+16\nu^2\right)+H_3^2\left(-32+209\nu-298\nu^2+\right.\right.$$
$$\left.\left.\left.\left.121\nu^3\right)\right\}\right]\right\}$$

$$p_5 = 2H_1H_3\nu\left\{-2H_2^6(\nu-1)^5+2H_1^6\left(\nu^2-\nu+1\right)-H_2^4(\nu-1)^2\nu\left[(\nu-1)^2(11+4\nu)+H_3^2\left(5-10\nu+\right.\right.\right.$$
$$\left.\left.6\nu^2\right)\right]-2\nu^3\left[40(\nu-1)^4(4\nu-1)+H_3^6\left(\nu^2-\nu+1\right)+2H_3^2(\nu-1)^2\left(27-79\nu+56\nu^2\right)+H_3^4\left(-15+46\nu-\right.\right.$$
$$\left.\left.53\nu^2+22\nu^3\right)\right]-H_2^2(\nu-1)\nu^2\left[-8(\nu-1)^3(16\nu-1)+H_3^4\left(6\nu^2-8\nu+5\right)+H_3^2\left(-33+131\nu-146\nu^2+\right.\right.$$
$$\left.\left.48\nu^3\right)\right]+H_1^4\left[H_2^2\left(6-14\nu+13\nu^2-5\nu^3\right)+\nu\left\{14-61\nu+92\nu^2-45\nu^3-2H_3^2\left(6-7\nu+6\nu^2\right)\right\}\right]+$$
$$H_1^2\left[H_2^4(\nu-1)^2\left(6-10\nu+5\nu^2\right)+H_2^2(\nu-1)\nu\left\{-3+41\nu-56\nu^2+18\nu^3+H_3^2\left(17-26\nu+17\nu^2\right)\right\}+\right.$$
$$\left.\left.2\nu^2\left\{H_3^4\left(6\nu^2-7\nu+6\right)+2(\nu-1)^2\left(57\nu^2-64\nu+11\right)+H_3^2\left(67\nu^3-155\nu^2+125\nu-37\right)\right\}\right]\right\}$$

$$p_6 = -\{H_2^8(\nu-1)^6 - 2H_2^6[H_3^2(1-2\nu)+8(\nu-1)^2]\nu(\nu-1)^4 + H_1^8(\nu^2+1) -$$
$$2H_2^2(\nu-1)\nu^2[-84H_3^2(\nu-1)^2+128(\nu-1)^5+17H_3^4(\nu-1)\nu+H_3^6(-1+\nu-2\nu^2)]+$$
$$\nu^4(-4+H_3^2+4\nu)^2[-17H_3^2(\nu-1)^2+16(\nu-1)^4+H_3^4(1+\nu^2)]+H_2^4(\nu-1)^2\nu^2[96(\nu-1)^4 -$$
$$3H_3^2(\nu-1)^2(3+8\nu)+H_3^4(2-6\nu+6\nu^2)]+H_1^6[-2H_2^2(-2+3\nu-2\nu^2+\nu^3)+\nu\{8-25\nu+42\nu^2 -$$
$$-25\nu^3-4H_3^2(4-3\nu+4\nu^2)\}]-H_1^2[2H_2^6(\nu-2)(\nu-1)^4+H_2^4(\nu-1)^2\nu\{3(\nu-1)^2(8+3\nu)+$$
$$H_3^2(20-34\nu+20\nu^2)\}+H_2^2(\nu-1)\nu^2\{-168(\nu-1)^3\nu+H_3^4(30-40\nu+34\nu^2)+17H_3^2(-7+$$
$$+27\nu-27\nu^2+7\nu^3)\}+2\nu^3\{8(\nu-1)^4(25\nu-8)+H_3^6(8-6\nu+8\nu^2)+4H_3^2(\nu-1)^2(67-122\nu+$$
$$67\nu^2)+H_3^4(-104+247\nu-230\nu^2+87\nu^3)\}]+2H_1^4[H_2^4(\nu-1)^2(3-3\nu+\nu^2)+H_2^2(\nu-1)\nu\{-17\nu$$
$$(\nu-1)+H_3^2(17-20\nu+15\nu^2)\}+\nu^2\{2H_3^4(9-8\nu+9\nu^2)+4(\nu-1)^2(4-21\nu+21\nu^2)+$$
$$H_3^2(-87+230\nu-247\nu^2+104\nu^3)\}]\}$$

$$p_7 = -2H_1H_3\{-2H_2^6(\nu-1)^5+2H_1^6(\nu^2-\nu+1)-H_2^4(\nu-1)^2\nu[-(\nu-1)^2(4+11\nu)+$$
$$H_3^2(5-10\nu+6\nu^2)]-H_2^2(\nu-1)\nu^2[-8(\nu-16)(\nu-1)^3+H_3^4(5-8\nu+6\nu^2)+H_3^2(-18+56\nu-$$
$$41\nu^2+3\nu^3)]-\nu^3[80(\nu-4)(\nu-1)^4+2H_3^6(\nu^2-\nu+1)+4H_3^2(\nu-1)^2(57-64\nu+11\nu^2)+$$
$$H_3^4(-45+92\nu-61\nu^2+14\nu^3)]+H_1^4[H_2^2(6-14\nu+13\nu^2-5\nu^3)-2\nu\{-22+53\nu-46\nu^2+$$
$$15\nu^3+H_3^2(6-7\nu+6\nu^2)\}]+H_1^2[H_2^4(\nu-1)^2(6-10\nu+5\nu^2)+H_2^2(\nu-1)\nu\{-48+146\nu-$$
$$131\nu^2+33\nu^3+H_3^2(17-26\nu+17\nu^2)\}+2\nu^2\{H_3^4(6-7\nu+6\nu^2)+2(\nu-1)^2(56-79\nu+27\nu^2)+$$
$$H_3^2(-67+155\nu-125\nu^2+37\nu^3)\}]\}$$

$$p_8 = -\{H_1^8+H_1^6[H_2^2(3-4\nu+\nu^2)+2\{H_3^2(3-8\nu+3\nu^2)-4(2-5\nu+3\nu^2)\}]+$$
$$H_3^2[H_2^6(\nu-1)^4+\nu^4\{H_3^2-(\nu-1)^2\}(-4+H_3^2+4\nu)^2-H_2^4(\nu-1)^2\nu\{H_3^2(2-3\nu)+$$
$$(\nu-1)^2(8+\nu)\}+H_2^2(\nu-1)\nu^2\{8(\nu-1)^3(2+\nu)+H_3^4(3\nu-1)+H_3^2(17-19\nu+4\nu^2-2\nu^3)\}]-$$
$$H_1^4[H_2^4(\nu-1)^2(2\nu-3)+H_2^2(\nu-1)\{-32+64\nu-49\nu^2+17\nu^3+H_3^2(13-33\nu+14\nu^2)\}+$$
$$\nu\{-16(\nu-1)^2(9\nu-8)+H_3^4(16-37\nu+16\nu^2)+2H_3^2(-76+187\nu-134\nu^2+23\nu^3)\}]+$$
$$H_1^2[H_2^6(\nu-1)^4+2H_2^4(\nu-1)^2\{H_3^2(4-9\nu+4\nu^2)-4(2-3\nu+\nu^3)\}+H_2^2(\nu-1)\nu\{16(\nu-1)^3$$
$$(8+\nu)+H_3^4(14-33\nu+13\nu^2)+H_3^2(-64+98\nu-38\nu^2+4\nu^3)\}+\nu^2\{-256(\nu-1)^4+$$
$$2H_3^6(3-8\nu+3\nu^2)+8H_3^2(\nu-1)^2(52-63\nu+7\nu^2)+H_3^4(-121+298\nu-209\nu^2+32\nu^3)\}]\}$$

$$p_9 = -2H_1H_3\{2H_1^6+H_3^2[-H_2^4(\nu-1)^2(2\nu-1)+H_2^2(\nu-1)\nu\{-16+H_3^2(1-4\nu)+16\nu-$$
$$\nu^2+\nu^3\}+\nu^3\{-2H_3^4+4(\nu-5)(\nu-1)^2+H_3^2(13-14\nu+\nu^2)\}]+H_1^4[H_2^2(4-5\nu+\nu^2)+$$
$$2\{-8+22\nu-14\nu^2+H_3^2(1-6\nu+\nu^2)\}]-H_1^2[H_2^4(\nu-2)(\nu-1)^2+H_2^2(\nu-1)\{-16+16\nu-\nu^2+\nu^3+$$
$$H_3^2(3-13\nu+3\nu^2)\}+2\nu\{-8(\nu-1)^2(5\nu-4)+H_3^4(1-6\nu+\nu^2)+H_3^2(-18+53\nu-38\nu^2+3\nu^3)\}]\}$$

$$p_{10} = -H_1^2H_3^2\{6H_1^4 + H_2^4(\nu-1)^2 - H_2^2(\nu-1)[(\nu^3-\nu^2+16\nu-16)+H_3^2(1-7\nu)]+\nu^2[6H_3^4 + +16(\nu-1)^2 - H_3^2(\nu^2-26\nu+25)]+H_1^2[H_3^2(\nu^2-16\nu+1)+H_2^2(\nu^2-8\nu+7)-8(5\nu^2-7\nu+2)]\}$$

$$p_{11} = -2H_1^3H_3^3\{2H_1^2 - H_2^2(\nu-1) - 2\nu(H_3^2+2\nu-2)\}$$

$$p_{12} = -H_1^4H_3^4$$

All the above polynomials will be referred as (5.22).

Now the purpose is to find the real zeros of equation (5.21) with coefficient's (5.22). The number of real zeros from (5.21) is directly related with the number of equilibria positions.

The zeros from (5.21) are calculated as fallows, is fixed a value of ν and H_3, then in the plane H_1H_2 is built an 10^{-4} average mesh in which for each point the number of zeros is calculated, then is listed each point where the number of zeros change for graphical representation. The software made for this analysis in described in Appendix A.2.

For each zero x can be found from first equation (5.17) two values of y, each root is then tested on second equation of (5.17). From the two values of y one do not satisfy the second equation (5.17) and is disregarded. Then for each (x, y) can be determined from last equation of system (5.16) two values of a_{33}, then with (5.14) the values of a_{31} and a_{32}. Thus, each real root of the algebraic equation corresponds to two sets of values of a_{31}, a_{32} and a_{33} which, by virtue of (5.15), uniquely determines the remaining direction cosines a_{11}, a_{12}, a_{13}, a_{21}, a_{22} and a_{23}. Because exists two values of a_{33}, can be presumed that for each real root exists two equilibria positions.

From numerical calculations and confirmed in *Sarychev and Al.* [11] to [19], can be verified that equation (5.21) do not have more than 12 and no less than 4 real zeros, and because exists two values of a_{33}, when the gyrostatic moment is small enough, the gyrostat general case cannot have more than 24 equilibria positions, which means that the gyrostat satellite is close to the satellite rigid body. When the gyrostatic moment is big enough, there is no less than 8 equilibria positions, which means that the vector of the gyrostatic moment must be perpendicular to the orbital plane.

The following pictures of equilibria, Figure 5.1 to Figure 5.8, represent the several regions where the number of real-zeros of equation (5.21) are the same, this means that the several regions have the same number of equilibria.

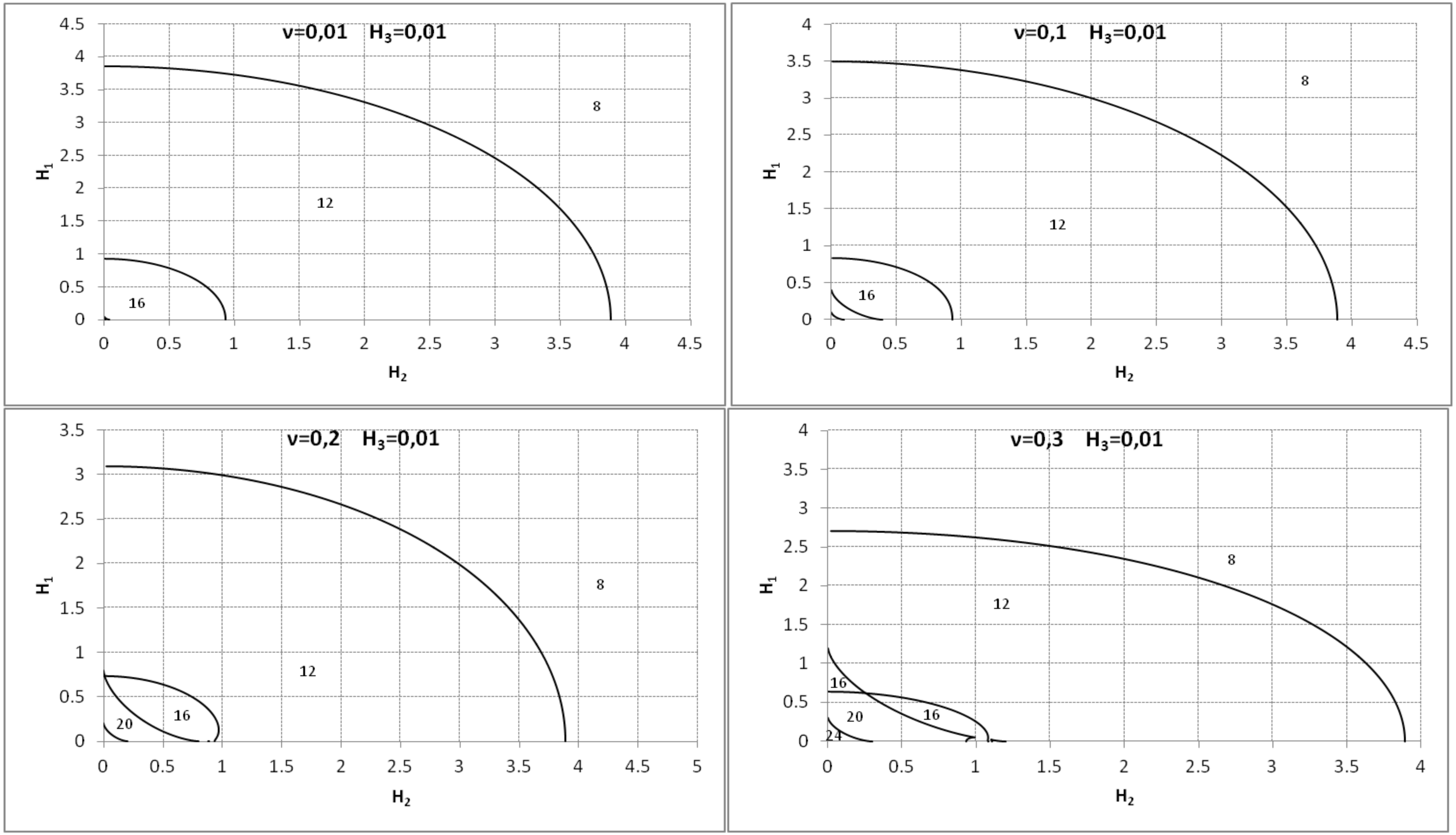


Figure 5.1 – ν =0.01 to ν =0.3 and H_3=0.01

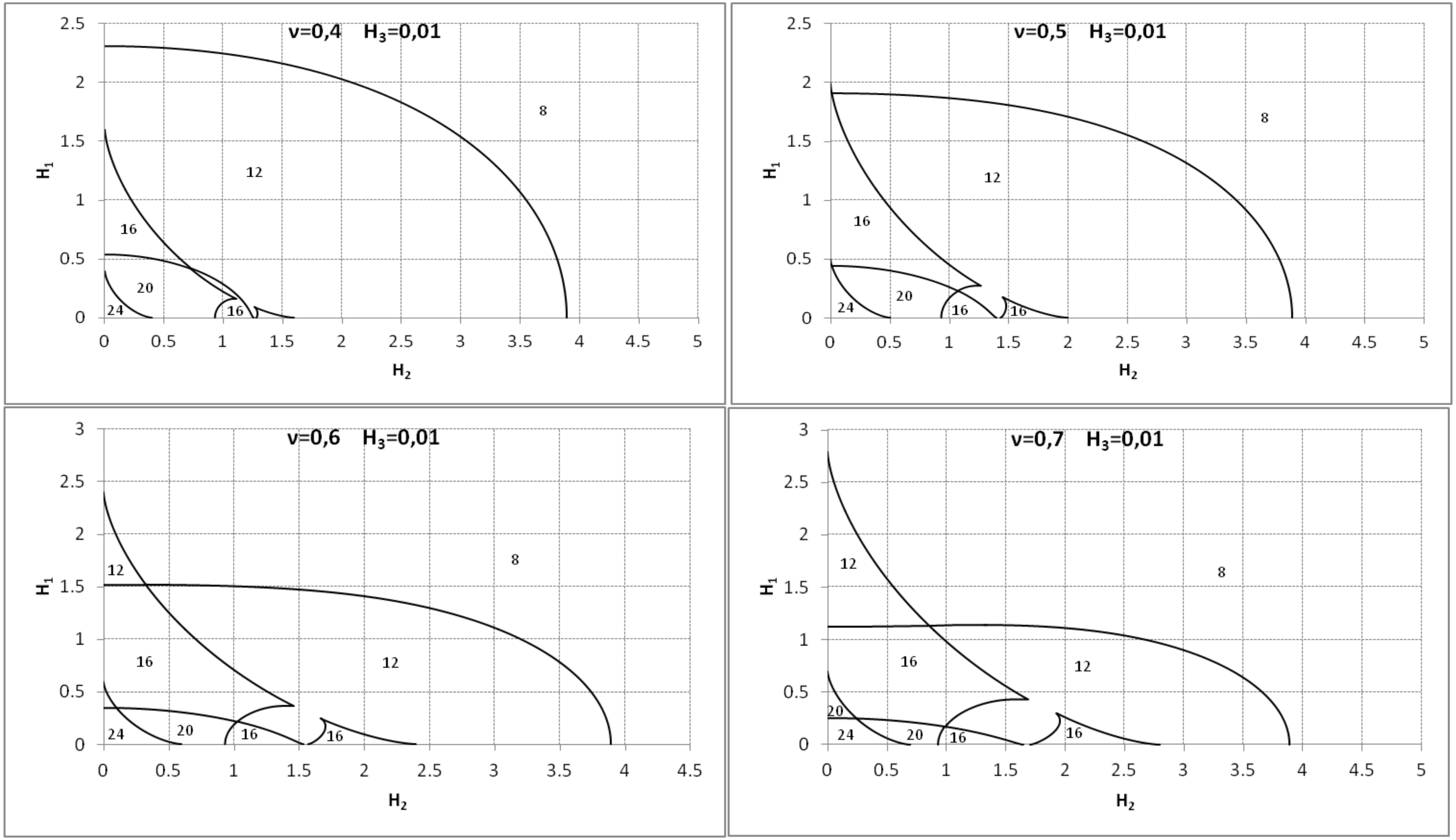


Figure 5.2 – ν =0.4 to ν =0.7 and H_3=0.01

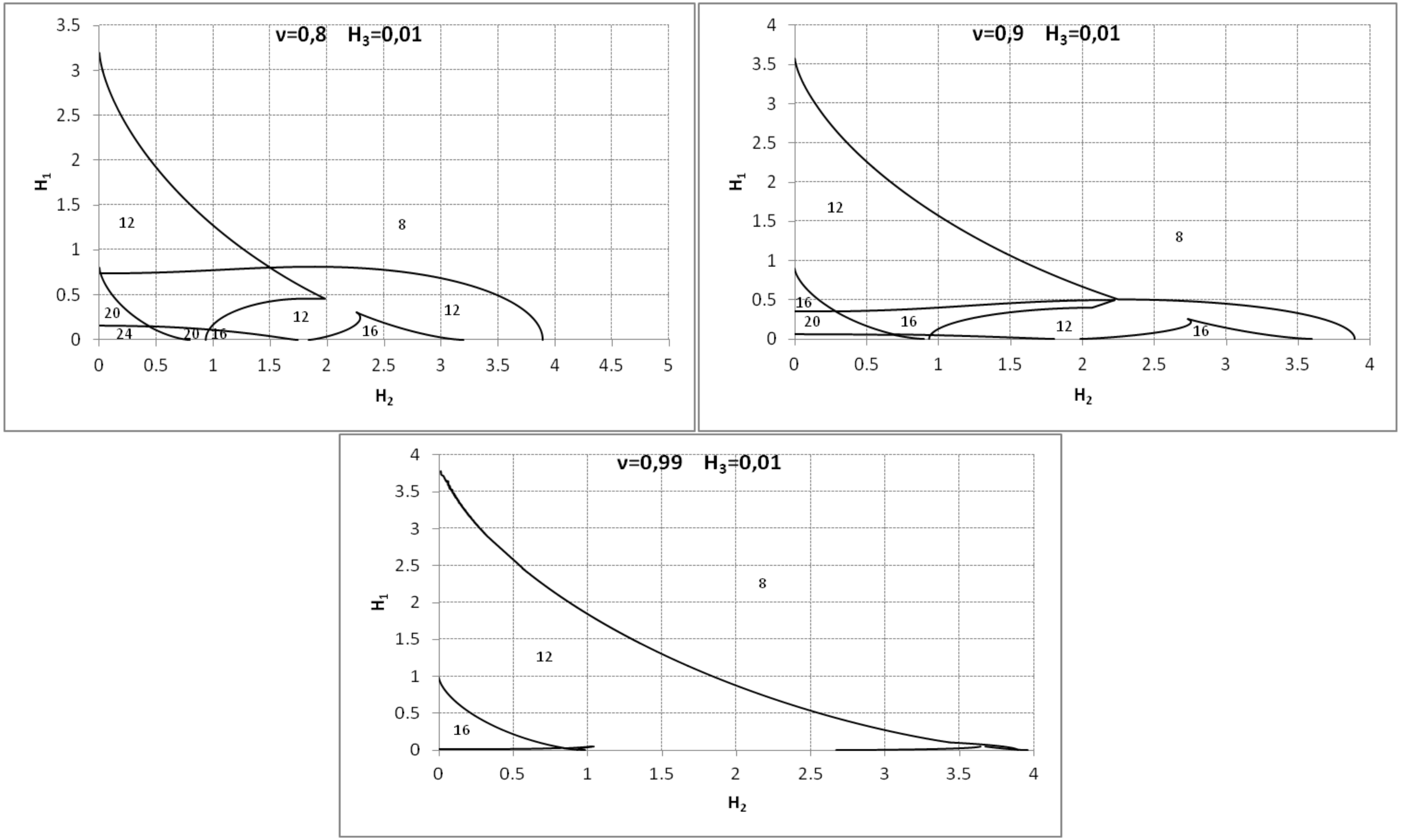


Figure 5.3 – ν =0.8 to ν =0.99 and H_3=0.01

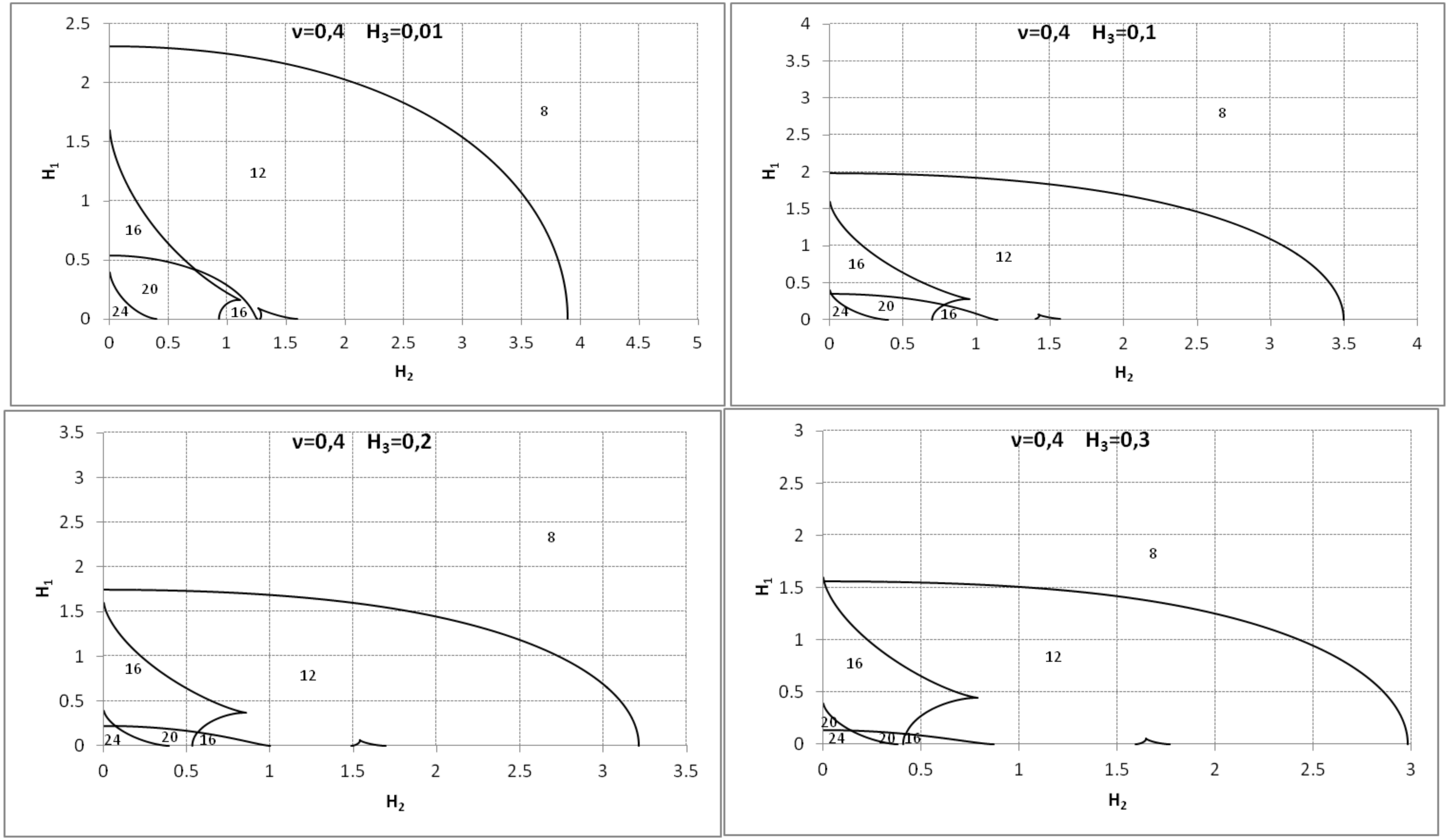


Figure 5.4- ν =0.4 and H_3=0.01 to H_3=0.3

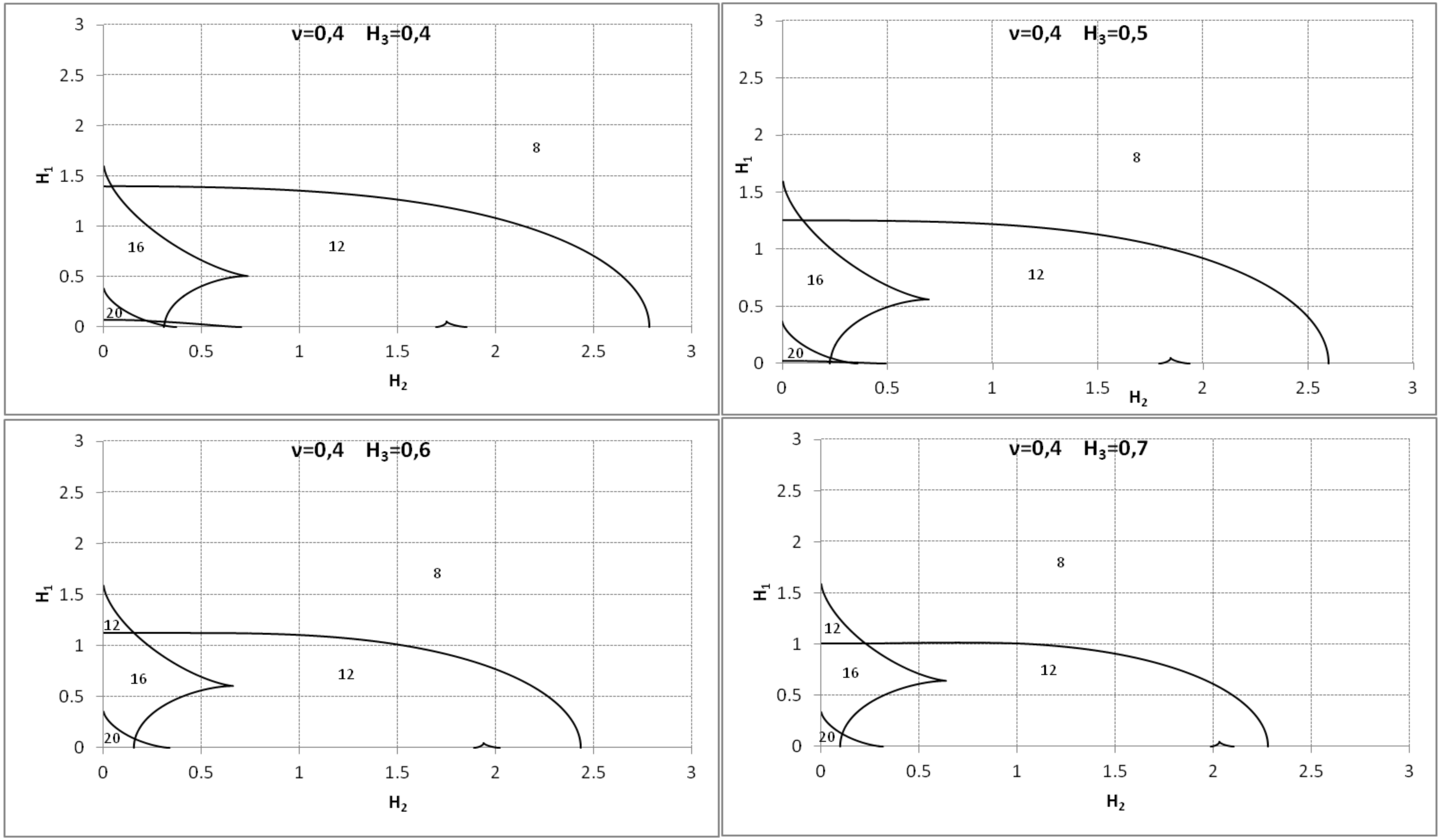


Figure 5.5- ν =0.4 and H_3=0.4 to H_3=0.7

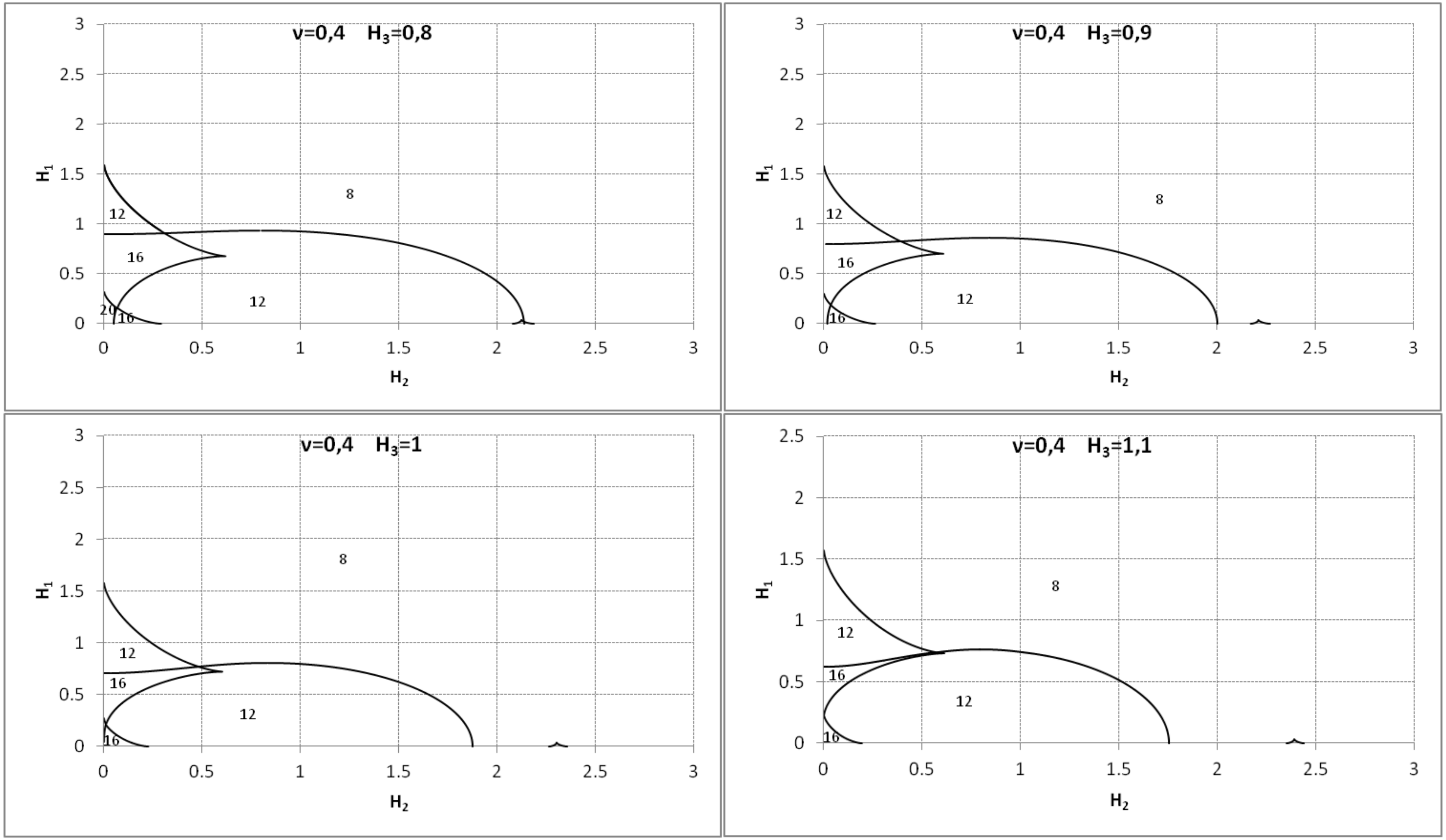


Figure 5.6- ν =0.4 and H_3=0.4 to H_3=0.7

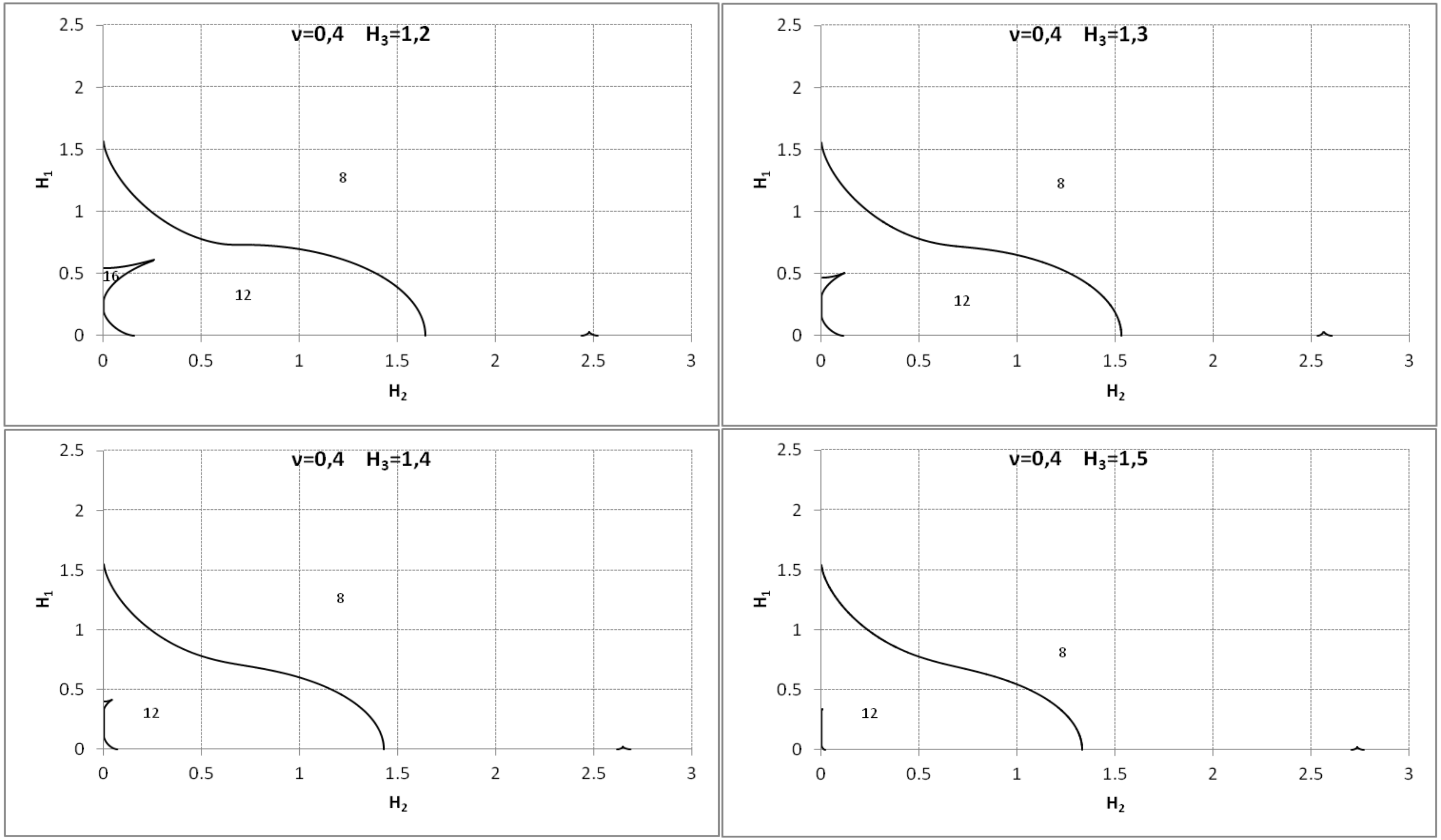


Figure 5.7- ν =0.4 and H_3=1.2 to H_3=1.5

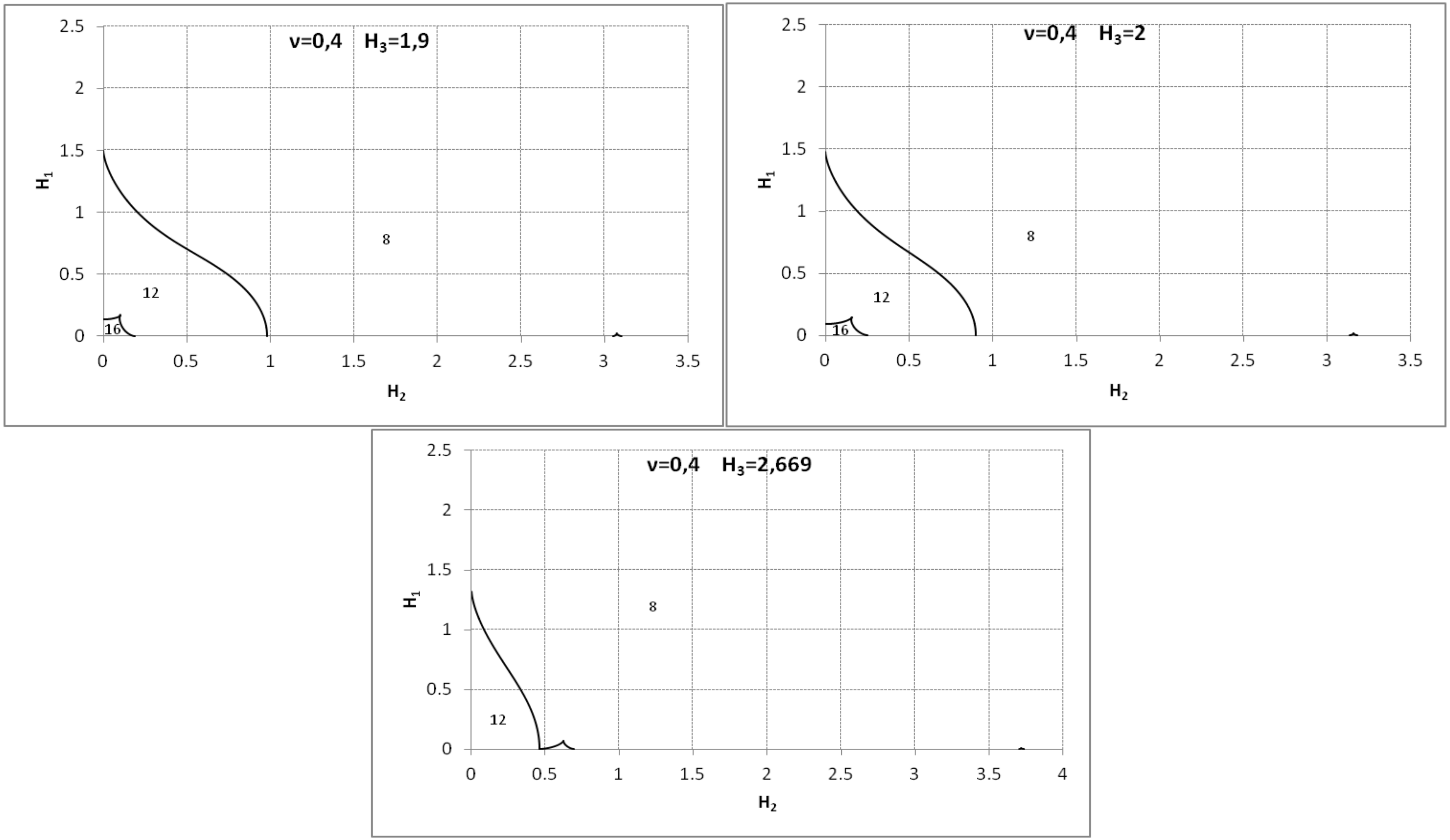


Figure 5.8- ν =0.4 and H_3=1.9 to H_3=2.669

From the analysis of all the calculated inertia configurations, it can be easily seen that with the increase of H_3 the several regions of equilibria become narrowed until they completely disappear, the point when a region of equilibria vanishes is called bifurcation point, which is more deeply studied in Chapter 6.1.

It can also be confirmed that in regions with small values of H_1 and H_2 exists 24 positions of equilibria, as we increase the values from H_1 and H_2, the 24 equilibria region will give place to a 20 equilibria region, then to a 16 equilibria region, then to a 12 equilibria region and finally only to a 8 equilibria.

There also can be found small regions of 16 and 12 equilibria outside their main region, these regions were completely unknown to date and at first were mistaken by "noise data" due to the fact that are located very close to $H_1 = 0$. The calculations near the axes can be rather problematic because it was assumed H_1, H_2, H_3, h_1, h_2 and h_3 were not equal to zero, so the regions near H_1 and H_2 axes have special peculiarities, when using a very high precision for H_2 near the axis H_1 exists a huge difference between coefficients of 12th degree polynomial. So, all numerical algorithms for solving algebraic equations use a procedure with first step to normalize first coefficient from POX^{12} to $PO = 1$ by dividing all polynomial coefficients on PO. As known PO is proportional to $H_i^{10}(i = 1,2,3)$, so the precision of numerical algorithm to calculate roots is very sensitive to huge difference between the values of polynomial coefficients.
The method used to disregard the "noise data" is a Mathematica software algorithm given in Appendix A.4 which consist in a point to point verification.

The previous mentioned 16 and 12 small equilibria regions can be found near $H_1 = 0$ and in relativity high values of H_2, and as we increase H_3, eventually the small region of 16 equilibria vanish, giving place to a second small region of 12 equilibria outside their main region. As we increase the value of H_3 both these two 12 equilibria become smaller and narrowed, but they never vanish, with the increase of H_3 also increases their position in H_2.

With the increase of ν, the regions of 24 and 20 equilibria start very small and growing in size until approximately ν=0.5, decreasing then again to smaller areas, this behaviour is compatible with the axially symmetric satellite. The remaining regions also present a similar behaviour, nevertheless they don't disappear as the 24 and 20 equilibria regions.

With the increase of H_3, the several equilibria regions become narrow and boundaries of the different equilibria regions almost touch each other, the fact is that $\left|\vec{H}\right| = \sqrt{H_1^2 + H_2^2 + H_3^2}$, so with the increase of H_3 the vector comes close to the radius vector and with that the equilibria will be narrowed with the growth of these parameters. The same phenomena happen with the increase of H_1 or H_2.

Regarding the areas where 2 points of different equilibria are very close, in mathematical meaning: $H_1' - H_1'' \approx 0 \rightarrow \dfrac{h_1'}{B-C} - \dfrac{h_1''}{B-C} \approx 0 \rightarrow h_1' - h_1'' \approx 0.$

Further studies are needed to explore the consequences of having a gyrostat system designed in the vicinity of such similar inertial configuration.
It can be seen from coefficients of equation (5.21) that depends on 4 dimensionless parameters ν, H_1, H_2, H_3. The system of stationary equation (5.5) depends on 6 dimensional parameters A, B, C, h_1, h_2, h_3. For numerical calculations, is very essential to decrease of the number of system parameters. It is possible to show that the number of real roots of equation (5.21) does not depend on the sign of the parameters H_1, H_2, H_3, it is easy to see that coefficients of equation (5.21) with odd x degree depend only on odd degree of the parameters H_1, H_2, H_3. For the coefficients with even x degree it is possible to represent them using factorization to the form of two factors, one factor equals $H_1 H_3$, and the second factor depending only on odd degree of the parameters $H_1 H_2$ and H_3. Thus, changing sign of H_1, H_2 and H_3, will change only the sign of the factor $H_1 H_3$ and therefore sign of real root of polynomial (5.21). Therefore, the number of real roots does not change. Hence, the numerical analysis of the number of real roots of the equation (5.21) is possible to do with positive values of H_1, H_2, H_3 and $0 < \nu < 1$. Which for the numerical investigation of real roots of equation (5.21) will be simplified.

6 GENERAL GYROSTAT PARTICULAR ASPECTS

6.1 H_3 BIFURCATION POINTS

As already mentioned, a bifurcation point is when a region of equilibrium vanishes per complete giving place to other region with different number of equilibria. The bifurcation analysis is perhaps one of the most important inputs that a designer should have in the preliminary phase of a satellite construction.

Regions of equilibria ν	24/20	20/16	16/12	12/8
0,001	H_1=0,0001 H_2=0,0001 H_3=0,999	H_1=0,0001 H_2=0,0001 H_3=1,000	H_1=0,0001 H_2=0,0156 H_3=3,995	H_1=0,0001 H_2=0,049 H_3=4.651
0,01	H_1=0,0001 H_2=0,0001 H_3=0,990	H_1=0,0065 H_2=0,0001 H_3=0,999	H_1=0,0001 H_2=0,0006 H_3=3,959	H_1=0,0001 H_2=0,0001 H_3=3,995
0,1	H_1=0,0001 H_2=0,0001 H_3=0,900	H_1=0,0740 H_2=0,0001 H_3=1,021	H_1=0,0001 H_2=0,0682 H_3=3,610	H_1=0,0001 H_2=0,0001 H_3=3,995
0,2	H_1=0,0001 H_2=0,0001 H_3=0,800	H_1=0,1400 H_2=0,0001 H_3=1,048	H_1=0,0001 H_2=0,1809 H_3=3,264	H_1=0,0001 H_2=0,0022 H_3=3,960
0,3	H_1=0,0001 H_2=0,0001 H_3=0,700	H_1=0,197 H_2=0,0001 H_3=1,082	H_1=0,0001 H_2=0,3154 H_3=2,950	H_1=0,0001 H_2=0,0056 H_3=3,926
0,4	H_1=0,0001 H_2=0,0001 H_3=0,600	H_1=0,231 H_2=0,0001 H_3=1,124	H_1=0,0001 H_2=0,4603 H_3=2,669	H_1=0,0007 H_2=0,0087 H_3=3,998
0,5	H_1=0,0001 H_2=0,0001 H_3=0,500	H_1=0,224 H_2=0,0001 H_3=1,182	H_1=0,0001 H_2=0,6132 H_3=2,412	H_1=0,0039 H_2=0,0001 H_3=3,995
0,6	H_1=0,0001 H_2=0,0001 H_3=0,400	H_1=0,1236 H_2=0,0001 H_3=1,186	H_1=0,0001 H_2=0,7778 H_3=2,167	H_1=0,0041 H_2=0,0001 H_3=3,995
0,7	H_1=0,0001 H_2=0,0001 H_3=0,300	H_1=0,0144 H_2=0,0001 H_3=1,105	H_1=0,0001 H_2=0,9675 H_3=1,915	H_1=0,0017 H_2=0,0001 H_3=3,996
0,8	H_1=0,0001 H_2=0,0001 H_3=0,200	H_1=0,0001 H_2=0,0155 H_3=0,909	H_1=0,0001 H_2=1,2107 H_3=1,629	H_1=0,0034 H_2=0,0001 H_3=3,995
0,9	H_1=0,0001 H_2=0,0001 H_3=0,100	H_1=0,0004 H_2=0,0989 H_3=0,676	H_1=0,0001 H_2=1,5915 H_3=1,245	H_1=0,0037 H_2=0,0001 H_3=3,990
0,99	H_1=0,0001 H_2=0,0001 H_3=0,010	H_1=0,0001 H_2=0,2521 H_3=0,168	H_1=0,0030 H_2=0,0001 H_3=0,997	H_1=0,0016 H_2=0,0001 H_3=3,985
0,999	H_1=0,0001 H_2=0,0001 H_3=0,001	H_1=0,0001 H_2=0,7713 H_3=0,028	H_1=0,0002 H_2=0,0008 H_3=0,969	H_1=0,0001 H_2=0,0171 H_3=3,984

Table 6-1: Bifurcation points from several gyrostat configurations

All the bifurcation points from Table 6-1 were calculated numerically, and can be seen that the H_3 bifurcation points from the 24 equilibria region vanish in accordance with equation $H_3 = 1 - \nu$.

For the 20 equilibria regions, the H_3 bifurcation points increase linearly with the increase of ν up to $\nu = 0{,}6$, decreasing after that with the decrease of ν in what appears to be a second degree equation.

For the 16 equilibria regions, the H_3 bifurcation points always decrease with the increase of ν, and can be approximated to $H_3 = 4 - 3\nu$.

The 12 equilibria "main region" the bifurcation points can be described in the vicinity of equation $H_3 = 4$ for $\nu \geq 0.01$.

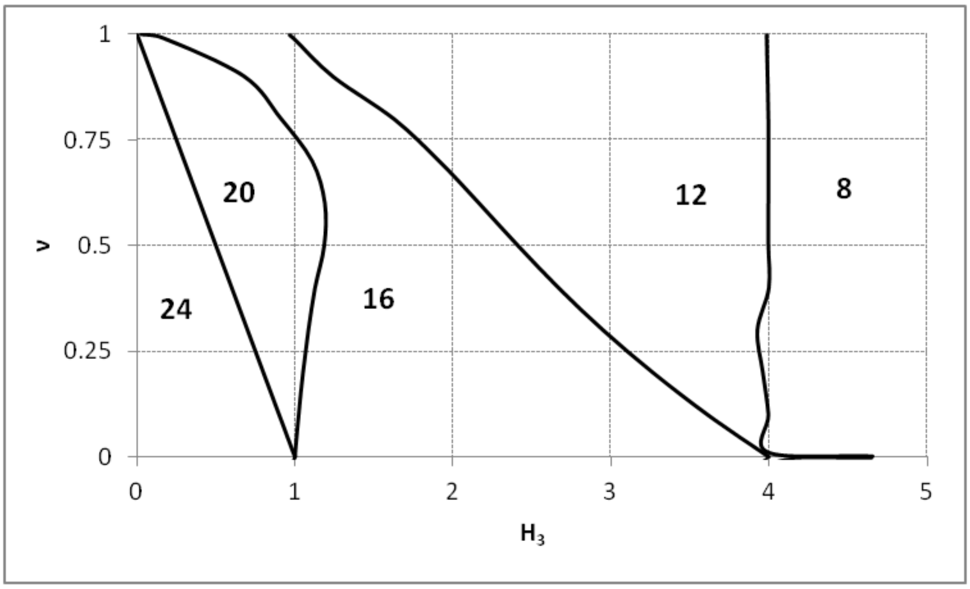


Figure 6.1 – 1st Quadrant General case bifurcation of equilibria

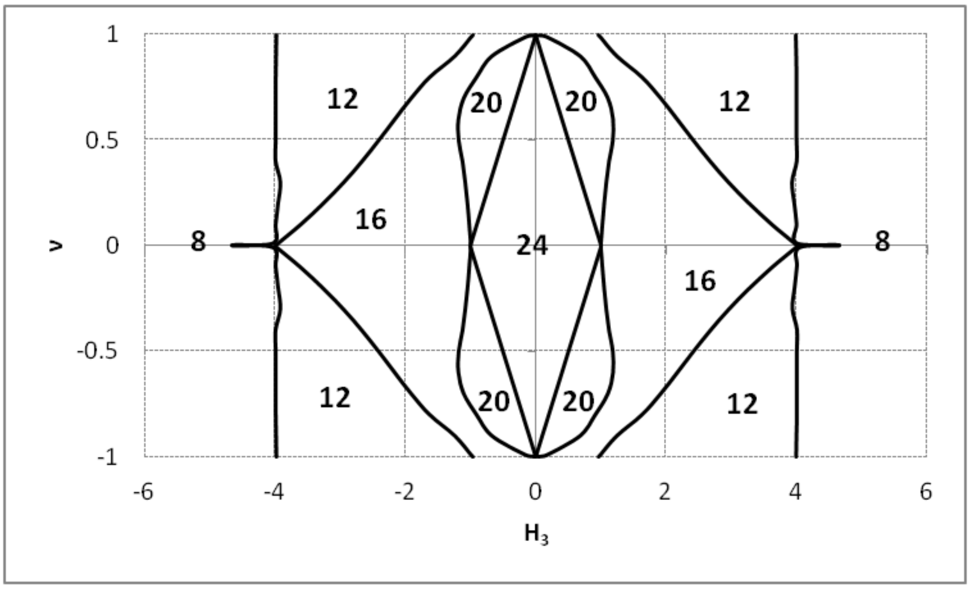


Figure 6.2 – 4 Quadrants General case bifurcation of equilibria

The bifurcation results are very interesting and a very good disclosure, taking into consideration the size and complexity of equation (5.21) with polynomials (5.22), such simpler result was not expected.

This new results permit an all new approach to the satellite construction, new inertial configurations more optimized can be applied, as well the designers can have a new insights from the vicinity from the desired systems.

6.2 4-QUADRANT PICTURE

When analyzing the coefficients from equation (5.21), the coefficients with odd x degree p_{2k} depend only from odd degree parameters H_1, H_2 and H_3. For the coefficients with even x degree p_{2k+1} we can transfer them, using factorize function to such form $p_{2k+1} = H_1 H_2 a_{2k+1}$, where factor a_{2k+1} depends from only odd degree parameters H_1, H_2 and H_3. So when is changed the sign for H_1 or H_2, it changes only the sign of real root of the polynomial but the number of real roots dos not changes. As so, changing the sign of the real roots has no impact on the number of equilibria regions, and because so all the pictures are perfectly symmetrical in both H_1 and H_2 axis, which is a huge simplification to the numerical simulations which can be made only in one quadrant.

Because the distribution of equilibria is symmetric, the software could be tested to ensure no flaws were detected, and so, to validate the numerical simulations software.

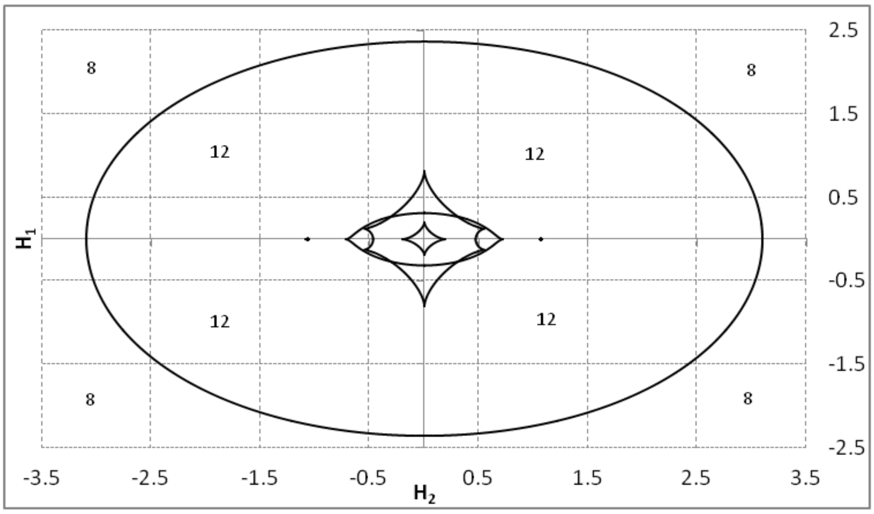


Figure 6.3 – 4-Quadrant View from ν =0.2 and H3=0.25 – Global View

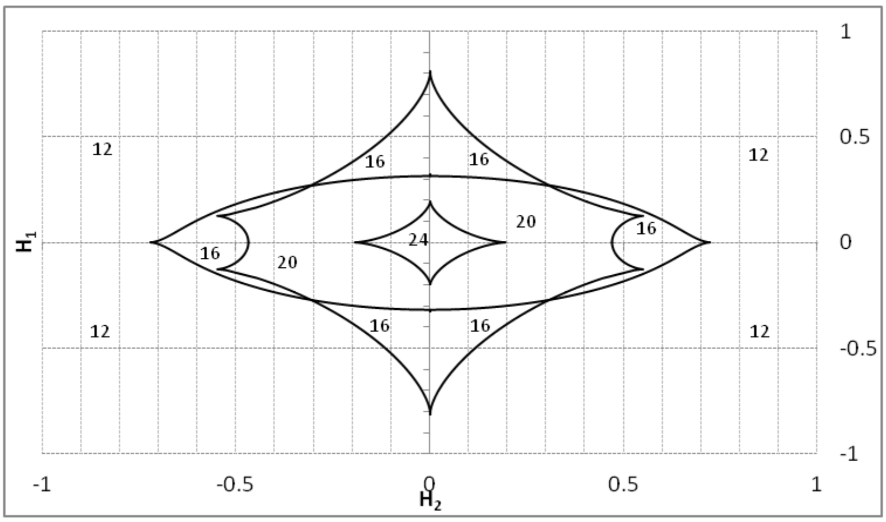
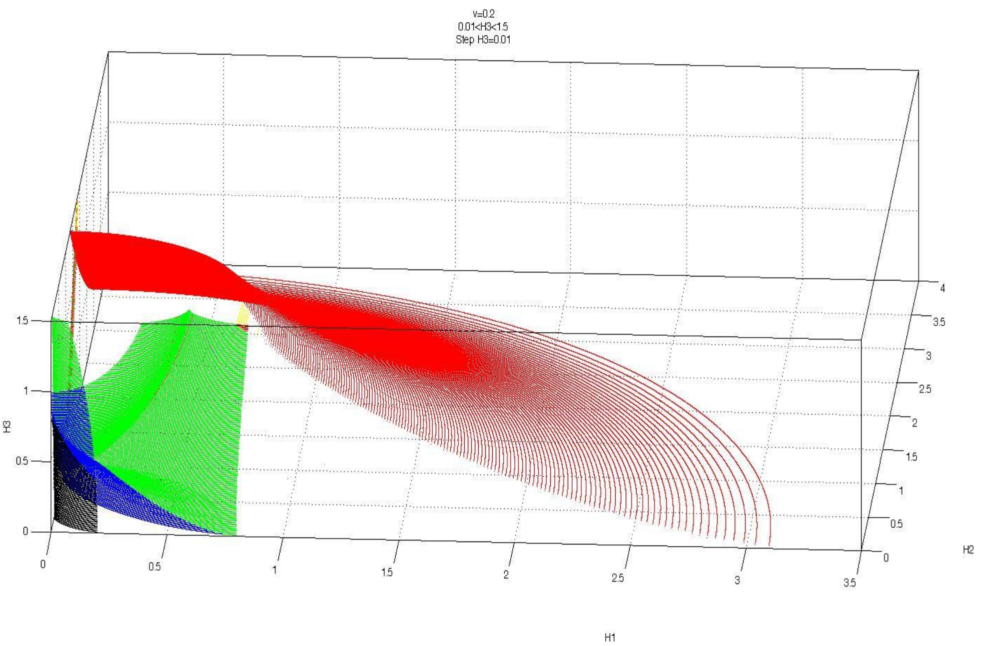


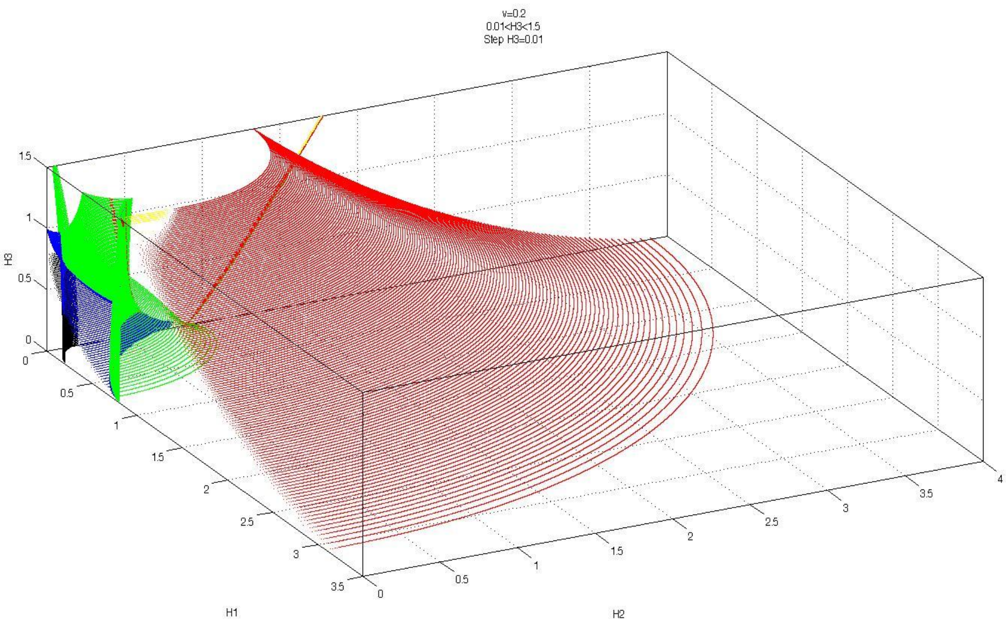
Figure 6.4 – 4-Quadrant View from ν =0.2 and H3=0.25 – Inner View

6.3 3-DIMENSIONAL PICTURE

Even that in terms of qualitatively a 3-D analysis is not interesting, in the point of view of equilibria calculations, a 3-D picture is a very good way to visualize how the several equilibria regions are distributed and respective evolutions into the spatial grid of H_1-H_2-H_3.

From the bifurcation of equilibria up to date there were no good 3-D graphical illustrations. These new way of illustrating the bifurcation of equilibria will not bring new insights, but is a great instrument to be both academically and industrially exploited, with this tool can be easily seen which is the convergence of a certain system, as well to obtain a certain degree of predictability.





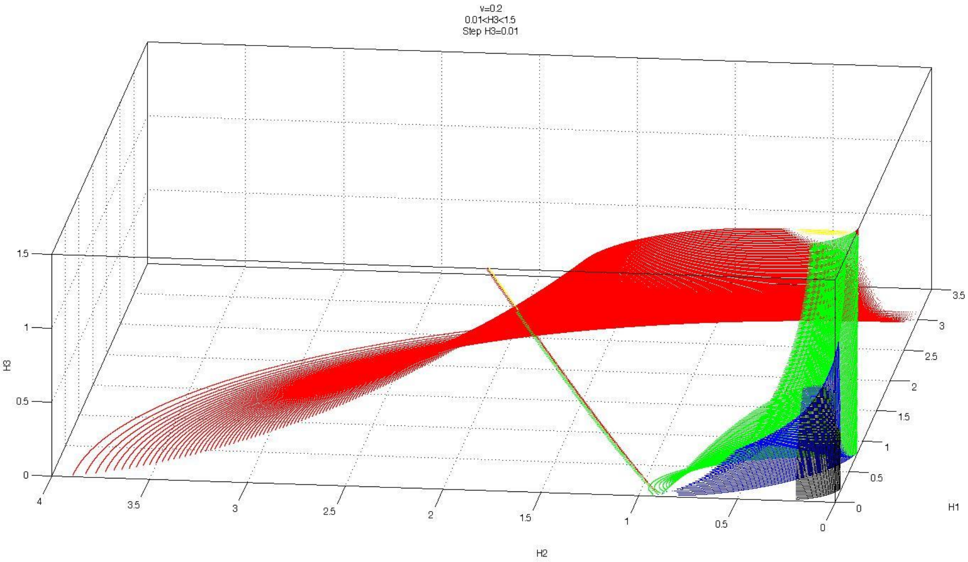


Figure 6.5 – 3D Representation for ν =0.2

Here is perfectly clear all the concepts enunciated before. It can be visualized that all regions get smaller and narrow with the increase of H_3, also can be seen the bifurcation points from the 24 and 20 equilibria regions shown in black and blue respectively, and as well in green and red the regions of 16 and 12 equilibria respectively.

A very interesting fact can be appreciated in this set of figures, can be seen the already referred small 16 and 12 equilibria region which were unknown to date. These small regions are illustrated in red and green and appear near $H_1 = 0$. Is important to mention that the regions appear to have a linear evolution, which should be confirmed by further studies in order to obtain the predictability of the evolution.

7 SUFFICIENT CONDITIONS OF STABILITY OF EQUILIBRIUM ORIENTATIONS FOR A GENERAL GYROSTAT

Before continuing developing the several concepts involved in the stability calculations, is important to clarify what stability is and how it is calculated.
There are a lot of definitions for stability, and none of them completely clarify or make it complete it clear. Part of the reason of the proliferation of stability theory is the diversity of applications where stability is applied and studied. Perhaps one of the more accepted definition of stability in celestial mechanics is given by Lagrange in [27], he stated that a system is stable if none of the point masses escapes (i.e., it reaches an infinite distance from the other point mass), further ahead in [27] Lyapunov's add to Lagrange definition that it can be stable just inside of a specific interval, i.e., the motion remains always within a specific interval.
If a Lyapunov's function is not dependent from time, it can be said that it remain time-invariant, stationary or even uniform.

The elegant feature of Lyapunov's method is reminiscent of the Routh-Hurwitz[14] approach to the stability of linear stationary systems, this method is from a quite simple study and implementation, which is largely detailed in authors in [18], [20], [22] and [23] and summarized below.

Let's consider a system:

$$\dot{x} = f(x) \tag{7.1}$$

Where f is not necessary a linear function of x.

Let define that if $\bar{x}$ is an equilibrium point of system (7.1) then $f(\bar{x}) = 0$.

Regarding this, a system is consider stable at an equilibrium point if it responds to small changes from the equilibrium with only small changes in its subsequent states.

Then, an equilibrium point, $\bar{x}$ is said to be stable if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$, such that:

$$|x(t_0) - \bar{x}| < \delta \Rightarrow |x(t) - \bar{x}| < \varepsilon, \quad \forall t > t_0 \tag{7.2}$$

The Lyapunov's second method is one of the most effective technique to investigate stability. The method here summarized provides the sufficient conditions to check the stability of an equilibrium point of a dynamic system.

Generally, as it can be seen later, this theorem permits to detect the stability (or instability) at a glance. However a set of sufficient conditions for stability is very desirable and this criterion permit to do just that.

[14] English mathematician Edward John Routh (1831-1907) developed a mathematical test that is a necessary and sufficient condition for the stability of a linear time invariant. The Routh test is an efficient recursive algorithm to determine whether all the roots of the characteristic polynomial of a linear system have negative real parts. German mathematician Adolf Hurwitz (1859-1919) independently proposed to arrange the coefficients of the polynomial into a square matrix, called the Hurwitz matrix, and showed that the polynomial is stable if and only if the sequence of determinants of its principal sub-matrices are all positive.

Let's considerer the system:

$$A = \begin{vmatrix} A_{\psi\psi} & A_{\psi\vartheta} & A_{\psi\varphi} \\ A_{\vartheta\psi} & A_{\vartheta\vartheta} & A_{\vartheta\varphi} \\ A_{\varphi\psi} & A_{\varphi\vartheta} & A_{\varphi\varphi} \end{vmatrix}$$

Which for the current configurations $A_{\psi\vartheta} = A_{\vartheta\psi}$, $A_{\psi\varphi} = A_{\varphi\psi}$ and $A_{\vartheta\varphi} = A_{\varphi\vartheta}$, then the sufficient stability conditions can be determined as:

$$\left|A_{\psi\psi}\right| > 0 \qquad \begin{vmatrix} A_{\psi\psi} & A_{\psi\vartheta} \\ A_{\psi\vartheta} & A_{\vartheta\vartheta} \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} A_{\psi\psi} & A_{\psi\vartheta} & A_{\psi\varphi} \\ A_{\psi\vartheta} & A_{\vartheta\vartheta} & A_{\vartheta\varphi} \\ A_{\psi\varphi} & A_{\vartheta\varphi} & A_{\varphi\varphi} \end{vmatrix} > 0 \qquad (7.3)$$

Resuming, the following conditions must be meet in order to achieve the criterion of sufficient stability conditions:

$$\begin{cases} A_{\psi\psi} > 0 \\ A_{\psi\psi}A_{\vartheta\vartheta} - A^2{}_{\psi\vartheta} > 0 \\ A_{\psi\psi}A_{\vartheta\vartheta}A_{\varphi\varphi} + 2A_{\psi\vartheta}A_{\vartheta\varphi}A_{\psi\varphi} - A_{\psi\psi}A^2{}_{\vartheta\varphi} - A_{\vartheta\vartheta}A^2{}_{\psi\varphi} - A_{\varphi\varphi}A^2{}_{\psi\vartheta} > 0 \end{cases} \qquad (7.4)$$

The first logical step is to locate the equilibria, that is, to identify conditions under which the system is either at rest or in uniform motion. Each such equilibria suggests a potential attitude stabilization scheme, especially if the equilibrium is stable.

Since the Hamiltonian is constant it can be used as a Lyapunov's function in the basic Lyapunov's stability theorem. Thus, a sufficient condition for an equilibrium orientation to be Lyapunov's stable, the correspondent Hamiltonian must represents a positive definite function as mentioned in (7.3) and (7.4).

Getting back to the integral of energy (4.15):

$$\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{3}{2}\omega_0^{\,2}\left[(A-C)a_{31}^{\,2} + (B-C)a_{32}^{\,2}\right] + \frac{1}{2}\omega_0^{\,2}\left[(B-A)a_{21}^{\,2} + (B-C)a_{23}^{\,2}\right] -$$
$$-\omega_0(\overline{h}_1 a_{21} + \overline{h}_2 a_{22} + \overline{h}_3 a_{23}) = H$$

And reminding:

$\nu = \dfrac{B-A}{B-C}$, $H_i = \dfrac{h_i}{B-C}$, and $h_i = \dfrac{\overline{h}_i}{\omega_0}$

Can be derived:

$$\begin{cases} h_i = \dfrac{\overline{h}_i}{\omega_0} \\ H_i = \dfrac{h_i}{(B-C)} \end{cases} \Leftrightarrow \begin{cases} - \\ h_i = H_i(B-C) \end{cases} \Leftrightarrow \begin{cases} H_i(B-C) = \dfrac{\overline{h}_i}{\omega_0} \\ - \end{cases} \Leftrightarrow \begin{cases} \overline{h}_i = H_i\omega_0(B-C) \\ - \end{cases}$$

Conveniently manipulating (4.15), the integral of energy can be presented as:

$$\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{1}{2}\omega_0^{\;2}(B-C)\left\{3\left[(1-\nu)a_{31}^2 + a_{32}^2\right] + \left(\nu\, a_{21}^2 + a_{23}^2\right) - 2\left(H_1a_{21} + H_2a_{22} + H_3a_{23}\right)\right\} = H$$

The above equation is referred as (7.5).

It will be introduced in the set of equations small variations in the direction angles, i.e. small displacements from the equilibria positions, or more exactly interpreted as small orbital disturbances. Now the main purpose is to verify how the system will respond to this disturbances in the vicinity of $\overline{\psi}$, $\overline{\vartheta}$ and $\overline{\varphi}$. Following this, the angles will then be present as:

$$\psi = \psi_0 + \overline{\psi},\ \vartheta = \vartheta_0 + \overline{\vartheta},\ \varphi = \varphi_0 + \overline{\varphi}, \tag{7.6}$$

Where ψ, ϑ, φ are small deviations from the equilibrium position $\psi = \psi_0 = const$, $\vartheta = \vartheta_0 = const$, and $\varphi = \varphi_0 = const$, satisfying the system of equations (3.7). Then, integral of energy (4.15) can be presented as follows:

$$\frac{1}{2}(A\overline{p}^2 + B\overline{q}^2 + C\overline{r}^2) + \frac{1}{2}\omega_0^{\;2}(B-C)\left(A_{\psi\psi}\overline{\psi}^2 + A_{\vartheta\vartheta}\overline{\vartheta}^2 + A_{\varphi\varphi}\overline{\varphi}^2 + 2A_{\psi\vartheta}\overline{\psi}\overline{\vartheta} + 2A_{\psi\varphi}\overline{\psi}\overline{\varphi} + 2A_{\vartheta\varphi}\overline{\vartheta}\overline{\varphi}\right) + \Sigma = const.$$

The above equation is referred as (7.7).

The Lyapunov's theorem tell that stability exists in case when Lyapunov's matrix be positively defined, i.e., all the square triangular from our matrix must be positives.

Expanding the direction cosines according a Taylor Series:

$$a_{ij}(\psi,\theta,\varphi) = a_{ij}(\psi_0 + \overline{\psi}, \vartheta_0 + \overline{\vartheta}, \varphi_0 + \overline{\varphi}) = a_{ij}(\psi_0, \vartheta_0, \varphi_0) + (\frac{\partial \overline{a}_{ij}}{\partial \psi_0}\overline{\psi} + \frac{\partial \overline{a}_{ij}}{\partial \vartheta_0}\overline{\vartheta} + \frac{\partial \overline{a}_{ij}}{\partial \varphi_0}\overline{\varphi}) +$$
$$+\frac{1}{2}(\frac{\partial^2 \overline{a}_{ij}}{\partial \psi_0^{\;2}}\overline{\psi}^2 + \frac{\partial^2 \overline{a}_{ij}}{\partial \vartheta_0^{\;2}}\overline{\vartheta}^2 + \frac{\partial^2 \overline{a}_{ij}}{\partial \varphi_0^{\;2}}\overline{\varphi}^2 + 2\frac{\partial^2 \overline{a}_{ij}}{\partial \psi_0 \partial \vartheta_0}\overline{\psi}\overline{\vartheta} + 2\frac{\partial^2 \overline{a}_{ij}}{\partial \psi_0 \partial \varphi_0}\overline{\psi}\overline{\varphi} + 2\frac{\partial^2 \overline{a}_{ij}}{\partial \vartheta_0 \partial \varphi_0}\overline{\vartheta}\overline{\varphi}) \tag{7.8}$$

Then, to study the stability of small displacements, it must be applied the expanded Taylor Series to the system of direction cosines (3.5), when applied the small displacements described in (7.6), system (3.5) is transformed into:

$$\begin{cases} \bar{a}_{11} = \cos\psi_0 \cos\varphi_0 - \sin\psi_0 \cos\vartheta_0 \sin\varphi_0 \\ \bar{a}_{12} = -\cos\psi_0 \sin\varphi_0 - \sin\psi_0 \cos\vartheta_0 \cos\varphi_0 \\ \bar{a}_{13} = \sin\psi_0 \sin\vartheta_0 \\ \bar{a}_{21} = \sin\psi_0 \cos\varphi_0 + \cos\psi_0 \cos\vartheta_0 \sin\varphi_0 \\ \bar{a}_{22} = -\sin\psi_0 \sin\varphi_0 + \cos\psi_0 \cos\vartheta_0 \cos\varphi_0 \\ \bar{a}_{23} = -\cos\psi_0 \sin\vartheta_0 \\ \bar{a}_{31} = \sin\vartheta_0 \sin\varphi_0 \\ \bar{a}_{32} = \sin\vartheta_0 \cos\varphi_0 \\ \bar{a}_{33} = \cos\vartheta_0 \end{cases} \tag{7.9}$$

Now let's apply the Taylor Series described in (7.8) to the system of direction cosines (7.9). We need to calculate the expanded Taylor Series to each direction cosine expression as follows:

For $\bar{a}_{11} = \cos\psi_0 \cos\varphi_0 - \sin\psi_0 \cos\vartheta_0 \sin\varphi_0$:

$\dfrac{\partial \bar{a}_{11}}{\partial \psi_0} = -\bar{a}_{21}$	$\dfrac{\partial \bar{a}_{11}}{\partial \vartheta_0} = \bar{a}_{13} \cos\varphi_0$	$\dfrac{\partial \bar{a}_{11}}{\partial \varphi_0} = \bar{a}_{12}$
$\dfrac{\partial^2 \bar{a}_{11}}{\partial {\psi_0}^2} = -\bar{a}_{11}$	$\dfrac{\partial^2 \bar{a}_{11}}{\partial {\vartheta_0}^2} = \sin\psi_0 \cos\vartheta_0 \sin\varphi_0$	$\dfrac{\partial^2 \bar{a}_{11}}{\partial {\varphi_0}^2} = -\bar{a}_{11}$
$\dfrac{\partial^2 \bar{a}_{11}}{\partial \psi_0 \partial \vartheta_0} = -\bar{a}_{23} \cos\varphi_0$	$\dfrac{\partial^2 \bar{a}_{11}}{\partial \psi_0 \partial \varphi_0} = -\bar{a}_{22}$	$\dfrac{\partial^2 \bar{a}_{11}}{\partial \vartheta_0 \partial \varphi_0} = \bar{a}_{13} \cos\varphi$

Replacing the above coefficients into equation (7.8):

$$a_{11} = \bar{a}_{11} + \left[-\bar{a}_{21}\psi + \vartheta(\bar{a}_{13} \sin\varphi_0) + \bar{a}_{12}\varphi\right] + \frac{1}{2}\left[-\bar{a}_{11}\psi^2 + (\bar{a}_{33} \sin\psi_0 \sin\varphi_0)\vartheta^2 - \bar{a}_{11}\varphi^2 - 2\bar{a}_{23}\psi\vartheta\cos\varphi_0 - 2\bar{a}_{22}\psi\varphi + 2\bar{a}_{13}\vartheta\varphi\cos\varphi_0\right]$$

For $\bar{a}_{12} = -\cos\psi_0 \sin\varphi_0 - \sin\psi_0 \cos\vartheta_0 \cos\varphi_0$:

$\frac{\partial \bar{a}_{12}}{\partial \psi_0} = -\bar{a}_{22}$	$\frac{\partial \bar{a}_{12}}{\partial \vartheta_0} = \bar{a}_{13} \cos\varphi_0$	$\frac{\partial \bar{a}_{12}}{\partial \varphi_0} = -\bar{a}_{11}$
$\frac{\partial^2 \bar{a}_{12}}{\partial {\psi_0}^2} = -\bar{a}_{12}$	$\frac{\partial^2 \bar{a}_{12}}{\partial {\vartheta_0}^2} = \bar{a}_{33} \sin\psi_0 \sin\varphi_0$	$\frac{\partial^2 \bar{a}_{12}}{\partial {\varphi_0}^2} = -\bar{a}_{12}$
$\frac{\partial^2 \bar{a}_{12}}{\partial \psi_0 \partial \vartheta_0} = \bar{a}_{13} \cos\varphi_0$	$\frac{\partial^2 \bar{a}_{12}}{\partial \psi_0 \partial \varphi_0} = \bar{a}_{21}$	$\frac{\partial^2 \bar{a}_{12}}{\partial \vartheta_0 \partial \varphi_0} = -\bar{a}_{13} sen\varphi_0$

Replacing the above coefficients into equation (7.8):

$$a_{12} = \bar{a}_{12} + \left(-\bar{a}_{22}\psi + \bar{a}_{13}\vartheta\cos\varphi_0 - \bar{a}_{11}\varphi\right) + \frac{1}{2}\left(-\bar{a}_{12}\psi^2 + \bar{a}_{33}\vartheta^2 \sin\psi_0 \cos\varphi_0 - \bar{a}_{12}\varphi^2 + \right.$$
$$\left. + 2\bar{a}_{13}\psi\vartheta\cos\varphi + 2\bar{a}_{21}\psi\varphi - 2\bar{a}_{13}\vartheta\varphi\sin\varphi_0\right)$$

For $\bar{a}_{13} = \sin\psi_0 \sin\vartheta_0$:

$\frac{\partial \bar{a}_{13}}{\partial \psi_0} = -\bar{a}_{23}$	$\frac{\partial \bar{a}_{13}}{\partial \vartheta_0} = \bar{a}_{33} \sin\psi_0$	$\frac{\partial \bar{a}_{13}}{\partial \varphi_0} = 0$
$\frac{\partial^2 \bar{a}_{13}}{\partial {\psi_0}^2} = -\bar{a}_{13}$	$\frac{\partial^2 \bar{a}_{13}}{\partial {\vartheta_0}^2} = -\bar{a}_{13}$	$\frac{\partial^2 \bar{a}_{13}}{\partial {\varphi_0}^2} = 0$
$\frac{\partial^2 \bar{a}_{13}}{\partial \psi_0 \partial \vartheta_0} = \bar{a}_{33} \cos\psi_0$	$\frac{\partial^2 \bar{a}_{13}}{\partial \psi_0 \partial \varphi_0} = 0$	$\frac{\partial^2 \bar{a}_{13}}{\partial \vartheta_0 \partial \varphi_0} = 0$

Replacing the above coefficients into equation (7.8):

$$a_{13} = \bar{a}_{13} + \left(-\bar{a}_{23}\psi + \bar{a}_{33}\vartheta\sin\psi_0\right) + \frac{1}{2}\left(-\bar{a}_{13}\psi^2 - \bar{a}_{13}\vartheta^2 + 2\bar{a}_{33}\psi\vartheta\cos\psi_0\right)$$

For $\bar{a}_{21} = \sin\psi_0 \cos\varphi_0 + \cos\psi_0 \cos\vartheta_0 \sin\varphi_0$:

$\dfrac{\partial \bar{a}_{21}}{\partial \psi_0} = \bar{a}_{11}$	$\dfrac{\partial \bar{a}_{21}}{\partial \vartheta_0} = \bar{a}_{23} \sin\varphi_0$	$\dfrac{\partial \bar{a}_{21}}{\partial \varphi_0} = \bar{a}_{22}$
$\dfrac{\partial^2 \bar{a}_{21}}{\partial {\psi_0}^2} = -\bar{a}_{21}$	$\dfrac{\partial^2 \bar{a}_{21}}{\partial {\vartheta_0}^2} = -\bar{a}_{33} \cos\psi_0 \sin\varphi_0$	$\dfrac{\partial^2 \bar{a}_{21}}{\partial {\varphi_0}^2} = -\bar{a}_{21}$
$\dfrac{\partial^2 \bar{a}_{21}}{\partial \psi_0 \partial \vartheta_0} = \bar{a}_{13} \sin\varphi_0$	$\dfrac{\partial^2 \bar{a}_{21}}{\partial \psi_0 \partial \varphi_0} = \bar{a}_{12}$	$\dfrac{\partial^2 \bar{a}_{21}}{\partial \vartheta_0 \partial \varphi_0} = \bar{a}_{23} \cos\varphi_0$

For $\bar{a}_{22} = -\sin\psi_0 \sin\varphi_0 + \cos\psi_0 \cos\vartheta_0 \cos\varphi_0$:

$\dfrac{\partial \bar{a}_{22}}{\partial \psi_0} = \bar{a}_{12}$	$\dfrac{\partial \bar{a}_{22}}{\partial \vartheta_0} = \bar{a}_{23} \cos\varphi_0$	$\dfrac{\partial \bar{a}_{22}}{\partial \varphi_0} = -\bar{a}_{21}$
$\dfrac{\partial^2 \bar{a}_{22}}{\partial {\psi_0}^2} = -\bar{a}_{22}$	$\dfrac{\partial^2 \bar{a}_{22}}{\partial {\vartheta_0}^2} = -\bar{a}_{33} \cos\psi_0 \cos\varphi_0$	$\dfrac{\partial^2 \bar{a}_{22}}{\partial {\varphi_0}^2} = -\bar{a}_{22}$
$\dfrac{\partial^2 \bar{a}_{22}}{\partial \psi_0 \partial \vartheta_0} = \bar{a}_{13} \cos\varphi_0$	$\dfrac{\partial^2 \bar{a}_{22}}{\partial \psi_0 \partial \varphi_0} = -\bar{a}_{11}$	$\dfrac{\partial^2 \bar{a}_{22}}{\partial \vartheta_0 \partial \varphi_0} = -\bar{a}_{23} \sin\varphi_0$

Replacing the above coefficients into equation (7.8):

$$a_{22} = \bar{a}_{22} + \left(\bar{a}_{12}\psi + \bar{a}_{23}\vartheta \cos\varphi_0 - \bar{a}_{21}\varphi\right) + \frac{1}{2}\left(-\bar{a}_{22}\psi^2 - \bar{a}_{33}\vartheta^2 \cos\psi_0 \cos\varphi_0 - \bar{a}_{22}\varphi^2 + \right.$$
$$\left. + 2\bar{a}_{13}\psi\vartheta \cos\varphi_0 - 2\bar{a}_{11}\psi\varphi - 2\bar{a}_{23}\vartheta\varphi \sin\varphi_0\right)$$

For $\bar{a}_{23} = -\cos\psi_0 sen\vartheta_0$:

$\dfrac{\partial \bar{a}_{23}}{\partial \psi_0} = \bar{a}_{13}$	$\dfrac{\partial \bar{a}_{23}}{\partial \vartheta_0} = -\bar{a}_{33}\cos\psi_0$	$\dfrac{\partial \bar{a}_{23}}{\partial \varphi_0} = 0$
$\dfrac{\partial^2 \bar{a}_{23}}{\partial {\psi_0}^2} = -\bar{a}_{23}$	$\dfrac{\partial^2 \bar{a}_{23}}{\partial {\vartheta_0}^2} = -\bar{a}_{23}$	$\dfrac{\partial^2 \bar{a}_{23}}{\partial {\varphi_0}^2} = 0$
$\dfrac{\partial^2 \bar{a}_{23}}{\partial \psi_0 \partial \vartheta_0} = \bar{a}_{33}\sin\psi_0$	$\dfrac{\partial^2 \bar{a}_{23}}{\partial \psi_0 \partial \varphi_0} = 0$	$\dfrac{\partial^2 \bar{a}_{23}}{\partial \vartheta_0 \partial \varphi_0} = 0$

Replacing the above coefficients into equation (7.8):

$$a_{23} = \bar{a}_{23} + \left(\bar{a}_{13}\psi + \bar{a}_{33}\vartheta\cos\psi_0\right) + \frac{1}{2}\left(-\bar{a}_{23}\psi^2 - \bar{a}_{23}\vartheta^2 + 2\bar{a}_{33}\psi\vartheta\sin\psi_0\right)$$

For $\bar{a}_{31} = \sin\vartheta_0 \sin\varphi_0$:

$\dfrac{\partial \bar{a}_{31}}{\partial \psi_0} = 0$	$\dfrac{\partial \bar{a}_{31}}{\partial \vartheta_0} = \bar{a}_{33}\sin\varphi_0$	$\dfrac{\partial \bar{a}_{31}}{\partial \varphi_0} = \bar{a}_{32}$
$\dfrac{\partial^2 \bar{a}_{31}}{\partial {\psi_0}^2} = 0$	$\dfrac{\partial^2 \bar{a}_{31}}{\partial {\vartheta_0}^2} = -\bar{a}_{31}$	$\dfrac{\partial^2 \bar{a}_{31}}{\partial {\varphi_0}^2} = -\bar{a}_{31}$
$\dfrac{\partial^2 \bar{a}_{31}}{\partial \psi_0 \partial \vartheta_0} = 0$	$\dfrac{\partial^2 \bar{a}_{31}}{\partial \psi_0 \partial \varphi_0} = 0$	$\dfrac{\partial^2 \bar{a}_{31}}{\partial \vartheta_0 \partial \varphi_0} = \bar{a}_{33}\cos\varphi_0$

Replacing the above coefficients into equation (7.8):

$$a_{31} = \bar{a}_{31} + \left(\bar{a}_{33}\vartheta\sin\varphi_0 + \bar{a}_{32}\varphi\right) + \frac{1}{2}\left(-\bar{a}_{31}\vartheta^2 - \bar{a}_{31}\varphi^2 + 2\bar{a}_{33}\vartheta\varphi\cos\varphi_0\right)$$

For $\bar{a}_{32} = \sin\vartheta_0 \cos\varphi_0$:

$\frac{\partial \bar{a}_{32}}{\partial \psi_0} = 0$	$\frac{\partial \bar{a}_{32}}{\partial \vartheta_0} = \bar{a}_{33} \cos\varphi_0$	$\frac{\partial \bar{a}_{32}}{\partial \varphi_0} = -\bar{a}_{31}$
$\frac{\partial^2 \bar{a}_{32}}{\partial {\psi_0}^2} = 0$	$\frac{\partial^2 \bar{a}_{32}}{\partial {\vartheta_0}^2} = -\bar{a}_{32}$	$\frac{\partial^2 \bar{a}_{32}}{\partial {\varphi_0}^2} = -\bar{a}_{32}$
$\frac{\partial^2 \bar{a}_{32}}{\partial \psi_0 \partial \vartheta_0} = 0$	$\frac{\partial^2 \bar{a}_{32}}{\partial \psi_0 \partial \varphi_0} = 0$	$\frac{\partial^2 \bar{a}_{32}}{\partial \vartheta_0 \partial \varphi_0} = -\bar{a}_{33} \sin\varphi_0$

Replacing the above coefficients into equation (7.8):

$$a_{32} = \bar{a}_{32} + \left(\bar{a}_{33}\vartheta \cos\varphi_0 - \bar{a}_{31}\varphi\right) + \frac{1}{2}\left(-\bar{a}_{32}\vartheta^2 - \bar{a}_{32}\varphi^2 - 2\bar{a}_{33}\vartheta\varphi \sin\varphi_0\right)$$

For $\bar{a}_{33} = \cos\vartheta_0$:

$\frac{\partial \bar{a}_{33}}{\partial \psi_0} = 0$	$\frac{\partial \bar{a}_{33}}{\partial \vartheta_0} = \sin\vartheta_0$	$\frac{\partial \bar{a}_{33}}{\partial \varphi_0} = 0$
$\frac{\partial^2 \bar{a}_{33}}{\partial {\psi_0}^2} = 0$	$\frac{\partial^2 \bar{a}_{33}}{\partial {\vartheta_0}^2} = -\bar{a}_{33}$	$\frac{\partial^2 \bar{a}_{33}}{\partial {\varphi_0}^2} = 0$
$\frac{\partial^2 \bar{a}_{33}}{\partial \psi_0 \partial \vartheta_0} = 0$	$\frac{\partial^2 \bar{a}_{33}}{\partial \psi_0 \partial \varphi_0} = 0$	$\frac{\partial^2 \bar{a}_{33}}{\partial \vartheta_0 \partial \varphi_0} = 0$

Replacing the above coefficients into equation (7.8):

$$a_{33} = \bar{a}_{33} + \left(-\vartheta \sin\vartheta_0\right) + \frac{1}{2}\left(-\bar{a}_{33}\vartheta^2\right)$$

Adding the above calculated expanded Taylor coefficients and equations (4.7) into integral of energy (4.15):

$$\frac{1}{2}(A\overline{p}^2+B\overline{q}^2+C\overline{r}^2)+\frac{1}{2}\omega_0{}^2(B-C)\Big\{3\Big[(1-\nu)\Big[\overline{a}_{31}+(\overline{a}_{33}\vartheta\sin\varphi_0+\overline{a}_{32}\varphi)+\frac{1}{2}\left(-\overline{a}_{31}\vartheta^2-\overline{a}_{31}\varphi^2+2\overline{a}_{33}\vartheta\varphi\cos\varphi_0\right)\Big]^2+$$

$$\Big[\overline{a}_{32}+(\overline{a}_{33}\vartheta\cos\varphi_0-\overline{a}_{31}\varphi)+\frac{1}{2}\left(-\overline{a}_{32}\vartheta^2-\overline{a}_{32}\varphi^2-2\overline{a}_{33}\vartheta\varphi\sin\varphi_0\right)\Big]^2\Big]+\Big[\nu\Big(\overline{a}_{21}+(\overline{a}_{11}\psi+\overline{a}_{23}\vartheta\sin\varphi_0+\overline{a}_{22}\varphi)+$$

$$+\frac{1}{2}\left(-\overline{a}_{21}\psi^2-\overline{a}_{33}\vartheta^2\cos\psi_0\sin\varphi_0-\overline{a}_{21}\varphi^2+2\overline{a}_{13}\psi\vartheta+2\overline{a}_{12}\psi\varphi+2\overline{a}_{23}\vartheta\varphi\cos\varphi_0\right)\Big)^2+\Big(\overline{a}_{23}+(\overline{a}_{13}\psi+\overline{a}_{33}\vartheta\cos\psi_0)+$$

$$+\frac{1}{2}\left(-\overline{a}_{23}\psi^2-\overline{a}_{23}\vartheta^2+2\overline{a}_{33}\psi\vartheta\sin\psi_0\right)\Big)^2\Big]-2\Big[H_1\Big(\overline{a}_{21}+(\overline{a}_{11}\psi+\overline{a}_{23}\vartheta\sin\varphi_0+\overline{a}_{22}\varphi)+$$

$$+\frac{1}{2}\left(-\overline{a}_{21}\psi^2-\overline{a}_{33}\vartheta^2\cos\psi_0\sin\varphi_0-\overline{a}_{21}\varphi^2+2\overline{a}_{13}\psi\vartheta+2\overline{a}_{12}\psi\varphi+2\overline{a}_{23}\vartheta\varphi\cos\varphi_0\right)\Big)+H_2\Big(\overline{a}_{22}+(\overline{a}_{12}\psi+\overline{a}_{23}\theta\cos\varphi_0-\overline{a}_{21}\varphi)+$$

$$+\frac{1}{2}\left(-\overline{a}_{22}\psi^2-\overline{a}_{33}\vartheta^2\cos\psi_0\cos\varphi_0-\overline{a}_{22}\varphi^2+2\overline{a}_{13}\psi\vartheta\cos\varphi_0-2\overline{a}_{11}\psi\varphi-2\overline{a}_{23}\vartheta\varphi\sin\varphi_0\right)\Big)+$$

$$+H_3\left(\overline{a}_{23}+(\overline{a}_{13}\psi+\overline{a}_{33}\vartheta\cos\psi_0)+\frac{1}{2}\left(-\overline{a}_{23}\psi^2-\overline{a}_{23}\vartheta^2+2\overline{a}_{33}\psi\vartheta\sin\psi_0\right)\right)\Big]\Big\}$$

Re-arranging the above integral of energy in the following format:

$$\frac{1}{2}(A\overline{p}^2+B\overline{q}^2+C\overline{r}^2)+\frac{1}{2}\omega_0{}^2(B-C)\left(A_{\psi\psi}\overline{\psi}^2+A_{\vartheta\vartheta}\overline{\vartheta}^2+A_{\varphi\varphi}\overline{\varphi}^2+2A_{\psi\vartheta}\overline{\psi}\overline{\vartheta}+2A_{\psi\varphi}\overline{\psi}\overline{\varphi}+2A_{\vartheta\varphi}\overline{\vartheta}\overline{\varphi}\right)+\Sigma=const.$$

(7.10)

Where:

$$A_{\psi\psi}=\nu\,(a_{11}^2-a_{21}^2)+(a_{13}^2-a_{23}^2)+H_1a_{21}+H_2a_{22}+H_3a_{23}$$

$$A_{\vartheta\vartheta}=(3+\cos^2\psi_0)(1-\nu\,\sin^2\varphi_0)\cos 2\vartheta_0-\frac{1}{4}\nu\sin 2\psi_0\,cos\vartheta_0\sin 2\varphi_0+$$
$$+(H_1\sin\varphi_0+H_2\cos\varphi_0)\cos\psi_0\cos\vartheta_0+H_3a_{23}$$

$$A_{\varphi\varphi}=\nu\,[(a_{22}^2-a_{21}^2)-3(a_{32}^2-a_{31}^2)]+H_1a_{21}+H_2a_{22} \tag{7.11}$$

$$A_{\psi\vartheta}=-\frac{1}{2}\sin 2\psi_0\sin 2\vartheta_0+\nu\,(a_{11}a_{23}+a_{13}a_{21})-\sin\psi_0(H_1a_{31}+H_2a_{32}+H_3a_{33})$$

$$A_{\psi\varphi}=\nu\,(a_{11}a_{22}+a_{12}a_{21})-H_1a_{12}+H_2a_{11}$$

$$A_{\vartheta\varphi}=-\frac{3}{2}\nu\,\sin 2\vartheta_0\sin 2\varphi_0+\nu\,(a_{21}\cos\varphi_0+a_{22})a_{23}-(H_1\cos\varphi_0-H_2\sin\varphi_0)a_{23}$$

Symbol Σ designates the terms of higher than the second order of smallness relative to $\overline{\psi},\overline{\vartheta},\overline{\varphi}$.

Then, if the quadratic form:

$$\frac{1}{2}(A\overline{p}^2+B\overline{q}^2+C\overline{r}^2)+\frac{1}{2}\omega_0^{\,2}(B-C)\left(A_{\psi\psi}\overline{\psi}^2+A_{\vartheta\vartheta}\overline{\vartheta}^2+A_{\varphi\varphi}\overline{\varphi}^2+2A_{\psi\vartheta}\overline{\psi}\,\overline{\vartheta}+2A_{\psi\varphi}\overline{\psi}\,\overline{\varphi}+2A_{\vartheta\varphi}\overline{\vartheta}\,\overline{\varphi}\right) \quad (7.12)$$

is definitely positive for some equilibrium solutions, then for these solutions the sufficient conditions of stability will be fulfilled.

Numerically the stability is calculated as follows, first we set the system parameters (H_1, H_2, H_3 and ν), then calculate the real roots with equation (5.21), then for each real root and using the first equation of system (5.18) is calculated the two corresponding values of y_1 and y_2. After is tested the values y_1 and y_2 into the second equation of system (5.18), one of the values does not satisfy the equation and is disregarded. Now, is known the values x and y, and is rather easy to calculate the two values of a_{33} from the third equation of system (5.16), and then the values of a_{31} and a_{32} with the system of equations (5.14), and therefore find the two sets of values for a_{31}, a_{32} and a_{33}. After is necessary to prove the orthogonally conditions using the system of equations (3.10), and after find the two sets of values $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}$ and a_{23} with the help of the system of equations (5.15).
To achieve the coefficients of equations described in (7.11) of the quadratic form (7.10) we should define values of $\sin(\psi)$, $\cos(\psi)$, $\sin(\vartheta)$, $\cos(\vartheta)$, $\sin(\varphi)$, $\cos(\varphi)$, of the Euler angles such $(0<\psi<2\pi),(0<\vartheta<\pi),(0<\varphi<2\pi)$.

Using direction cosines (3.7) it can be defined $tg(\varphi)=\dfrac{a_{31}}{a_{32}}=\dfrac{x_1}{y_1}$. Then because φ is a periodic function with period π, if the value (or trend line) is negative we define $\varphi+\pi$ just for a more practical graphical representation.

The angle $\vartheta(0\le\vartheta\le\pi)$ can be defined from the last equation of (3.7) $(\cos\vartheta=a_{33})$. The angle $\psi(0\le\psi\le 2\pi)$ is uniquely determined with the help of relations $\sin\psi=\dfrac{a_{13}}{\sin\vartheta}$ and $\cos\psi=-\dfrac{a_{23}}{\sin\vartheta}$. Thus, it can uniquely be determined the orientation angles $\psi_0, \vartheta_0, \varphi_0$ and calculate the coefficients of quadratic form (7.12) and finally check if they are positive defined in accordance with (7.4).

Due to $0\le\varphi\le 2\pi$, each real root $tg\varphi=\dfrac{x_1}{y_1}$ corresponds to two values of angle φ (φ_1 and $\varphi_2=\varphi_1+\pi$). From the features of the quadratic form coefficients (7.11) it follows that the sufficient stability conditions (7.4) for the values φ_1 and φ_2 are equal.
In addition, it is possible to prove that conditions (7.4) do not depend on the sign of the parameters H_1, H_2, H_3. Hence, the numerical analysis of the sufficient stability conditions of equilibrium solutions of equations (5.11), are possible to achieve with only positive values of H_1, H_2, H_3 and $0<\nu<1$, and also a single value of φ (φ_1 or φ_2) corresponding to each real root of equations (5.18) and (5.21).

Each figure in Appendix D present the dependence of φ from H_1 and a variation in the parameters ν, H_3 and H_2. Dashed lines indicate curves where stability conditions (7.4) are valid and the full lines where stability conditions are not fulfilled.

Since the sufficient stability conditions (7.4) for the values φ_1 and $\varphi_1 + \pi$ $(0 \le \varphi \le 2\pi)$ are the same, the numerical results in the figures from Appendix D are presented only for the range $0 \le \varphi \le \pi$.
The stability calculations were made for the inertia parameters $\nu = 0.01$, $\nu = 0.1$, $\nu = 0.2$, $\nu = 0.3$, $\nu = 0.4$, $\nu = 0.5$, $\nu = 0.6$, $\nu = 0.7$, $\nu = 0.8$, $\nu = 0.9$ and $\nu = 0.99$. The results are shown in figures from Appendix D.

From the analysis of all numerical calculations it follows that for the H_3 parameter values less then $1-\nu$ and for small H_1, H_2 there are 24 equilibrium arrangements of which 4 are stable. There are also 2 stable equilibria for $\nu > 0.5$ and $H_3 \ge 1-\nu$.
When the values of parameter H_1 increase, sequential mergers of equilibrium curves occurs at points that correspond to the points of intersection of the straight lines $H_2 = const.$ with the borders of regions with fixed number of equilibria. For example, in figure where $(\nu = 0.2, H_3 = 0.4)$ there are 4 points of intersection of the straight line $H_2 = 0.1$ with the borders of regions with fixed number of equilibria $H_1 = 0.039, H_1 = 0.17, H_1 = 0.531$ and $H_1 = 2.077$; in the figure $(\nu = 0.2, H_2 = 0.1, H_3 = 0.4)$ these points the equilibrium curves are merging.
When values of parameters H_1, H_2, H_3 of the gyrostatic torque are greater than or equal to 4, there are 8 equilibrium solutions, but only 2 of them are stable. For large values of the parameters H_1, H_2, H_3 the equilibrium values of φ are close to the trivial solutions, where some axis of the orbital coordinate system and some axes of the body coordinate system coincide.

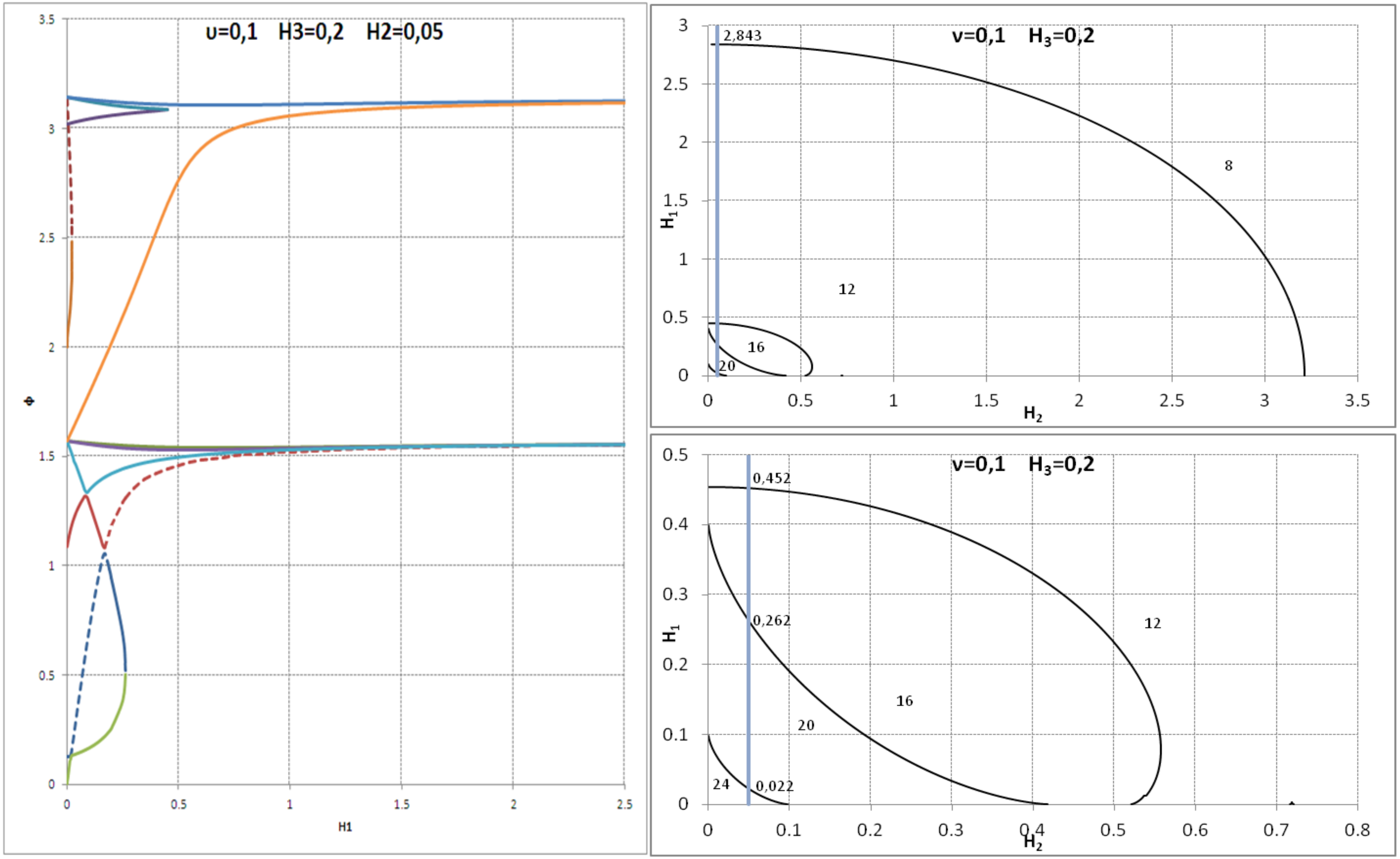


Figure 7.1 – Stability and respective equilibria picture for ν =0.1, H_3=0.2 and H_2=0.05

From the stability pictures of Appendix D we draw the following observations:

ν =Constant, H_3=Constant and H_2=Varying

- With the increasing of H_2 the stable line decrease its maximum value of φ, also the unstable lines decrease in φ with the increase of H_2. The regions of 12 and 10, described as "protuberances" with the increase of H_2 decrease in H_1 and φ until completely disappear.

ν =Constant, H_3=Varying and H_2=Constant

- With the increasing of H_3 the position of the stable and unstable lines remain approximately constant in φ with the exception of the "protuberances". The upper "protuberance" decrease in size with the increasing of H_3 until it finally disappear. Meanwhile the lower "protuberance" with the increase of H_3 increases in size of φ and H_1 up to a certain point then starts decreasing in φ and H_1 until it finally disappears.

ν =Varying, H_3=Constant and H_2=Constant

- With the increasing of ν the stable and the unstable lines remain approximately constant in φ values. The relevant point with the increase of the parameter ν is that the "protuberances" increase in size of H_1 and φ.

8 AXIALLY SYMMETRIC GYROSTAT EQUILIBRIA ANALYSIS

An axially symmetric gyrostat satellite is like in the first approach of this work, a solid body with statically and dynamically balanced rotors located inside it, however unlike in the first part, this kind of axially symmetric spacecraft has an inertial symmetry axis. In this second approach two of the three principal central moments of inertia will be the same.

The study of the axially symmetric case is important to better understand what happens when ν assumes values very close to zero or very close to one, and also important, to validate our computer simulation model, because since equations were programmed on the software, small errors could occur, but an as can be seen further ahead, the general case model was validated setting ν very close to zero and to one and compare with the well-known case of axially symmetrical case.

The analysis of the axially symmetrical cases can also improve the determination of the boundary conditions and the behaviour of this kind of systems; it can also describe better the transitory solutions between the almost axially symmetric and the completely axially symmetric cases.
Relating this chapter with the study of the general case, it can be seen that with $\nu = \frac{B-A}{B-C}$ and for the general case $B > A > C$ the value of ν can be described as $0 < \nu < 1$. From this arise a very important characteristic, the case where $\nu = 0.01$ is very close to the situation of $\nu = 0$ (axially symmetrical satellite were $A = B$), and the case were $\nu = 0.99$ is very close to the situation of ν =1 (the axially symmetrical satellite where $A = C$).

As in the general case study of equilibria described in chapter 5, the study of equilibria for the axially symmetric gyrostat can also be defined as the identification of the conditions under which the considered system is either at rest or in uniform motion. Each such equilibria position suggests a potential attitude stabilization scheme, especially if the equilibrium is stable.

Because the axially symmetric case is a simplification of the general case, we can start from equation (5.6) and develop it to reflect the axially symmetric case.
For the studied axisymmetric case is defined as $A \neq B = C$ the system of equations (5.6) is transformed into:

$$\begin{cases} 4(A-B)a_{21}a_{31} + (h_1 a_{31} + h_2 a_{32} + h_3 a_{33}) = 0 \\ (A-B)a_{11}a_{31} = 0 \\ (A-B)a_{11}a_{21} + (h_1 a_{11} + h_2 a_{12} + h_3 a_{13}) = 0 \end{cases} \tag{8.1}$$

Replacing into (8.1) the expressions for direction cosines from (8.2), is obtained three equations with unknowns α_0, β_0 and γ_0. A convenient method of closing equations (8.1) consists to add six conditions of orthogonally of the direction cosines (3.10).

$$\begin{cases} a_{11} = \cos\alpha\cos\beta \\ a_{12} = \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ a_{13} = \sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma \\ a_{21} = \sin\beta \\ a_{22} = \cos\beta\cos\gamma \\ a_{23} = -\cos\beta\sin\gamma \\ a_{31} = -\sin\alpha\cos\beta \\ a_{32} = \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma \\ a_{33} = \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma \end{cases} \qquad (8.2)$$

Thus, proceeding to investigate the equilibrium positions of an axially symmetric gyrostat satellite, using systems (8.1) and system of equations (3.10) and (3.11) depending on parameters $A-B$, h_1, h_2, h_3 (the direct problem). From this arises two consequences from the second equation of system of equations (8.1), or $a_{11} = 0$ or $a_{31} = 0$.

For the first case:

$$\begin{cases} a_{11} = 0 \\ 4(A-B)a_{21}a_{31} + h_1 a_{31} + h_2 a_{32} + h_3 a_{33} = 0 \\ h_2 a_{12} + h_3 a_{13} = 0 \\ a_{12}^2 + a_{13}^2 = 1 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \\ a_{12}a_{22} + a_{13}a_{23} = 0 \\ a_{12}a_{32} + a_{13}a_{33} = 0 \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0 \end{cases} \qquad (8.3)$$

For the second case:

$$\begin{cases} a_{31} = 0 \\ h_2 a_{32} + h_3 a_{33} = 0 \\ (A-B)a_{11}a_{21} + h_1 a_{11} + h_2 a_{12} + h_3 a_{13} = 0 \\ a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ a_{32}^2 + a_{33}^2 = 1 \\ a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0 \\ a_{12}a_{32} + a_{13}a_{33} = 0 \\ a_{22}a_{32} + a_{23}a_{33} = 0 \end{cases} \qquad (8.4)$$

From the third and fourth equations on (8.3) is obtained:

$$\begin{cases} a_{12} = \pm \dfrac{h_3}{\sqrt{h_2^2 + h_3^2}} \\ a_{13} = \pm \dfrac{h_2}{\sqrt{h_2^2 + h_3^2}} \end{cases}$$

Taking into account the above two equations and the adapted equations (3.10) and (3.11), the system of equations (8.3) can be presented as:

$$\begin{cases} a_{11} = 0 \\ a_{12} = \pm \dfrac{h_3}{\sqrt{h_2^2 + h_3^2}} \\ a_{13} = \pm \dfrac{h_2}{\sqrt{h_2^2 + h_3^2}} \\ a_{22} = -a_{13}a_{31} \\ a_{23} = a_{12}a_{31} \\ a_{32} = a_{13}a_{21} \\ a_{33} = -a_{12}a_{21} \\ a_{21}^2 + a_{31}^2 = 1 \\ 4(A - B)a_{21}a_{31} + h_1 a_{31} \pm a_{21}\sqrt{h_2^2 + h_3^2} = 0 \end{cases} \qquad (8.5)$$

Determining then the direction cosines a_{21} and a_{31} from the last two equations of (8.5), the solutions from system (8.3) are achieved. Considering the last two equations on (8.5), it can be re-written in the following form:

$$\begin{cases} 4a_{21}a_{31} + a_{31}m \pm a_{21}n = 0 \\ a_{21}^2 + a_{31}^2 = 1 \end{cases} \qquad (8.6)$$

Where:

$m = \dfrac{h_1}{(A - B)}$ and $n = \dfrac{\sqrt{h_2^2 + h_3^2}}{(A - B)}$

Thus, system (8.6) can again be re-written in a slightly different form:

$$\begin{cases} a_{31} = \dfrac{\pm n a_{21}}{4a_{21} + m} \\ 16a_{21}^4 + 8m a_{21}^3 + (m^2 + n^2 - 16)a_{21}^2 - 8m a_{21} - m^2 = 0 \end{cases} \qquad (8.7)$$

From system (8.7) it follows that the second equation can have no more than four real roots a_{21} depending on m and n. Considering the first equation, the number of real solutions a_{21}, a_{31} to system (8.7) cannot therefore exceed 8.

The first equation of system (8.6) represents the equation of an hyperbola, with one branch passing through the origin of the coordinate system ($a_{21}=0$, $a_{31}=0$) in the plane of variables a_{21}, a_{31}, the second equation describes a unit circle in this plane.

The number of real solutions to system (8.6) depends on the character of the intersections of the hyperbola branches with the circle.

It is clear that the two branches of the hyperbola, which pass through the origin of coordinates, will obviously intersect with the circle at four points. If the two other branches also intersect with the circle too, we have four additional solutions. In the case when hyperbola branches touch the circle, four solutions merge into two (there are two multiple roots). Three different variants of mutual positions of hyperbola branches and the circle are shown in figures 8.1, 8.2, and 8.3. Thus, system (8.6) and, hence, system (8.5) have either eight or four solutions.

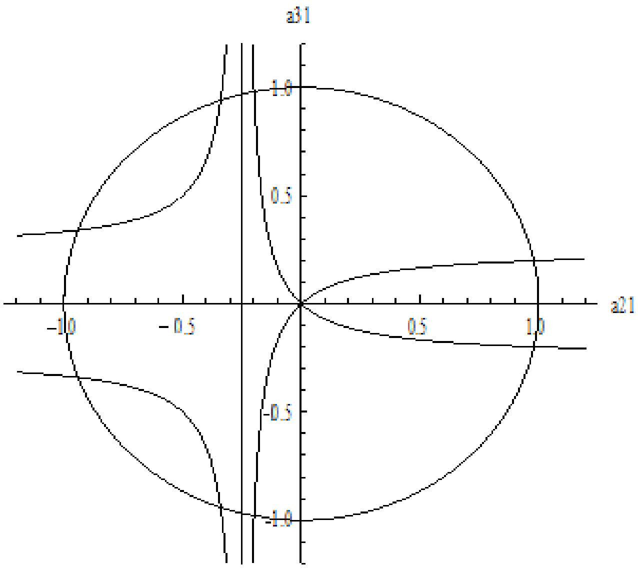


Figure 8.1 – Mutual positions of the circle and hyperbole from system (8.6) were $m=n=1$, which represents 8 real solutions

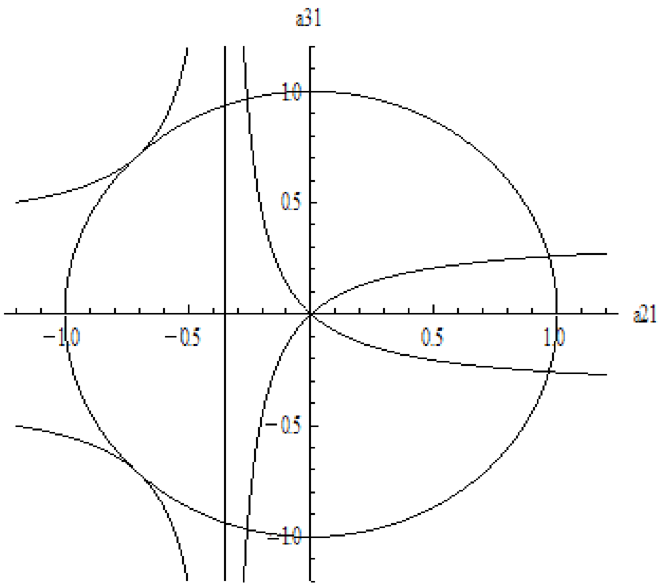


Figure 8.2 – Mutual positions of the circle and hyperbole from system (8.6) were $m=n=\sqrt{2}$, which represents 6 real solutions

For picture 8.3, the value for $m = n = \sqrt{2}$ satisfy the astroid equation (8.6).

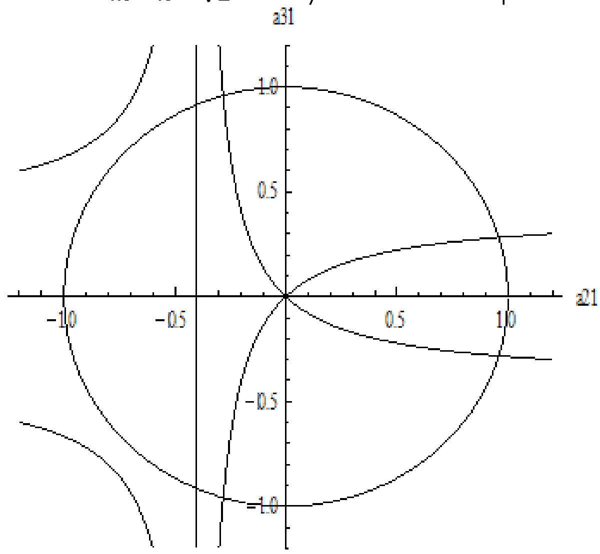


Figure 8.3 – Mutual positions of the circle and hyperbole from system (8.6) were $m = n = 1.6$, which represents 4 real solutions

Let's determine the boundaries in the parameter plane (m, n) separating regions with different numbers of solutions to system (8.6). Bifurcation points are the points of the plane (m, n) belonging simultaneously to branches of hyperbolas, which do not pass through the origin of coordinates and circle; tangents to the hyperbola and circle coincide at bifurcation points. The condition of coincidence of the tangents has the form $\frac{\partial a_{31}}{\partial a_{21}} = \frac{-4a_{31} \pm n}{(4a_{21} + m)} = -\frac{a_{21}}{a_{31}}$ or $4\left(a_{21}^2 - a_{31}^2\right) + ma_{21} \pm na_{31} = 0$
(8.8)

From the first equation of (8.6) it can be obtained:

$$a_{31} = \frac{\pm na_{21}}{4a_{21} + m}$$

Replacing the above expression for a_{31} into the second equation of (8.6) and equation (8.8), the following system can be obtained:

$$\begin{cases} \dfrac{n^2 a_{21}^2}{(4a_{21} + m)^2} = 1 - a_{21}^2 \\ \dfrac{mn^2}{(4a_{21} + m)^2} = -(4a_{21} + m) \end{cases} \qquad (8.9)$$

Dividing the left side of the first equation of system (8.9) by the left side of the second equation of system (8.9) and the right side of the first equation by the right side of the second equation of system (8.9) we get:

$$m^{2/3} + n^{2/3} = 4^{2/3} \qquad (8.10)$$

The equation (8.10) represents an astroid, and together with equations (8.8) we can check that eight solutions exist inside the region $m^{2/3} + n^{2/3} \leq 4^{2/3}$, and four solutions in the region $m^{2/3} + n^{2/3} > 4^{2/3}$.
Also, the astroid equation (8.10) represents the tangents points between the hyperbole and circle from system of equations (8.6).

Investigating now the second case, from the second and sixth equations on (8.3) is obtained:

$$\begin{cases} a_{32} = \dfrac{\pm h_3}{\sqrt{h_2^2 + h_3^2}} \\ a_{33} = \dfrac{\pm h_2}{\sqrt{h_2^2 + h_3^2}} \end{cases}$$

Taking into account the previous equations and (3.10), (3.11), the system (8.4) can be presented as:

$$\begin{cases} a_{12} = -a_{21}a_{33} \\ a_{13} = a_{21}a_{32} \\ a_{22} = a_{11}a_{33} \\ a_{23} = -a_{11}a_{32} \\ a_{31} = 0 \\ a_{32} = \pm\dfrac{h_3}{\sqrt{h_2^2 + h_3^2}} \\ a_{33} = \pm\dfrac{h_2}{\sqrt{h_2^2 + h_3^2}} \\ 4(A-B)a_{11}a_{21} + h_1 a_{11} \pm a_{21}\sqrt{h_2^2 + h_3^2} = 0 \\ a_{11}^2 + a_{21}^2 = 1 \end{cases} \tag{8.11}$$

When determining the direction cosines a_{11} and a_{21} from the last two equations of (8.11), the solutions from system (8.4) are achieved. Considering the last two equations of (8.11), they can be re-written in the following form:

$$\begin{cases} a_{11}a_{21} + a_{11}m \pm a_{21}n = 0 \\ a_{11}^2 + a_{21}^2 = 1 \end{cases} \tag{8.12}$$

Where:

$m = \dfrac{h_1}{(A-B)}$ and $n = \dfrac{\sqrt{h_2^2 + h_3^2}}{(A-B)}$

Then can be re-written again in a slightly different format:

$$\begin{cases} a_{11} = \pm \dfrac{a_{21} n}{a_{21} + m} \\ a_{21}^4 + 2a_{21}^3 m + a_{21}^2 \left(n^2 + m^2 - 1\right) - 2a_{21} m - m^2 = 0 \end{cases} \qquad (8.13)$$

From system (8.13) it follows that the second equation can have no more than four real roots a_{21} depending on m and n. Regarding first equation, the number of real solutions a_{11}, a_{21} to system (8.13) cannot exceed 8.

For the first equation of system (8.12) for both signs before the last term in the left hand side represents the equation of a hyperbola, whose one branch passes through the origin of the coordinate system ($a_{11} = 0, a_{21} = 0$) in the plane of variables a_{11}, a_{21}, and the second equation describes in this plane a unit circle.
The number of real solutions of system (8.12) depends on the character of intersections of hyperbolas and the circle.
It is clear that two branches of hyperbolas, which pass through the origin of coordinates, obviously intersect with a circle at four points. If two other branches intersect with the circle too, we have four additional solutions. In the case when hyperbola branches touch the circle, four solutions merge in two (there are two multiple roots). Three different variants of mutual positions of hyperbola branches and the circle are shown in figures 8.4, 8.5, and 8.6. Thus, system (8.12) and, hence, system (8.11) have either eight or four solutions.

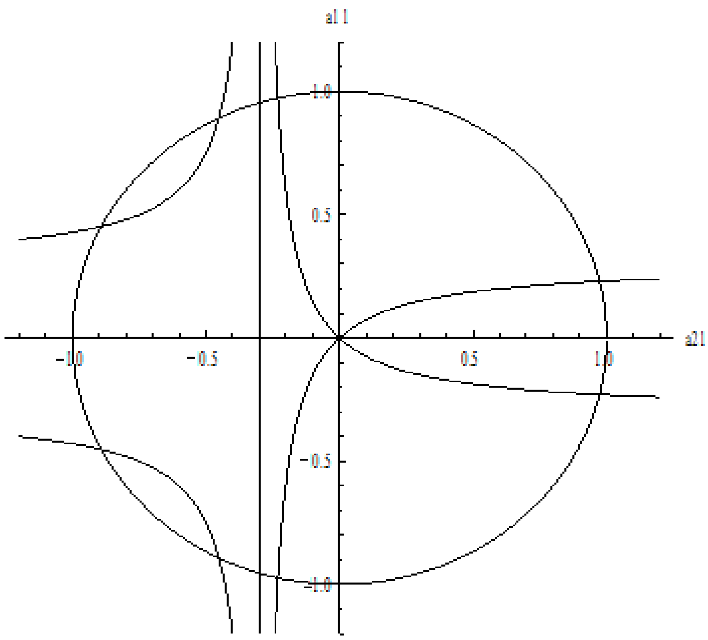


Figure 8.4 – Mutual positions of the circle and hyperbole from system (8.13) were $m = n = 0.3$, which represents 8 real solutions

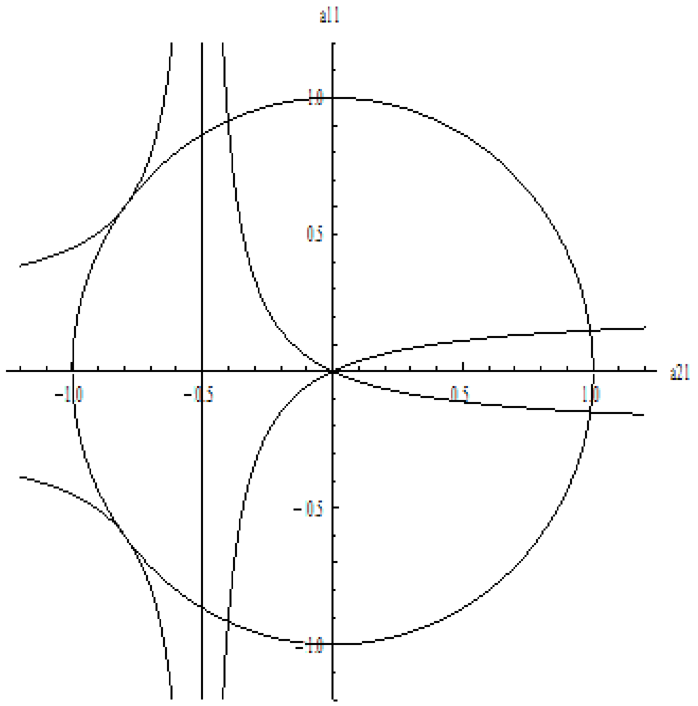


Figure 8.5 – Mutual positions of the circle and hyperbole from system (8.13) were $m = 0.5$ and $n = \left(1 - 0.5^{2/3}\right)^{3/2}$, which represents 6 real solutions

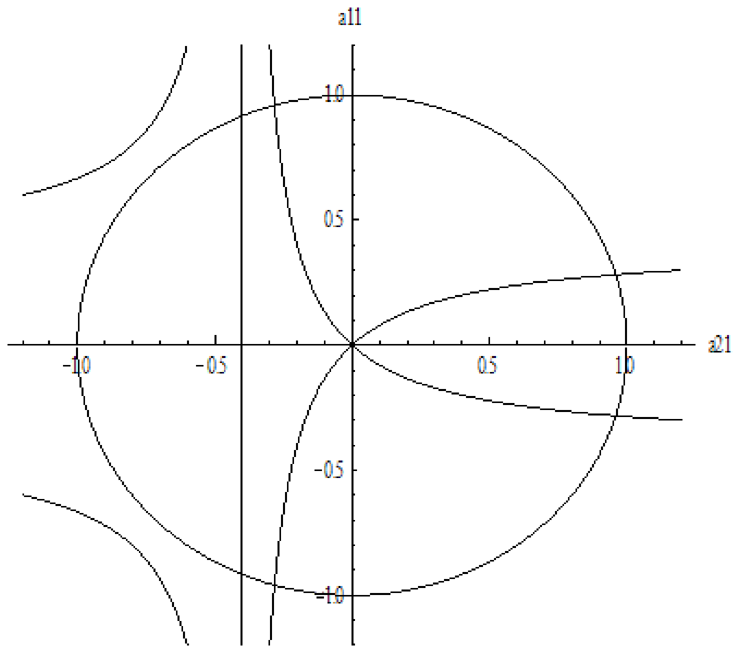


Figure 8.6 – Mutual positions of the circle and hyperbole from system (8.13) were $m = n = 0.4$, which represents 4 real solutions

Is now determined the boundaries in the parameter plane (m, n) which separates regions with different numbers of solutions to system (8.13). Bifurcation points are the points of the plane (m, n) belonging simultaneously to branches of hyperbolas, which do not pass through the origin of coordinates and circle; tangents to the hyperbola and circle coincide at bifurcation points. The condition of coincidence of the tangents has the form:

$$\frac{\partial a_{11}}{\partial a_{21}} = \frac{-a_{11} \pm n}{(a_{21}+m)} = -\frac{a_{21}}{a_{11}} \text{ or } \left(a_{21}^2 - a_{11}^2\right) + a_{21}m \pm a_{11}n = 0 \qquad (8.14)$$

From the first equation of (8.12) it can be obtained:

$$a_{11} = \pm \frac{a_{21}n}{a_{21}+m}$$

Replacing the previous expression for a_{11} into the second equation of (8.12) and equation (8.14), the following system can be obtained:

$$\begin{cases} \dfrac{a_{21}^2 n^2}{(a_{21}+m)^2} = 1 - a_{21}^2 \\ \dfrac{mn^2}{(a_{21}+m)^2} = -(a_{21}+m) \end{cases} \qquad (8.15)$$

Dividing the left side of the first equation of system (8.15) by the left side of the second equation of system (8.15) and the right side of the first equation by the right side of the second equation of system (8.15) we get:

$$m^{2/3} + n^{2/3} = 1 \qquad (8.16)$$

The equation (8.16) represents an astroid, and together with equations (8.15) it can be verified that exists 8 solutions inside the region $m^{2/3} + n^{2/3} \leq 1$, and then 4 solutions in the region $m^{2/3} + n^{2/3} > 1$.

A very interesting analysis can now be made comparing the axially symmetric case with the previous solutions for the general case. Taking into consideration the case $\nu = 0$ which corresponds to the case where $A = B$, we have the fixed regions of equilibria:

$$\begin{cases} h_1^2 + h_2^2 = \left(4^{2/3} - h_3^{2/3}\right)^3 \\ h_1^2 + h_2^2 = \left(1 - h_3^{2/3}\right)^3 \end{cases}$$

Taking into consideration the case $\nu = 1$ which corresponds to the case where $A = C$, we have the fixed regions of equilibria:

$$\begin{cases} h_2^{2/3} + \left(h_1^2 + h_3^2\right)^{1/3} = 4^{2/3} \\ h_2^{2/3} + \left(h_1^2 + h_3^2\right)^{1/3} = 1 \end{cases}$$

Comparing the above two extreme cases with the general case, it can be verified graphically in the following figure 8.7 and figure 8.8, that in the general case exists smaller regions, these regions corresponds to higher equilibria not available in the axially symmetric due to the parameter ν be different from zero and from one.
From the analysis of figure 8.7 and figure 8.8 the numerical model can also be validated. The resemblance between the axially symmetric and the asymmetrical are notorious.

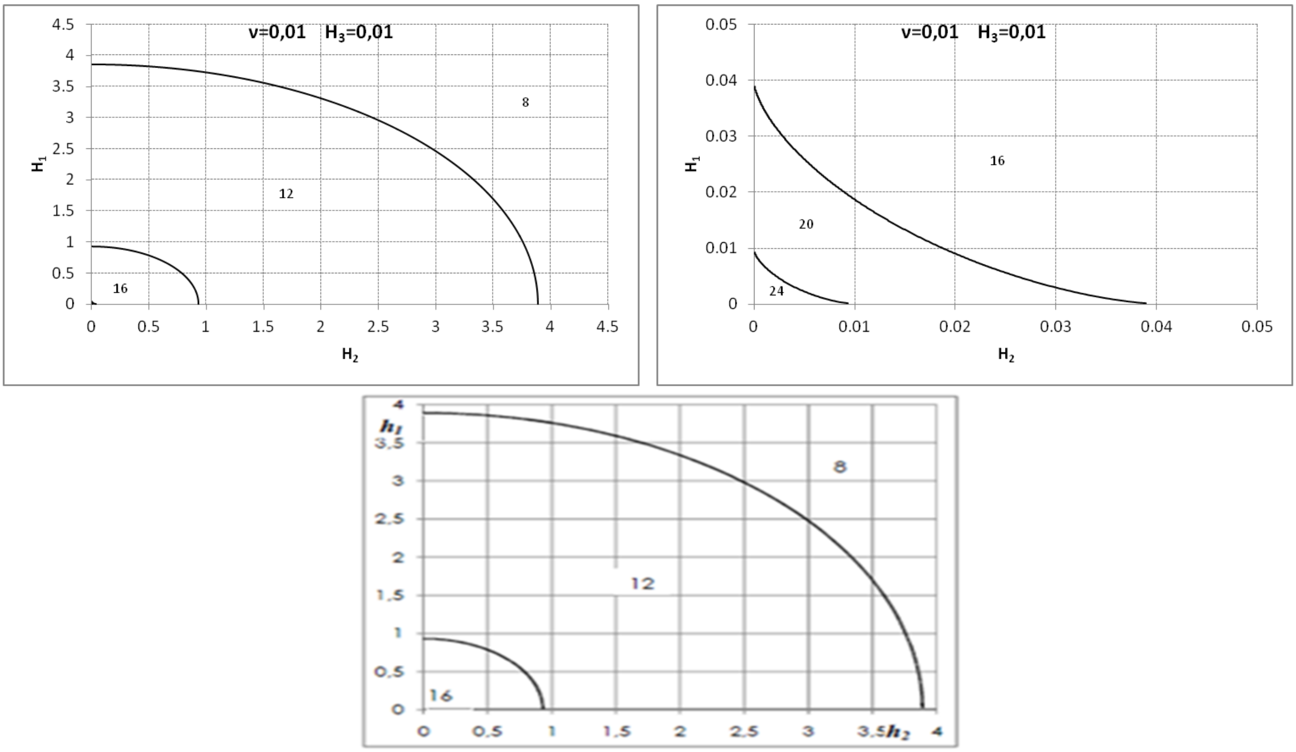


Figure 8.7 – Above the two Equilibria picture for the General case were $\nu = 0.01 / H_3 = 0.01$ and below the picture for Axially Simmetric case were $\nu = 0 / H_3 = 0.01$

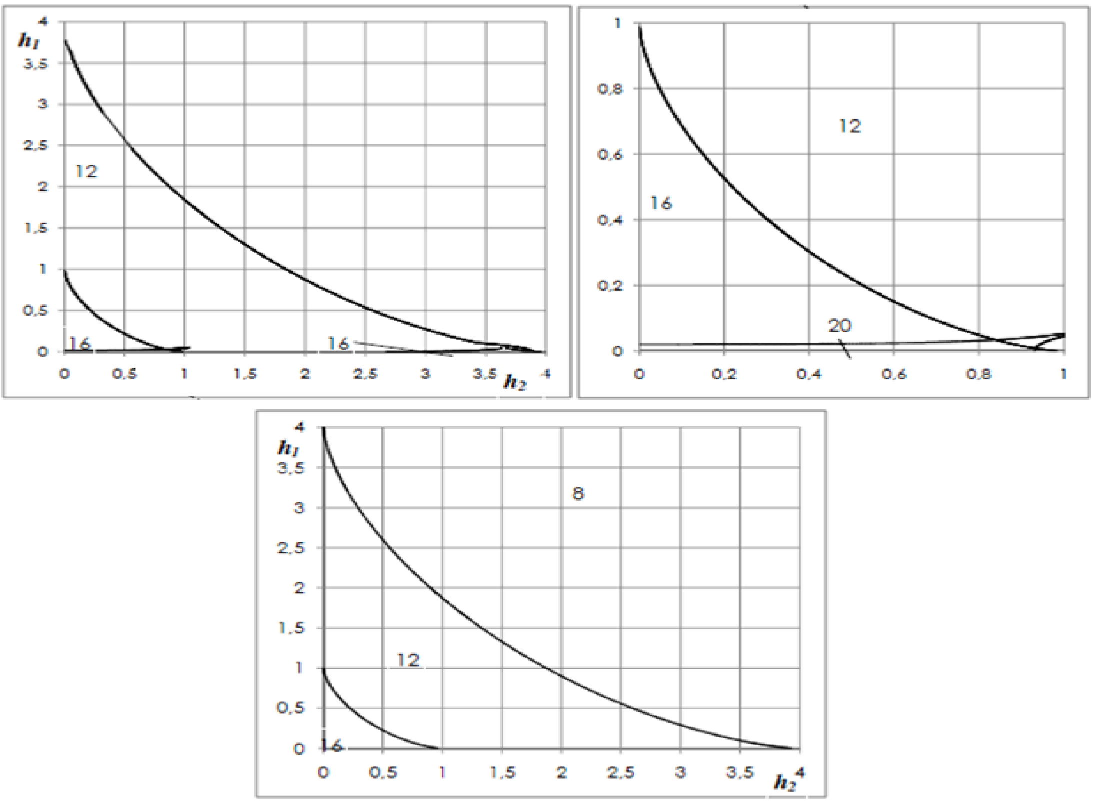


Figure 8.8 – Above the two Equilibria picture for the General case were $\nu = 0.99 / H_3 = 0.01$ and below the picture for Axially Simmetric case were $\nu = 1 / H_3 = 0.01$

The astroids (8.10) and (8.16) separating in the plane m, n three regions with different numbers of equilibrium positions of an axially symmetric gyrostat satellite represent the equations of the bifurcation points. There are 16 equilibria solutions in the region $m^{2/3} + n^{2/3} \leq 1$, 12 equilibria solutions in $1 < m^{2/3} + n^{2/3} \leq 4^{2/3}$ and finally 8 equilibria solutions in the region $m^{2/3} + n^{2/3} > 4^{2/3}$.
Taking into account equations (8.10) and equations (8.16) we can plot the following picture:

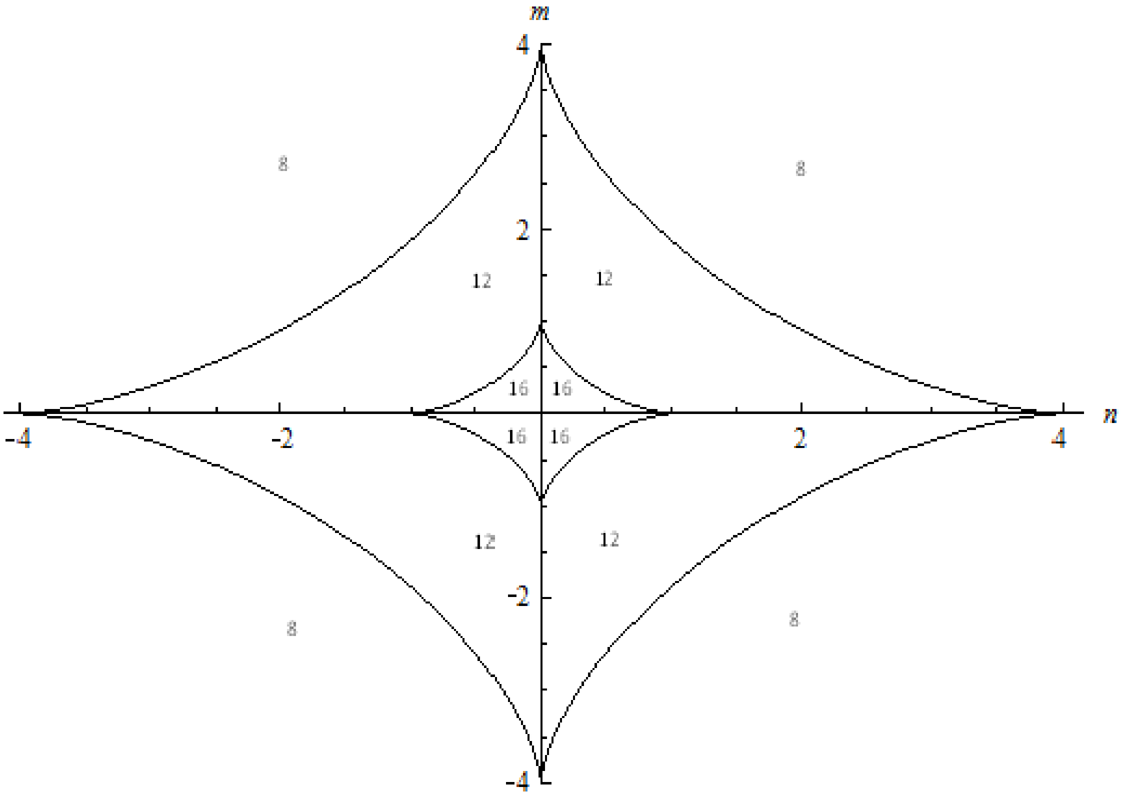


Figure 8.9 – Astroids (9.14) and (9.26) which represents the bifurcation equations of our system

9 SUFFICIENT CONDITIONS OF STABILITY OF EQUILIBRIA ORIENTATIONS FOR A AXIALLY SYMMETRIC GYROSTAT

As in chapter 7, we will use the same Lyapunov's method and the integral of energy as a Lyapunov's function, and for an axially symmetric satellite the integral of energy is:

$$\overline{H} = \frac{1}{2}\left[A\overline{p}^2 + B\left(\overline{q}^2 + \overline{r}^2\right)\right] + \frac{1}{2}(A-B)\left(3a_{31}^2 - a_{21}^2\right) - \left(h_1 a_{21} + h_2 a_{22} + h_3 a_{23}\right) = const. \tag{9.1}$$

Lets present α, β and γ as follows:

$$\alpha = \alpha_0 + \overline{\alpha} \qquad \beta = \beta_0 + \overline{\beta} \qquad \gamma = \gamma_0 + \overline{\gamma}$$

Where $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ are small deviations from the equilibrium position $\alpha = \alpha_0 = const.$, $\beta = \beta_0 = const.$ and $\gamma = \gamma_0 = const.$, satisfying the system of equations (3.10) and (8.1). Then the integral of energy (9.1) can be presented as follows:

$$\frac{1}{2}\left[A\overline{p}^2 + B\left(\overline{q}^2 + \overline{r}^2\right)\right] + \frac{1}{2}\Big\{3(B-A)\left(a_{31}^2 - a_{11}^2\right)\overline{\alpha}^2 + \left[(B-A)\left(3\sin^2\alpha_0 + 1\right)\cos 2\beta_0 + h_1 a_{21} + h_2 a_{22} + h_3 a_{23}\right]\overline{\beta}^2 - 2a_{21}\left(h_2\sin\gamma_0 + h_3\cos\gamma_0\right)\overline{\beta}\overline{\gamma} + \left(h_2 a_{22} + h_3 a_{23}\right)\overline{\gamma}^2\Big\} + \Sigma = const. \tag{9.2}$$

Where Σ represents the terms of higher than second order of smallness with respect to $\overline{\alpha}, \overline{\beta}$ and $\overline{\gamma}$.

For case 1, $a_{11} = \cos\alpha_0 \sin\alpha_0 = 0$ and hence, $\cos\alpha_0 = 0$. From the second and third equations of system (8.3) the following relations are obtained:

$$\begin{cases} 4(A-B)\sin\beta_0\cos\beta_0 + h_1\cos\beta_0 - \sin\beta_0\left(h_2\cos\gamma_0 - h_3\sin\gamma_0\right) = 0 \\ h_2\sin\gamma_0 + h_3\cos\gamma_0 = 0 \end{cases} \tag{9.3}$$

Eliminating with the help of system (9.3), expressions $h_2\sin\gamma_0 + h_3\cos\gamma_0$ and $h_2\cos\gamma_0 - h_3\sin\gamma_0$ from the integral of energy (9.2) the following expression is obtained:

$$\frac{1}{2}\left[A\overline{p}^2 + B\left(\overline{q}^2 + \overline{r}^2\right)\right] + \frac{3}{2}(B-A)\cos^2\beta_0\overline{\alpha}^2 + \frac{1}{2}\left[\frac{4(A-B)\sin^3\beta_0 + h_1}{\sin\beta}\right]\overline{\beta}^2 + \\ + \frac{1}{2}\frac{\left[4(A-B)\sin\beta_0 + h_1\right]\cos^2\beta_0}{\sin\beta_0}\overline{\gamma}^2 + \Sigma = const. \tag{9.4}$$

From (9.4) is obtained the sufficient stability conditions of equilibrium position for case 1.

$$\begin{cases} (B-A) > 0 \\ \left[4(A-B)\sin^3\beta_0 + h_1\right]\sin\beta_0 > 0 \\ \left[4(A-B)\sin\beta_0 + h_1\right]\sin\beta_0 > 0 \end{cases}$$

Because $(B-A)>0$ and $m=\frac{h_1}{(A-B)}$ the above system is transformed into:

$$\begin{cases}(B-A)>0\\(m+4\sin^3\beta_0)\sin\beta_0<0\\(m+4\sin\beta_0)\sin\beta_0<0\end{cases}\tag{9.5}$$

For case 2, $a_{31}=-\sin\alpha_0\cos\beta_0=0$ and hence, $\sin\alpha_0=0$. From the second and third equations of system (8.4) the following relations are obtained:

$$\begin{cases}(A-B)\cos\beta_0\sin\beta_0+h_1\cos\beta_0-\sin\beta_0(h_2\cos\gamma_0-h_3\sin\gamma_0)=0\\h_2\sin\gamma_0+h_3\cos\gamma_0=0\end{cases}\tag{9.6}$$

Eliminating with the help of system (9.6), expressions $h_2\sin\gamma_0+h_3\cos\gamma_0$ and $h_2\cos\gamma_0-h_3\sin\gamma_0$ from the integral of energy (9.2) the following expression is obtained:

$$\frac{1}{2}\left[A\bar{p}^2+B(\bar{q}^2+\bar{r}^2)\right]+\frac{3}{2}(A-B)\cos^2\beta_0\bar{\alpha}^2+\frac{1}{2}\left[\frac{(A-B)\sin^3\beta_0+h_1}{\sin\beta}\right]\bar{\beta}^2+$$
$$+\frac{1}{2}\frac{[(A-B)\sin\beta_0+h_1]\cos^2\alpha_0}{\sin\beta_0}\bar{\gamma}^2+\Sigma=const.\tag{9.7}$$

From (9.7) is obtained the sufficient stability conditions of equilibrium position for case 2.

$$\begin{cases}(A-B)>0\\\left[(A-B)\sin^3\beta_0+h_1\right]\sin\beta_0>0\\\left[(A-B)\sin\beta_0+h_1\right]\sin\beta_0>0\end{cases}$$

Introducing the parameter $m=\frac{h_1}{A-B}$ the above system is transformed into:

$$\begin{cases}(A-B)>0\\(m+\sin^3\beta_0)\sin\beta_0>0\\(m+\sin\beta_0)\sin\beta_0>0\end{cases}\tag{9.8}$$

The stability conditions (9.5) for case 1 were $a_{11}=0$ represents the following curves:

$$\begin{cases}m+4\sin^3\beta_0=0\\m+4\sin\beta_0=0\end{cases}\Leftrightarrow\begin{cases}m=-4\sin^3\beta_0\\m=-4\sin\beta_0\end{cases}$$

Which represent the following curves:

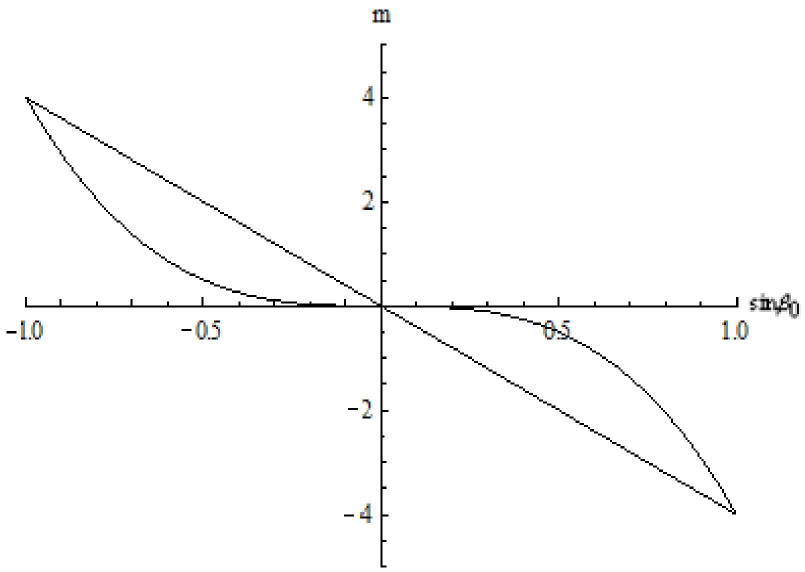


Figure 9.1 – Curves for the stability conditions of case 1 were a_{11}=0

Investigating the following conditions:

- For $0 < \sin\beta_0 < 1$

$$\begin{cases} m + 4\sin^3\beta_0 < 0 \\ m + 4\sin\beta_0 < 0 \end{cases}$$

- For $-1 < \sin\beta_0 < 0$

$$\begin{cases} m + 4\sin^3\beta_0 > 0 \\ m + 4\sin\beta_0 > 0 \end{cases}$$

Resuming we have:

Case 1 - a_{11}=0	$-1 < \sin\beta_0 < 0$	$0 < \sin\beta_0 < 1$
	$m + 4\sin^3\beta_0 > 0$	$m + 4\sin\beta_0 < 0$
	$m + 4\sin\beta_0 > 0$	$m + 4\sin^3\beta_0 < 0$

Table 9-1: Stability Conditions for Case 1 - a_{11}=0

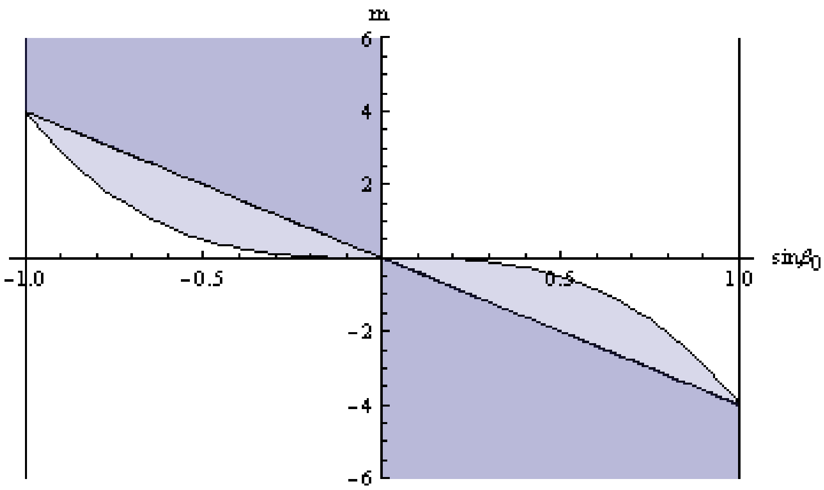


Figure 9.2 – Stability Conditions for case 1 were a_{11}=0

With the help figure 9.2 we can see that the mutual stability conditions are:

Case 1 - a_{11}=0	$-1 < \sin\beta_0 < 0$	$0 < \sin\beta_0 < 1$
	$m + 4\sin\beta_0 > 0$	$m + 4\sin\beta_0 < 0$

Table 9-2: Stability Conditions for Case 1 - a_{11}=0

The stability conditions (9.8) for case 2 were a_{31}=0 represents the following curves:

$$\begin{cases} m + \sin^3\beta_0 = 0 \\ m + \sin\beta_0 = 0 \end{cases} \Leftrightarrow \begin{cases} m = -\sin^3\beta_0 \\ m = -\sin\beta_0 \end{cases}$$

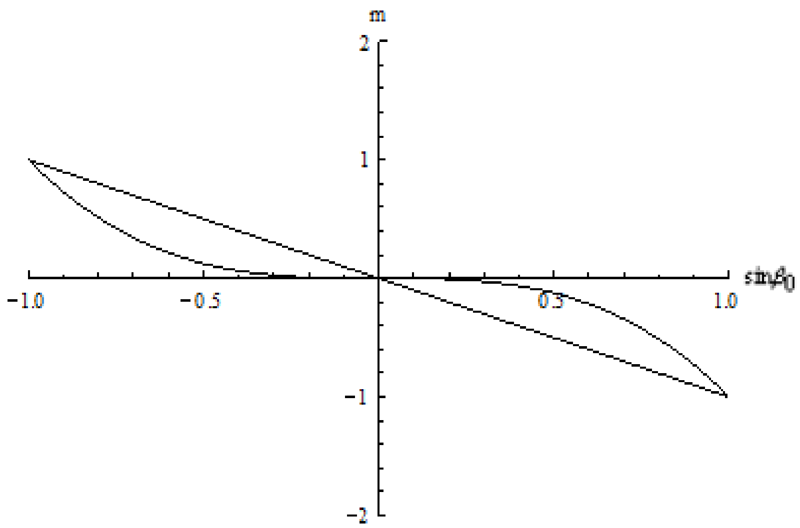


Figure 9.3 – Curves for the stability conditions of case 2 were a_{31}=0

Investigating the following conditions:

- For $0 < \sin\beta_0 < 1$

$$\begin{cases} m + \sin^3\beta_0 > 0 \\ m + \sin\beta_0 > 0 \end{cases}$$

- For $-1 < \sin\beta_0 < 0$

$$\begin{cases} m + \sin^3\beta_0 < 0 \\ m + \sin\beta_0 < 0 \end{cases}$$

Resuming we have:

Case 2 - a_{31}=0	$-1 < \sin\beta_0 < 0$	$0 < \sin\beta_0 < 1$
	$m + \sin^3\beta_0 < 0$	$m + \sin\beta_0 > 0$
	$m + \sin\beta_0 < 0$	$m + \sin^3\beta_0 > 0$

Table 9-3: Stability Conditions for Case 2 - a_{31}=0

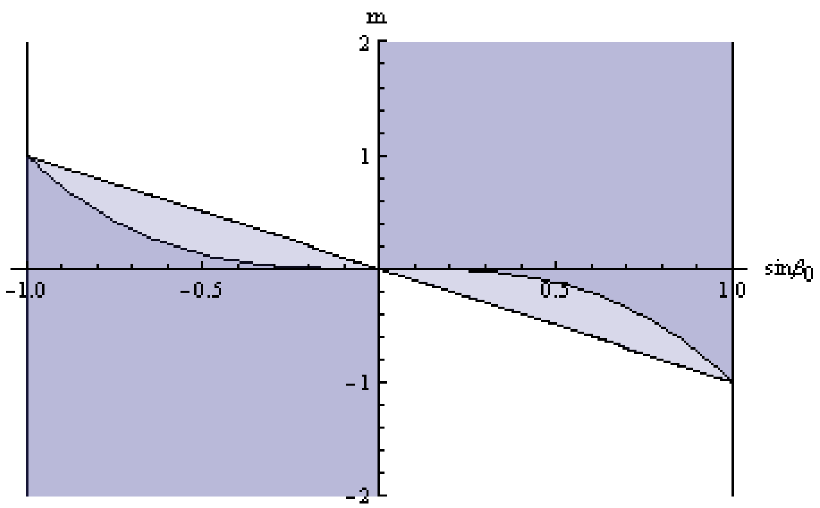


Figure 9.4 – Stability Conditions for case 2 were a_{31}=0

With the help of figure 9.4 we can see that the mutual stability conditions are:

Case 2 - a_{31}=0	$-1 < \sin\beta_0 < 0$	$0 < \sin\beta_0 < 1$
	$m + \sin^3\beta_0 < 0$	$m + \sin^3\beta_0 > 0$

Table 9-4: Stability Conditions for Case 2 - a_{31}=0

With the help of figure 9.2 and figure 9.4, it can be verified that the conditions which satisfy the stability conditions of an axially symmetric gyrostat are:

	$-1 < \sin\beta_0 < 0$	$0 < \sin\beta_0 < 1$
Case 1 - $a_{11}=0$	$m + \sin^3\beta_0 < 0$	$m + 4\sin\beta_0 < 0$
Case 2 - $a_{31}=0$	$m + 4\sin\beta_0 > 0$	$m + \sin^3\beta_0 > 0$

Table 9-5: Stability Conditions that satisfy an Axially Symmetric gyrostat

So the regions of fulfillment of the sufficient stability conditions of equilibrium orientations of an axially symmetric gyrostat satellite are illustrated by figure 9.5.

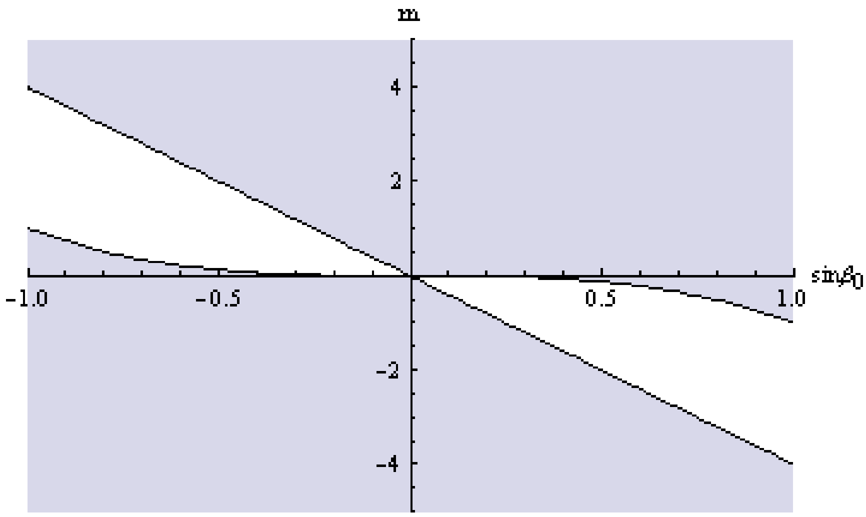


Figure 9.5 – Regions of fulfillment of the sufficient stability conditions of equilibrium orientations of an Axially Symmetric gyrostat satellite

The blue regions are the regions of fulfillment of the stability conditions of equilibrium orientations of an axially symmetric gyrostat satellite and the white regions the unstable regions.

10 CONCLUSIONS

The study conducted along these several chapters have led to new and very interesting results regarding the gyrostat dynamics on a circular orbit subjected to a gravitational torque. Due to the colossal task of working with huge equations like equation (5.21), scientists were discouraged to proceed further than (5.20), much because the complexity involved in such calculations and the most probable impossibility to find a simple analytical solution to the gyrostat satellite general case.
To overcome the problem to deal with the enormous equations, it was developed a numerical simulation software which was properly validated using to the well know case of the axially symmetric gyrostat, which after was used to obtain the numerical results from the general gyrostat case.

With the variation of the known parameters, the first results obtained already lead us to a certain predictability or at least to suspect that at least a small and simple rules could arise. The first results confirmed that with the increase of H_3, the various regions of equilibria narrow until they completely disappear.

With the increase of ν, the regions of 24 and 20 start very small, growing then in size until approximately ν=0.5, decreasing then again to smaller areas. It was also confirmed that the variation of the system with the parameter ν is compatible with the axially symmetric satellite.

Was also verified that in regions with small values of H_1 and H_2 exists 24 positions of equilibria, and when is increased the values from H_1 and H_2, the 24 equilibria region will give place to a 20 equilibria region, then to a 16 equilibria, then to a 12 equilibria and finally only to a 8 equilibria positions region.
Also unknown up to now, it was revealed small regions of 16 and 12 equilibria outside their main regions. These small regions can be found near $H_1 = 0$ and in relatively high values of H_2. Then, and with the increase of H_3, is noticed that eventually the small region of 16 equilibria vanish, giving place to a second small region of 12 equilibria outside their main region. As we further increase the value of H_3 these 12 equilibria become smaller and narrower, but they never disappear altogether, with the increase of H_3 also increases their position in H_2.

After conducting a huge an exhaustive numerical study, it is easy to find the bifurcation points for the evolution of the inertia parameters. After a carefully organization of the bifurcation points, surprizing results immediately arise, the numerical calculations show that the complex gyrostat general case has a surprisingly simple bifurcation distribution. For the 24 equilibria the bifurcation is described by $H_3 = 1 - \nu$. The 20 equilibria bifurcation region can be described by a first degree equation and after a certain point a second degree equation. The 16 equilibria bifurcation region can approximately be described by $H_3 = 4 - 3\nu$. And finally for the 12 equilibria bifurcation be described in the vicinity of $H_3 \approx 4$.

The next step is to study the stability of equilibria, after the unexpected simple results obtained in the study of equilibria, there is hope that such simple results can also be obtained on the stability calculations.

First results show that when the values of parameter H_1 increase, sequential mergers of equilibrium curves occurs at points that correspond to the points of intersection of the straight lines $H_2 = const.$ with the borders of regions with fixed number of equilibria.

After analysis of the numerical calculations, again surprising simple results arise from the rather complicated equations. It follows from the conducted numerical simulations that for H_3 values less then $1-\nu$ and for small H_1, H_2 there are 24 equilibrium arrangements which 4 are stable. Exist also 2 stable equilibria for $\nu > 0.5$ and $H_3 \geq 1-\nu$.

When values of parameters H_1, H_2, H_3 of the gyrostatic torque are greater than or equal to 4, there are 8 equilibrium solutions, but only 2 of them are stable. For large values of the parameters H_1, H_2, H_3 the equilibrium values of φ are close to the trivial solutions, where some axis of the orbital coordinate system and some axes of the body coordinate system coincide.

It was again confirmed that for a given gyrostat satellite subjected to a gravitational torque in a circular orbit, and when the gyrostatic moment is small enough, the gyrostat general case cannot have more than 24 equilibria positions, which means that the gyrostat satellite is close to the satellite rigid body. When the gyrostatic moment is big enough, there is no less than 8 equilibria positions, which means that the vector of the gyrostatic moment must be perpendicular to the orbital plane. Also, the stable equilibria are no less than 2 and no more than 4.

It was not expected such simple solutions for the system in study, much because the complexity of the equations involved. The results found for the bifurcation and stability are so simple that will permit for space applications designers to evaluate, even in the preliminary phase of a project, how many equilibrium positions such system will have, and how many will be stable.

11 FUTURE PROJECTS AND RECOMMENDATIONS

This work was rather complex but several issues were left behind. The first and perhaps the most important was the introduction of damping in our general system for the energy dissipation.

Another interesting study would be performing an equivalent analysis for an elliptic orbit, in such case, a gyrostat satellite will not have equilibrium orientations, but periodic solutions. This study will give even more general point of view for this kind of systems, and cover more possible spacecraft missions, much because of most space applications used today use a polar quasi-circular orbits.

Regarding the small regions of 16 and 12 equilibria, further analysis of how these regions change and how can they be managed would be very interesting, especially for projects where high values of H_3 will be needed. Also, and since the evolution of these small regions appears to obey a first order equation, to compute the evolution of these small regions would be a plus to the general understanding of the general case.

It would also be interesting to perform new calculations of the bifurcation points in function of H_1 / H_3 and H_2 / H_3, and of course to perform the stability calculations in function of the remaining configurations.

For a more complex study, and because this study was conducted having into account only the gravitational torque, it would be challenging to involve more forces, such as the magnetic, solar pressure and aerodynamic for instance.

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APPENDIX A – SOFTWARE

During the accomplishment of this investigation several software's were used to better and fast achieve, understand and also calculate the desired equations or set of equations.

The software for the calculation of the equilibria described in Appendix A.2, obtain the calculation inputs from keyboard, namely the initial and final values of H_1, H_2, and the constant values of H_3 and ν and also the steps (increase from the initial values) of H_1, H_2. With all this data, the software goes through all the equations until there is a change in the number of zeros is detected, after that records the coordinates of that change and the number of zeros present. In the end, presents all the changes of number of zeros with the correspondent number of zeros in a txt output format that can be easily exported into excel for a graphical analysis.

The software presented in Appendix A.3, does the stability calculations of pre given H_1 values and calculates if they are stable or unstable. With the output of this software in a txt format, it can be imported rather easily into excel for plotting the respective picture of stability.

In Appendix A.4 is software performed in Mathematica, which is a rather simple way to confirm if a certain value is stable or not.

In Appendix A.5 is the software used in the calculations of the Axially Symmetric case, this software was used to plot the pictures of the different equilibria and stability conditions.

In Appendix A.6 is a simple software performed in MSDOS in order to merge all the equilibria calculation files into only one txt file. This is useful for plotting the 3D picture. Then this unique file is converted into a Matlab readable format by the means of software described in Appendix A.7, and after plot the equilibria 3D picture with the software in Appendix A.8 or in Appendix A.9 depending which is the choice.

A.1 – C++ LIBRARIES

Besides the standard C++ libraries in this work was used the LIBBLAS32.lib and the liblapack32.lib. These libraries are auxiliary or linker libraries from the general LAPACK++ library. The LAPACK++ library is a library for high performance numerical linear algebra computations. This library includes support for solving linear systems using LU, Cholesky, QR matrix factorizations, and symmetric eigenvalues problems between other methods. The library liblapackpp.dll.a is a development library which is required to link applications that use the several specific and dynamic libraries.

A.2 – EQUILIBRIA COMPUTATION (C++)

```
#include <iostream>
#include <fstream>
#include <math.h>
#include <lapackpp/laslv.h>


using namespace std;


/* Main Program */
int main(){

        /* Global Variables*/
        double H1, H1_final, H2, H2_final, H2int, H3, v, escalon, escalonH1, p[13];
        int first_time=0, n_de_raices_actual = 0, n_de_raices_anterior = 0;
        double H2_dummy;
        ofstream fichero;
        LaVectorDouble real(12);
        LaVectorDouble imaginario(12);


        /* Create the file where the results will be stored*/
        fichero.open ("testdata.txt");

        /* Obtain values from keyboard */
        cout<<"H1 inicial: ";
        cin>>H1;
        cout<<"H1 final: ";
        cin>>H1_final;
        cout<<"H2 inicial: ";
        cin>>H2;
        cout<<"H2 final: ";
        cin>>H2_final;
        cout<<"H3: ";
        cin>>H3;
        cout<<"v: ";
        cin>>v;
        cout<<"Step H1: ";
        cin>>escalonH1;
        cout<<"Step H2:";
        cin>>escalon;

        /* store H2 value for future use */
        H2int=H2;

        /* Create a cicle for H1 and H2*/
        for (; H1 <= H1_final; H1 = H1 + escalonH1){

                for (; H2 <= H2_final; H2 = H2 + escalon){
                        /* Calculate the 12th grade equation */
                        p[0] = -16*pow(H1,4)*pow(H2,2)*pow(H3,4)*pow(v,6);
```

```
                              p[1] =
32*pow(H1,3)*pow(H2,2)*pow(H3,3)*pow(v,5)*(2*pow(H1,2)-pow(H2,2)*(v-1)-
2*v*(pow(H3,2)+2*v-2));
                              p[2] = -
16*pow(H1,2)*pow(H2,2)*pow(H3,2)*pow(v,4)*(6*pow(H1,4)+pow(H2,4)*pow((v-1),2)-
pow(H2,2)*(v-1)*(16*(pow(v,3)-pow(v,2))+(v-1)+pow(H3,2)*(1-7*v))+pow(H1,2)*((-
25*pow(v,2)+26*v-1)+pow(H3,2)*(pow(v,2)-16*v+1)+pow(H2,2)*(pow(v,2)-
8*v+7))+2*pow(v,2)*(3*pow(H3,4)+8*pow(v-1,2)-4*pow(H3,2)*(2*pow(v,2)-7*v+5)));
                              p[3] = 32*H1*pow(H2,2)*H3*pow(v,3)*(2*pow(H1,6)+pow(H1,4)*((-
13*pow(v,2)+14*v-1)+2*pow(H3,2)*(pow(v,2)-6*v+1)+pow(H2,2)*(pow(v,2)-
5*v+4))+pow(H3,2)*(-pow(H2,4)*pow(v-1,2)*(2*v-1)+pow(H2,2)*(v-1)*v*(pow(H3,2)*(1-
4*v)+(16*pow(v,3)-16*pow(v,2)+v-1))+2*pow(v,3)*(-pow(H3,4)+8*pow(v-1,2)*(4*v-
5)+2*pow(H3,2)*(7-11*v+4*pow(v,2))))-pow(H1,2)*(pow(H2,4)*(v-2)*pow(v-
1,2)+pow(H2,2)*(v-1)*((16*pow(v,3)-16*pow(v,2)+v-1)+pow(H3,2)*(3*pow(v,2)-
13*v+3))+2*v*(-2*pow(v-1,2)*(5*v-1)+pow(H3,4)*(pow(v,2)-
6*v+1)+pow(H3,2)*(18*pow(v,3)-53*pow(v,2)+38*v-3))));
                              p[4] = -16*pow(H2,2)*pow(v,2)*(pow(H1,8)+pow(H1,6)*(-1+10*v-
9*pow(v,2)+pow(H2,2)*(3-4*v+pow(v,2))+2*pow(H3,2)*(3-
8*v+3*pow(v,2)))+pow(H3,2)*(pow(H2,6)*pow(v-1,4)+(pow(H3,2)-16*pow(v-
1,2))*pow(v,4)*pow(-4+pow(H3,2)+4*v,2)+pow(H2,4)*pow(v-1,2)*v*(-8*pow(v-
1,2)*(2*v+1)+pow(H3,2)*(3*v-2))+pow(H2,2)*(v-1)*pow(v,2)*(pow(H3,4)*(3*v-
1)+16*pow(v-1,3)*(1+8*v)+pow(H3,2)*(17-49*v+64*pow(v,2)-
32*pow(v,3))))+pow(H1,2)*(pow(H2,6)*pow(v-1,4)+pow(H2,4)*pow(v-1,2)*(-1*pow(v-
1,2)*(1+8*v)+2*pow(H3,2)*(4-9*v+4*pow(v,2)))+pow(H2,2)*(v-1)*v*(8*pow(v-
1,3)*(1+2*v)+pow(H3,4)*(14-33*v+13*pow(v,2))+pow(H3,2)*(-4+38*v-
98*pow(v,2)+64*pow(v,3)))+2*pow(v,2)*(-8*pow(v-1,4)+pow(H3,6)*(3-
8*v+3*pow(v,2))+4*pow(H3,2)*pow(v-1,2)*(7-63*v+52*pow(v,2))+pow(H3,4)*(-23+134*v-
187*pow(v,2)+76*pow(v,3))))-pow(H1,4)*(pow(H2,4)*pow(v-1,2)*(2*v-3)+pow(H2,2)*(v-
1)*(-2+4*v-19*pow(v,2)+17*pow(v,3)+pow(H3,2)*(13-33*v+14*pow(v,2)))+v*(-8*pow(v-
1,2)*(3*v-1)+pow(H3,4)*(16-37*v+16*pow(v,2))+pow(H3,2)*(-32+209*v-
298*pow(v,2)+121*pow(v,3)))));
                              p[5] = 32*H1*pow(H2,2)*H3*v*(-2*pow(H2,6)*pow(v-
1,5)+2*pow(H1,6)*(pow(v,2)-v+1)-pow(H2,4)*pow(v-1,2)*v*(pow(v-
1,2)*(11+4*v)+pow(H3,2)*(5-10*v+6*pow(v,2)))-2*pow(v,3)*(40*pow(v-1,4)*(4*v-
1)+pow(H3,6)*(pow(v,2)-v+1)+2*pow(H3,2)*pow(v-1,2)*(27-
79*v+56*pow(v,2))+pow(H3,4)*(-15+46*v-53*pow(v,2)+22*pow(v,3)))-pow(H2,2)*(v-
1)*pow(v,2)*(-8*pow(v-1,3)*(16*v-1)+pow(H3,4)*(6*pow(v,2)-8*v+5)+pow(H3,2)*(-
33+131*v-146*pow(v,2)+48*pow(v,3)))+pow(H1,4)*(pow(H2,2)*(6-14*v+13*pow(v,2)-
5*pow(v,3))+v*(14-61*v+92*pow(v,2)-45*pow(v,3)-2*pow(H3,2)*(6-
7*v+6*pow(v,2))))+pow(H1,2)*(pow(H2,4)*pow(v-1,2)*(6-
10*v+5*pow(v,2))+pow(H2,2)*(v-1)*v*(-3+41*v-56*pow(v,2)+18*pow(v,3)+pow(H3,2)*(17-
26*v+17*pow(v,2)))+2*pow(v,2)*(pow(H3,4)*(6*pow(v,2)-7*v+6)+2*pow(v-
1,2)*(57*pow(v,2)-64*v+11)+pow(H3,2)*(67*pow(v,3)-155*pow(v,2)+125*v-37))));
                              p[6] = -16*pow(H2,2)*(pow(H2,8)*pow(v-1,6)-
2*pow(H2,6)*(pow(H3,2)*(1-2*v)+8*pow(v-1,2))*v*pow(v-1,4)+pow(H1,8)*(pow(v,2)+1)-
2*pow(H2,2)*(v-1)*pow(v,3)*(-84*pow(H3,2)*pow(v-1,3)+128*pow(v-
1,5)+17*pow(H3,4)*(v-1)*v+pow(H3,6)*(-1+v-2*pow(v,2)))+pow(v,4)*(pow(-
4+pow(H3,2)+4*v,2))*(-17*pow(H3,2)*pow(v-1,2)+16*pow(v-
1,4)+pow(H3,4)*(1+pow(v,2)))+pow(H2,4)*pow(v-1,2)*pow(v,2)*(96*pow(v-1,4)-
3*pow(H3,2)*pow(v-1,2)*(3+8*v)+pow(H3,4)*(2-6*v+6*pow(v,2)))+pow(H1,6)*(-
2*pow(H2,2)*(-2+3*v-2*pow(v,2)+pow(v,3))+v*(8-25*v+42*pow(v,2)-25*pow(v,3)-
4*pow(H3,2)*(4-3*v+4*pow(v,2))))-pow(H1,2)*(2*pow(H2,6)*(v-2)*pow(v-
1,4)+pow(H2,4)*pow(v-1,2)*v*(3*pow(v-1,2)*(8+3*v)+pow(H3,2)*(20-
34*v+20*pow(v,2)))+pow(H2,2)*(v-1)*pow(v,2)*(-168*pow(v-1,3)*v+pow(H3,4)*(30-
40*v+34*pow(v,2))+17*pow(H3,2)*(-7+27*v-
27*pow(v,2)+7*pow(v,3)))+2*pow(v,3)*(8*pow(v-1,4)*(25*v-8)+pow(H3,6)*(8-
6*v+8*pow(v,2))+4*pow(H3,2)*pow(v-1,2)*(67-122*v+67*pow(v,2))+pow(H3,4)*(-
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104+247*v-230*pow(v,2)+87*pow(v,3))))+2*pow(H1,4)*(pow(H2,4)*pow(v-1,2)*(3-
3*v+pow(v,2))+pow(H2,2)*(v-1)*v*(-17*v*(v-1)+pow(H3,2)*(17-
20*v+15*pow(v,2)))+pow(v,2)*(2*pow(H3,4)*(9-8*v+9*pow(v,2))+4*pow(v-1,2)*(4-
21*v+21*pow(v,2))+pow(H3,2)*(-87+230*v-247*pow(v,2)+104*pow(v,3)))));
                        p[7] = -32*H1*pow(H2,2)*H3*(-2*pow(H2,6)*pow(v-
1,5)+2*pow(H1,6)*(pow(v,2)-v+1)-pow(H2,4)*pow(v-1,2)*v*(-1*pow(v-
1,2)*(4+11*v)+pow(H3,2)*(5-10*v+6*pow(v,2)))-pow(H2,2)*(v-1)*pow(v,2)*(-8*(v-
16)*pow(v-1,3)+pow(H3,4)*(5-8*v+6*pow(v,2))+pow(H3,2)*(-18+56*v-
41*pow(v,2)+3*pow(v,3)))-pow(v,3)*(80*(v-4)*pow(v-1,4)+2*pow(H3,6)*(pow(v,2)-
v+1)+4*pow(H3,2)*pow(v-1,2)*(57-64*v+11*pow(v,2))+pow(H3,4)*(-45+92*v-
61*pow(v,2)+14*pow(v,3)))+pow(H1,4)*(pow(H2,2)*(6-14*v+13*pow(v,2)-5*pow(v,3))-
2*v*(-22+53*v-46*pow(v,2)+15*pow(v,3)+pow(H3,2)*(6-
7*v+6*pow(v,2))))+pow(H1,2)*(pow(H2,4)*pow(v-1,2)*(6-
10*v+5*pow(v,2))+pow(H2,2)*(v-1)*v*(-48+146*v-
131*pow(v,2)+33*pow(v,3)+pow(H3,2)*(17-
26*v+17*pow(v,2)))+2*pow(v,2)*(pow(H3,4)*(6-7*v+6*pow(v,2))+2*pow(v-1,2)*(56-
79*v+27*pow(v,2))+pow(H3,2)*(-67+155*v-125*pow(v,2)+37*pow(v,3)))));
                        p[8] = -16*pow(H2,2)*(pow(H1,8)+pow(H1,6)*(pow(H2,2)*(3-
4*v+pow(v,2))+2*(pow(H3,2)*(3-8*v+3*pow(v,2))-4*(2-
5*v+3*pow(v,2))))+pow(H3,2)*(pow(H2,6)*pow(v-1,4)+pow(v,4)*(pow(H3,2)-1*pow(v-
1,2))*pow((-4+pow(H3,2)+4*v),2)-pow(H2,4)*pow(v-1,2)*v*(pow(H3,2)*(2-3*v)+pow(v-
1,2)*(8+v))+pow(H2,2)*(v-1)*pow(v,2)*(8*pow(v-1,3)*(2+v)+pow(H3,4)*(3*v-
1)+pow(H3,2)*(17-19*v+4*pow(v,2)-2*pow(v,3))))-pow(H1,4)*(pow(H2,4)*pow(v-
1,2)*(2*v-3)+pow(H2,2)*(v-1)*(-32+64*v-49*pow(v,2)+17*pow(v,3)+pow(H3,2)*(13-
33*v+14*pow(v,2)))+v*(-16*pow(v-1,2)*(9*v-8)+pow(H3,4)*(16-
37*v+16*pow(v,2))+2*pow(H3,2)*(-76+187*v-
134*pow(v,2)+23*pow(v,3))))+pow(H1,2)*(pow(H2,6)*pow(v-1,4)+2*pow(H2,4)*pow(v-
1,2)*(pow(H3,2)*(4-9*v+4*pow(v,2))-4*(2-3*v+pow(v,3)))+pow(H2,2)*(v-1)*v*(16*pow(v-
1,3)*(8+v)+pow(H3,4)*(14-33*v+13*pow(v,2))+pow(H3,2)*(-64+98*v-
38*pow(v,2)+4*pow(v,3)))+pow(v,2)*(-256*pow(v-1,4)+2*pow(H3,6)*(3-
8*v+3*pow(v,2))+8*pow(H3,2)*pow(v-1,2)*(52-63*v+7*pow(v,2))+pow(H3,4)*(-121+298*v-
209*pow(v,2)+32*pow(v,3)))));
                        p[9] = -32*H1*pow(H2,2)*H3*(2*pow(H1,6)+pow(H3,2)*(-
1*pow(H2,4)*pow(v-1,2)*(2*v-1)+pow(H2,2)*(v-1)*v*(-16+pow(H3,2)*(1-4*v)+16*v-
pow(v,2)+pow(v,3))+pow(v,3)*(-2*pow(H3,4)+4*(v-5)*pow(v-1,2)+pow(H3,2)*(13-
14*v+pow(v,2))))+pow(H1,4)*(pow(H2,2)*(4-5*v+pow(v,2))+2*(-8+22*v-
14*pow(v,2)+pow(H3,2)*(1-6*v+pow(v,2))))-pow(H1,2)*(pow(H2,4)*(v-2)*pow(v-
1,2)+pow(H2,2)*(v-1)*(-16+16*v-pow(v,2)+pow(v,3)+pow(H3,2)*(3-
13*v+3*pow(v,2)))+2*v*(-8*pow(v-1,2)*(5*v-4)+pow(H3,4)*(1-6*v+pow(v,2))+pow(H3,2)*(-
18+53*v-38*pow(v,2)+3*pow(v,3)))));
                        p[10] = -
16*pow(H1,2)*pow(H2,2)*pow(H3,2)*(6*pow(H1,4)+pow(H2,4)*pow(v-1,2)-pow(H2,2)*(v-
1)*((pow(v,3)-pow(v,2)+16*v-16)+pow(H3,2)*(1-7*v))+pow(v,2)*(6*pow(H3,4)+16*pow(v-
1,2)-pow(H3,2)*(pow(v,2)-26*v+25))+pow(H1,2)*(pow(H3,2)*(pow(v,2)-
16*v+1)+pow(H2,2)*(pow(v,2)-8*v+7)-8*(5*pow(v,2)-7*v+2)));
                        p[11] = -32*pow(H1,3)*pow(H2,2)*pow(H3,3)*(2*pow(H1,2)-
pow(H2,2)*(v-1)-2*v*(pow(H3,2)+2*v-2));
                        p[12] = -16*pow(H1,4)*pow(H2,2)*pow(H3,4);

                        /* Creamos la matriz asociada del polinomio caracteristico */
                        double A[]={0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -p[12]/p[0],
                                    1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -p[11]/p[0],
                                    0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -p[10]/p[0],
                                    0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -p[9]/p[0],
                                    0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, -p[8]/p[0],
                                    0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -p[7]/p[0],
                                    0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -p[6]/p[0],
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                                    0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -p[5]/p[0],
                                    0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -p[4]/p[0],
                                    0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -p[3]/p[0],
                                    0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -p[2]/p[0],
                                    0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -p[1]/p[0]};

                 LaGenMatDouble matriz_asociada(A,12,12,true);



                 LaEigSolve(matriz_asociada,real,imaginario,matriz_asociada);

     /* check how many real toots exist*/
     /* root counter */
   n_de_raices_actual=0;
             for (int i=0; i < 12; i++){
              if (imaginario(i) == 0.0)
                        n_de_raices_actual++;
             }

                 /* if there is a jump in the number of real roots the value is stored*/
             if (first_time == 0){
              n_de_raices_anterior = n_de_raices_actual;
              first_time = 1;
             }
             else{

              if (n_de_raices_anterior != n_de_raices_actual){
               double H2_anterior = H2 - escalon;
    H2_dummy = H2;

    double escalon2=escalon/1000;
               H2 = H2_anterior;

               for(; H2 <= H2_dummy; H2 = H2 + escalon2) {

                   /* calculate the 12th grade equation */
                 p[0] = -16*pow(H1,4)*pow(H2,2)*pow(H3,4)*pow(v,6);
                 p[1] =
32*pow(H1,3)*pow(H2,2)*pow(H3,3)*pow(v,5)*(2*pow(H1,2)-pow(H2,2)*(v-1)-
2*v*(pow(H3,2)+2*v-2));
                 p[2] = -
16*pow(H1,2)*pow(H2,2)*pow(H3,2)*pow(v,4)*(6*pow(H1,4)+pow(H2,4)*pow((v-1),2)-
pow(H2,2)*(v-1)*(16*(pow(v,3)-pow(v,2))+(v-1)+pow(H3,2)*(1-7*v))+pow(H1,2)*((-
25*pow(v,2)+26*v-1)+pow(H3,2)*(pow(v,2)-16*v+1)+pow(H2,2)*(pow(v,2)-
8*v+7))+2*pow(v,2)*(3*pow(H3,4)+8*pow(v-1,2)-4*pow(H3,2)*(2*pow(v,2)-7*v+5)));
                 p[3] = 32*H1*pow(H2,2)*H3*pow(v,3)*(2*pow(H1,6)+pow(H1,4)*((-
13*pow(v,2)+14*v-1)+2*pow(H3,2)*(pow(v,2)-6*v+1)+pow(H2,2)*(pow(v,2)-
5*v+4))+pow(H3,2)*(-pow(H2,4)*pow(v-1,2)*(2*v-1)+pow(H2,2)*(v-1)*v*(pow(H3,2)*(1-
4*v)+(16*pow(v,3)-16*pow(v,2)+v-1))+2*pow(v,3)*(-pow(H3,4)+8*pow(v-1,2)*(4*v-
5)+2*pow(H3,2)*(7-11*v+4*pow(v,2))))-pow(H1,2)*(pow(H2,4)*(v-2)*pow(v-
1,2)+pow(H2,2)*(v-1)*((16*pow(v,3)-16*pow(v,2)+v-1)+pow(H3,2)*(3*pow(v,2)-
13*v+3))+2*v*(-2*pow(v-1,2)*(5*v-1)+pow(H3,4)*(pow(v,2)-
6*v+1)+pow(H3,2)*(18*pow(v,3)-53*pow(v,2)+38*v-3))));
                 p[4] = -16*pow(H2,2)*pow(v,2)*(pow(H1,8)+pow(H1,6)*(-1+10*v-
9*pow(v,2)+pow(H2,2)*(3-4*v+pow(v,2))+2*pow(H3,2)*(3-
8*v+3*pow(v,2)))+pow(H3,2)*(pow(H2,6)*pow(v-1,4)+(pow(H3,2)-16*pow(v-
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1,2))*pow(v,4)*pow(-4+pow(H3,2)+4*v,2)+pow(H2,4)*pow(v-1,2)*v*(-8*pow(v-1,2)*(2*v+1)+pow(H3,2)*(3*v-2))+pow(H2,2)*(v-1)*pow(v,2)*(pow(H3,4)*(3*v-1)+16*pow(v-1,3)*(1+8*v)+pow(H3,2)*(17-49*v+64*pow(v,2)-32*pow(v,3))))+pow(H1,2)*(pow(H2,6)*pow(v-1,4)+pow(H2,4)*pow(v-1,2)*(-1*pow(v-1,2)*(1+8*v)+2*pow(H3,2)*(4-9*v+4*pow(v,2)))+pow(H2,2)*(v-1)*v*(8*pow(v-1,3)*(1+2*v)+pow(H3,4)*(14-33*v+13*pow(v,2))+pow(H3,2)*(-4+38*v-98*pow(v,2)+64*pow(v,3)))+2*pow(v,2)*(-8*pow(v-1,4)+pow(H3,6)*(3-8*v+3*pow(v,2))+4*pow(H3,2)*pow(v-1,2)*(7-63*v+52*pow(v,2))+pow(H3,4)*(-23+134*v-187*pow(v,2)+76*pow(v,3))))-pow(H1,4)*(pow(H2,4)*pow(v-1,2)*(2*v-3)+pow(H2,2)*(v-1)*(-2+4*v-19*pow(v,2)+17*pow(v,3)+pow(H3,2)*(13-33*v+14*pow(v,2)))+v*(-8*pow(v-1,2)*(3*v-1)+pow(H3,4)*(16-37*v+16*pow(v,2))+pow(H3,2)*(-32+209*v-298*pow(v,2)+121*pow(v,3)))));

p[5] = 32*H1*pow(H2,2)*H3*v*(-2*pow(H2,6)*pow(v-1,5)+2*pow(H1,6)*(pow(v,2)-v+1)-pow(H2,4)*pow(v-1,2)*v*(pow(v-1,2)*(11+4*v)+pow(H3,2)*(5-10*v+6*pow(v,2)))-2*pow(v,3)*(40*pow(v-1,4)*(4*v-1)+pow(H3,6)*(pow(v,2)-v+1)+2*pow(H3,2)*pow(v-1,2)*(27-79*v+56*pow(v,2))+pow(H3,4)*(-15+46*v-53*pow(v,2)+22*pow(v,3)))-pow(H2,2)*(v-1)*pow(v,2)*(-8*pow(v-1,3)*(16*v-1)+pow(H3,4)*(6*pow(v,2)-8*v+5)+pow(H3,2)*(-33+131*v-146*pow(v,2)+48*pow(v,3)))+pow(H1,4)*(pow(H2,2)*(6-14*v+13*pow(v,2)-5*pow(v,3))+v*(14-61*v+92*pow(v,2)-45*pow(v,3)-2*pow(H3,2)*(6-7*v+6*pow(v,2))))+pow(H1,2)*(pow(H2,4)*pow(v-1,2)*(6-10*v+5*pow(v,2))+pow(H2,2)*(v-1)*v*(-3+41*v-56*pow(v,2)+18*pow(v,3)+pow(H3,2)*(17-26*v+17*pow(v,2)))+2*pow(v,2)*(pow(H3,4)*(6*pow(v,2)-7*v+6)+2*pow(v-1,2)*(57*pow(v,2)-64*v+11)+pow(H3,2)*(67*pow(v,3)-155*pow(v,2)+125*v-37))));

p[6] = -16*pow(H2,2)*(pow(H2,8)*pow(v-1,6)-2*pow(H2,6)*(pow(H3,2)*(1-2*v)+8*pow(v-1,2))*v*pow(v-1,4)+pow(H1,8)*(pow(v,2)+1)-2*pow(H2,2)*(v-1)*pow(v,3)*(-84*pow(H3,2)*pow(v-1,3)+128*pow(v-1,5)+17*pow(H3,4)*(v-1)*v+pow(H3,6)*(-1+v-2*pow(v,2)))+pow(v,4)*(pow(-4+pow(H3,2)+4*v,2))*(-17*pow(H3,2)*pow(v-1,2)+16*pow(v-1,4)+pow(H3,4)*(1+pow(v,2)))+pow(H2,4)*pow(v-1,2)*pow(v,2)*(96*pow(v-1,4)-3*pow(H3,2)*pow(v-1,2)*(3+8*v)+pow(H3,4)*(2-6*v+6*pow(v,2)))+pow(H1,6)*(-2*pow(H2,2)*(-2+3*v-2*pow(v,2)+pow(v,3))+v*(8-25*v+42*pow(v,2)-25*pow(v,3)-4*pow(H3,2)*(4-3*v+4*pow(v,2))))-pow(H1,2)*(2*pow(H2,6)*(v-2)*pow(v-1,4)+pow(H2,4)*pow(v-1,2)*v*(3*pow(v-1,2)*(8+3*v)+pow(H3,2)*(20-34*v+20*pow(v,2)))+pow(H2,2)*(v-1)*pow(v,2)*(-168*pow(v-1,3)*v+pow(H3,4)*(30-40*v+34*pow(v,2))+17*pow(H3,2)*(-7+27*v-27*pow(v,2)+7*pow(v,3)))+2*pow(v,3)*(8*pow(v-1,4)*(25*v-8)+pow(H3,6)*(8-6*v+8*pow(v,2))+4*pow(H3,2)*pow(v-1,2)*(67-122*v+67*pow(v,2))+pow(H3,4)*(-104+247*v-230*pow(v,2)+87*pow(v,3))))+2*pow(H1,4)*(pow(H2,4)*pow(v-1,2)*(3-3*v+pow(v,2))+pow(H2,2)*(v-1)*v*(-17*v*(v-1)+pow(H3,2)*(17-20*v+15*pow(v,2)))+pow(v,2)*(2*pow(H3,4)*(9-8*v+9*pow(v,2))+4*pow(v-1,2)*(4-21*v+21*pow(v,2))+pow(H3,2)*(-87+230*v-247*pow(v,2)+104*pow(v,3)))));

p[7] = -32*H1*pow(H2,2)*H3*(-2*pow(H2,6)*pow(v-1,5)+2*pow(H1,6)*(pow(v,2)-v+1)-pow(H2,4)*pow(v-1,2)*v*(-1*pow(v-1,2)*(4+11*v)+pow(H3,2)*(5-10*v+6*pow(v,2)))-pow(H2,2)*(v-1)*pow(v,2)*(-8*(v-16)*pow(v-1,3)+pow(H3,4)*(5-8*v+6*pow(v,2))+pow(H3,2)*(-18+56*v-41*pow(v,2)+3*pow(v,3)))-pow(v,3)*(80*(v-4)*pow(v-1,4)+2*pow(H3,6)*(pow(v,2)-v+1)+4*pow(H3,2)*pow(v-1,2)*(57-64*v+11*pow(v,2))+pow(H3,4)*(-45+92*v-61*pow(v,2)+14*pow(v,3)))+pow(H1,4)*(pow(H2,2)*(6-14*v+13*pow(v,2)-5*pow(v,3))-2*v*(-22+53*v-46*pow(v,2)+15*pow(v,3)+pow(H3,2)*(6-7*v+6*pow(v,2))))+pow(H1,2)*(pow(H2,4)*pow(v-1,2)*(6-10*v+5*pow(v,2))+pow(H2,2)*(v-1)*v*(-48+146*v-131*pow(v,2)+33*pow(v,3)+pow(H3,2)*(17-26*v+17*pow(v,2)))+2*pow(v,2)*(pow(H3,4)*(6-7*v+6*pow(v,2))+2*pow(v-1,2)*(56-79*v+27*pow(v,2))+pow(H3,2)*(-67+155*v-125*pow(v,2)+37*pow(v,3)))));

p[8] = -16*pow(H2,2)*(pow(H1,8)+pow(H1,6)*(pow(H2,2)*(3-4*v+pow(v,2))+2*(pow(H3,2)*(3-8*v+3*pow(v,2))-4*(2-

```
5*v+3*pow(v,2))))+pow(H3,2)*(pow(H2,6)*pow(v-1,4)+pow(v,4)*(pow(H3,2)-1*pow(v-
1,2))*pow((-4+pow(H3,2)+4*v),2)-pow(H2,4)*pow(v-1,2)*v*(pow(H3,2)*(2-3*v)+pow(v-
1,2)*(8+v))+pow(H2,2)*(v-1)*pow(v,2)*(8*pow(v-1,3)*(2+v)+pow(H3,4)*(3*v-
1)+pow(H3,2)*(17-19*v+4*pow(v,2)-2*pow(v,3))))-pow(H1,4)*(pow(H2,4)*pow(v-
1,2)*(2*v-3)+pow(H2,2)*(v-1)*(-32+64*v-49*pow(v,2)+17*pow(v,3)+pow(H3,2)*(13-
33*v+14*pow(v,2)))+v*(-16*pow(v-1,2)*(9*v-8)+pow(H3,4)*(16-
37*v+16*pow(v,2))+2*pow(H3,2)*(-76+187*v-
134*pow(v,2)+23*pow(v,3))))+pow(H1,2)*(pow(H2,6)*pow(v-1,4)+2*pow(H2,4)*pow(v-
1,2)*(pow(H3,2)*(4-9*v+4*pow(v,2))-4*(2-3*v+pow(v,3)))+pow(H2,2)*(v-1)*v*(16*pow(v-
1,3)*(8+v)+pow(H3,4)*(14-33*v+13*pow(v,2))+pow(H3,2)*(-64+98*v-
38*pow(v,2)+4*pow(v,3)))+pow(v,2)*(-256*pow(v-1,4)+2*pow(H3,6)*(3-
8*v+3*pow(v,2))+8*pow(H3,2)*pow(v-1,2)*(52-63*v+7*pow(v,2))+pow(H3,4)*(-121+298*v-
209*pow(v,2)+32*pow(v,3)))));
                p[9] = -32*H1*pow(H2,2)*H3*(2*pow(H1,6)+pow(H3,2)*(-
1*pow(H2,4)*pow(v-1,2)*(2*v-1)+pow(H2,2)*(v-1)*v*(-16+pow(H3,2)*(1-4*v)+16*v-
pow(v,2)+pow(v,3))+pow(v,3)*(-2*pow(H3,4)+4*(v-5)*pow(v-1,2)+pow(H3,2)*(13-
14*v+pow(v,2))))+pow(H1,4)*(pow(H2,2)*(4-5*v+pow(v,2))+2*(-8+22*v-
14*pow(v,2)+pow(H3,2)*(1-6*v+pow(v,2))))-pow(H1,2)*(pow(H2,4)*(v-2)*pow(v-
1,2)+pow(H2,2)*(v-1)*(-16+16*v-pow(v,2)+pow(v,3)+pow(H3,2)*(3-
13*v+3*pow(v,2)))+2*v*(-8*pow(v-1,2)*(5*v-4)+pow(H3,4)*(1-6*v+pow(v,2))+pow(H3,2)*(-
18+53*v-38*pow(v,2)+3*pow(v,3)))));
                p[10] = -
16*pow(H1,2)*pow(H2,2)*pow(H3,2)*(6*pow(H1,4)+pow(H2,4)*pow(v-1,2)-pow(H2,2)*(v-
1)*((pow(v,3)-pow(v,2)+16*v-16)+pow(H3,2)*(1-7*v))+pow(v,2)*(6*pow(H3,4)+16*pow(v-
1,2)-pow(H3,2)*(pow(v,2)-26*v+25))+pow(H1,2)*(pow(H3,2)*(pow(v,2)-
16*v+1)+pow(H2,2)*(pow(v,2)-8*v+7)-8*(5*pow(v,2)-7*v+2)));
                p[11] = -32*pow(H1,3)*pow(H2,2)*pow(H3,3)*(2*pow(H1,2)-
pow(H2,2)*(v-1)-2*v*(pow(H3,2)+2*v-2));
                p[12] = -16*pow(H1,4)*pow(H2,2)*pow(H3,4);

                /* Creamos la matriz asociada del polinomio caracteristico */
                double A[]={0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -p[12]/p[0],
                            1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -p[11]/p[0],
                            0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -p[10]/p[0],
                            0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -p[9]/p[0],
                            0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, -p[8]/p[0],
                            0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -p[7]/p[0],
                            0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -p[6]/p[0],
                            0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -p[5]/p[0],
                            0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -p[4]/p[0],
                            0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -p[3]/p[0],
                            0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -p[2]/p[0],
                            0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -p[1]/p[0]};

                LaGenMatDouble matriz_asociada(A,12,12,true);

                /* Calculate the eigen values of the matrix*/
                LaEigSolve(matriz_asociada,real,imaginario,matriz_asociada);

        /* check how many real roots exist*/
         n_de_raices_actual=0;
             for (int i=0; i < 12; i++){
               if (imaginario(i) == 0.0)
                     n_de_raices_actual++;
             }

                if (n_de_raices_anterior != n_de_raices_actual) {
```

```
                                H2_anterior = H2 - escalon2;
                                fichero << H1 << "\t" << H2_anterior << "\t" << H3 << "\t" << v << "\t"
<< n_de_raices_anterior << "\n";
                                       }
                  n_de_raices_anterior = n_de_raices_actual;
                    }//for
                H2 = H2_dummy;
            }//if
          }//else
          n_de_raices_anterior = n_de_raices_actual;
                       }//for2
                       H2=H2int;
        first_time=0;
        n_de_raices_anterior=0;
            }//for1

            /* Closet he file */
            fichero.close();
            return 0;
}
```

A.3 – STABILITY COMPUTATION (C++)

```
/*
 * main.cpp
 *

/* Dependencies */

#include <iostream>
#include <iomanip>
#include <fstream>
#include <math.h>
#include <lapackpp/laslv.h>

#include <stdlib.h>

/* Namespaces */

using namespace std;

/* Global Definitions */

#define SA_EPS_6      1.0E-6
#define SA_EPS_8      1.0E-8
#define SA_EPS_10     1.0E-10

typedef enum Y_FLAG {SA_ZERO=0, SA_ONE=1, SA_TWO=2} SA_y_flag;

SA_y_flag SA_Y_FLAG;


/* Types Definitions */

typedef int     SA_int;
typedef char    SA_char;
typedef double   SA_float;
typedef double  SA_double;

typedef ofstream  SA_ofstream;

/* Function Definitions */

SA_int SA_Coordinates(SA_double A[][3], SA_double h1, SA_double h2, SA_double h3,
SA_double v, SA_double x, SA_double y);
SA_int SA_Orthogonality_Proof(SA_double a11, SA_double a12, SA_double a13,
                       SA_double a21, SA_double a22, SA_double a23,
                       SA_double a31, SA_double a32, SA_double a33);
SA_int SA_Coef(SA_double a[], SA_double b[],  SA_double h1, SA_double h2, SA_double
h3, SA_double v, SA_double x);
SA_int SA_Polynomial(SA_double p[], SA_double H1, SA_double H2, SA_double H3,
SA_double v);
SA_int SA_Polynomial_Root(LaVectorDouble &Real_Part, LaVectorDouble &Img_Part,
                    SA_double h1, SA_double h2, SA_double h3, SA_double v);
SA_int SA_eq_quad(SA_double Out_Vect[2], SA_float a, SA_float b, SA_float c);
```

```
SA_int SA_Analyse_Root(LaVectorDouble &Real_Part, LaVectorDouble &Img_Part,
SA_double y[], SA_int y_status[],
                  SA_double h1, SA_double h2, SA_double h3, SA_double v);
SA_int SA_Stability_Analysis(SA_double Angles[], SA_double AA[], SA_double Cond[],
                      SA_double A[][3], SA_double h1, SA_double h2, SA_double h3,
SA_double v);
/* Main Program */
int main()
{

   /* Local Variables */

   int Nd =12;

   SA_double h1, h2, h3, v;

   SA_ofstream Out_File;
   SA_ofstream Out_File_II;
   SA_ofstream Out_File_III;

   LaVectorDouble Real_Part(Nd);
   LaVectorDouble Img_Part(Nd);

   SA_double y[Nd];
   SA_int    y_status[Nd];

   SA_double A_Coord[3][3];

   SA_double Angles[3];
   SA_double AA[6];
   SA_double Conditions[3];

   SA_int Stability_Status;


   SA_double
AP[]={0.0001,0.001,0.005,0.01,0.015,0.02,0.025,0.03,0.035,0.04,0.05,0.06,0.07,0.08,0.09,0.1,0
.11,0.12,0.13,0.14,0.15,0.16,0.17,0.18,0.19,0.2,0.25,0.3,0.35,0.4,0.45,0.5,0.55,0.6,0.65,0.7,0.75
,0.8,0.85,0.9,0.95,1,1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8,1.9,2,2.1,2.2,2.3,2.4,2.5};

   /*SA_double h1_start = 0.005;
   SA_double h1_end   = 0.04;
   SA_int    h1_N     = 7;

   /* Open and Configure Data File */
   Out_File.open ("DATA_I.txt");
   Out_File.setf(ios::left);
   Out_File.setf(ios::showpoint);
   Out_File.setf(ios::showpos);
   Out_File.setf(ios::fixed);
   Out_File.precision(5);

   Out_File_II.open ("DATA_II.txt");
   Out_File_II.setf(ios::left);
   Out_File_II.setf(ios::showpoint);
   Out_File_II.setf(ios::showpos);
   Out_File_II.setf(ios::fixed);
```

```
Out_File_II.precision(5);

Out_File_III.open ("DATA_III.txt");
Out_File_III.setf(ios::left);
Out_File_III.setf(ios::showpoint);
Out_File_III.setf(ios::showpos);
Out_File_III.setf(ios::fixed);
Out_File_III.precision(5);

/* Set Variables */
h2 = 0.5;
h3 = 0.01;
v = 0.1;

/*h1 = 0.005; */

/*for (int i=0; i<h1_N; i++) */
for (int i=0; i<57; i++)
{

   /*h1 += (h1_end - h1_start)/h1_N; */

   h1=AP[i];


   /* STEP I - Compute Roots of Equation (17) */

   SA_Polynomial_Root(Real_Part, Img_Part, h1, h2, h3, v);


   /* STEP II - Analysis Of Roots */

   SA_Analyse_Root(Real_Part, Img_Part, y, y_status, h1, h2, h3, v);

   /* STEP III -> IV  */

   Out_File<<h1<<"\t";
   Out_File_II<<h1<<"\t";
   Out_File_III<<h1<<"\t";

   for (int j=0; j<Nd; j++)
   {

      SA_Coordinates(A_Coord, h1, h2, h3, v, Real_Part(j), y[j]);

      if(fabs(Img_Part(j)) > 0.0)
      {
         Out_File_III<<"\t";
      }
      else
      {
         Out_File_III<<Real_Part(j);
      }

      Stability_Status = SA_Stability_Analysis(Angles, AA, Conditions, A_Coord, h1, h2,
h3, v);
```

```
          if (Stability_Status == 1)
          {
             cout<<"Stability : "<< " Achieved"<<endl;

             if(y_status[j] == SA_ONE )
             {
                Out_File<<Angles[2];
                Out_File_II<<"        ";
             }
             else
             {
                Out_File<<"        ";
//                Out_File<<Angles[2];
                Out_File_II<<"        ";
             }
          }
          else
          {
             cout<<"Stability : "<< " Not Achieved"<<endl;

             if(y_status[j] == SA_ONE)
             {
                Out_File<<"        ";
                Out_File_II<<Angles[2];
             }
             else
             {
                Out_File<<"        ";
                Out_File_II<<"        ";
             }


          }

          if (j == (Nd-1))
          {
             Out_File<<endl;
             Out_File_II<<endl;
             Out_File_III<<endl;
          }
          else
          {
             Out_File<<"\t";
             Out_File_II<<"\t";
             Out_File_III<<"\t";
          }
       }

    }

    /* Close File */
    Out_File.close();
    Out_File_II.close();
    Out_File_III.close();

    /* Use GnuPlot Bash File */
```

```
//   system("chmod +x GNUPlot.sh");
//   system("sh GNUPlot.sh");


   return (0);
}



/* */
SA_int SA_Coordinates(SA_double A[][3], SA_double h1, SA_double h2, SA_double h3,
SA_double v, SA_double x, SA_double y)
{

   SA_double a11, a12, a13;
   SA_double a21, a22, a23;
   SA_double a31, a32, a33;

   SA_int Proof_Check;

   /* First Root */

   a33 = sqrt(1/(1+x*x+y*y));

   a31=x*a33;
   a32=y*a33;

   a23=(4*a33*(pow(a31,2)*(1-v)+pow(a32,2)))/(h1*a31+h2*a32+h3*a33);
   a22=(-4*a32*(v*pow(a31,2)+pow(a33,2)))/(h1*a31+h2*a32+h3*a33);
   a11=(-4*a32*a33)/(h1*a31+h2*a32+h3*a33);
   a12=(4*(1-v)*a33*a31)/ (h1*a31+h2*a32+h3*a33);
   a13=(4*v*a31*a32)/ (h1*a31+h2*a32+h3*a33);
   a21=(4*a31*(v*pow(a32,2)-pow(a33,2)*(1-v)))/(h1*a31+h2*a32+h3*a33);

   Proof_Check = SA_Orthogonality_Proof(a11, a12, a13, a21, a22, a23, a31, a32, a33);

   if (Proof_Check == SA_ONE)
   {
      A[0][0] = a11;  A[0][1] = a12;  A[0][2] = a13;
      A[1][0] = a21;  A[1][1] = a22;  A[1][2] = a23;
      A[2][0] = a31;  A[2][1] = a32;  A[2][2] = a33;

      return (SA_ONE);
   }

   /* Second Root if First Root not verified*/

   a33 = -sqrt(1/(1+x*x+y*y));

   a31=x*a33;
   a32=y*a33;

   a23=(4*a33*(pow(a31,2)*(1-v)+pow(a32,2)))/(h1*a31+h2*a32+h3*a33);
   a22=(-4*a32*(v*pow(a31,2)+pow(a33,2)))/(h1*a31+h2*a32+h3*a33);
   a11=(-4*a32*a33)/(h1*a31+h2*a32+h3*a33);
   a12=(4*(1-v)*a33*a31)/ (h1*a31+h2*a32+h3*a33);
   a13=(4*v*a31*a32)/ (h1*a31+h2*a32+h3*a33);
```

```
    a21=(4*a31*(v*pow(a32,2)-pow(a33,2)*(1-v)))/(h1*a31+h2*a32+h3*a33);

    Proof_Check = SA_Orthogonality_Proof(a11, a12, a13, a21, a22, a23, a31, a32, a33);

    A[0][0] = a11;  A[0][1] = a12;  A[0][2] = a13;
    A[1][0] = a21;  A[1][1] = a22;  A[1][2] = a23;
    A[2][0] = a31;  A[2][1] = a32;  A[2][2] = a33;

    return (SA_ZERO);
}

SA_int SA_Orthogonality_Proof(SA_double a11, SA_double a12, SA_double a13,
                              SA_double a21, SA_double a22, SA_double a23,
                              SA_double a31, SA_double a32, SA_double a33)
{
//   SA_double Cond_I   = a11*a11 + a12*a12 +a13*a13 -1;
//   SA_double Cond_II  = a21*a21 + a22*a22 +a23*a23 -1;
//   SA_double Cond_III = a31*a31 + a32*a32 +a33*a33 -1;
//   SA_double Cond_IV  = a11*a21 + a12*a22 + a13*a23;
//   SA_double Cond_V   = a11*a31 + a12*a32 + a13*a33;
    SA_double Cond_VI  = a21*a31 + a22*a32 + a23*a33;

    if( (fabs(Cond_VI) < SA_EPS_6))
    {
        return (SA_ONE);
    }
    else
    {
        return (SA_ZERO);
    }

}

/* */
SA_int SA_Coef(SA_double a[], SA_double b[],  SA_double h1, SA_double h2, SA_double
h3, SA_double v, SA_double x)
{
    a[0]=h2*(h1-v*h3*x);
    a[1]=h1*h3+(4*v*(1-v)+pow(h1,2)-(1-v)*pow(h2,2)-v*pow(h3,2))*x-v*h1*h3*pow(x,2);
    a[2]=-h2*(1-v)*(h1*x+h3)*x;

    b[0]=pow(h2,2);
    b[1]=2*h2*(h1*x+h3);
    b[2]=(pow(h2,2)+pow(h3,2)-16)+2*h1*h3*x+(pow(h1,2)+pow(h2,2)-
16*pow(v,2))*pow(x,2);
    b[3]=2*h2*(h1*x+h3)*(1+pow(x,2));
    b[4]=pow(h1*x+h3,2)*(1+pow(x,2))-16*pow(x,2)*pow(1-v,2);

    return (0);

}

/* */
SA_int SA_Polynomial(SA_double p[], SA_double H1, SA_double H2, SA_double H3,
SA_double v)
{
```

```
    p[0] = -16*pow(H1,4)*pow(H2,2)*pow(H3,4)*pow(v,6);
    p[1] =  32*pow(H1,3)*pow(H2,2)*pow(H3,3)*pow(v,5)*(2*pow(H1,2)-pow(H2,2)*(v-1)-
2*v*(pow(H3,2)+2*v-2));
    p[2] = -
16*pow(H1,2)*pow(H2,2)*pow(H3,2)*pow(v,4)*(6*pow(H1,4)+pow(H2,4)*pow((v-1),2)-
pow(H2,2)*(v-1)*(16*(pow(v,3)-pow(v,2))+(v-1)+pow(H3,2)*(1-7*v))+pow(H1,2)*((-
25*pow(v,2)+26*v-1)+pow(H3,2)*(pow(v,2)-16*v+1)+pow(H2,2)*(pow(v,2)-
8*v+7))+2*pow(v,2)*(3*pow(H3,4)+8*pow(v-1,2)-4*pow(H3,2)*(2*pow(v,2)-7*v+5)));
    p[3] =  32*H1*pow(H2,2)*H3*pow(v,3)*(2*pow(H1,6)+pow(H1,4)*((-13*pow(v,2)+14*v-
1)+2*pow(H3,2)*(pow(v,2)-6*v+1)+pow(H2,2)*(pow(v,2)-5*v+4))+pow(H3,2)*(-
pow(H2,4)*pow(v-1,2)*(2*v-1)+pow(H2,2)*(v-1)*v*(pow(H3,2)*(1-4*v)+(16*pow(v,3)-
16*pow(v,2)+v-1))+2*pow(v,3)*(-pow(H3,4)+8*pow(v-1,2)*(4*v-5)+2*pow(H3,2)*(7-
11*v+4*pow(v,2))))-pow(H1,2)*(pow(H2,4)*(v-2)*pow(v-1,2)+pow(H2,2)*(v-
1)*((16*pow(v,3)-16*pow(v,2)+v-1)+pow(H3,2)*(3*pow(v,2)-13*v+3))+2*v*(-2*pow(v-
1,2)*(5*v-1)+pow(H3,4)*(pow(v,2)-6*v+1)+pow(H3,2)*(18*pow(v,3)-53*pow(v,2)+38*v-
3))));
    p[4] = -16*pow(H2,2)*pow(v,2)*(pow(H1,8)+pow(H1,6)*(-1+10*v-
9*pow(v,2)+pow(H2,2)*(3-4*v+pow(v,2))+2*pow(H3,2)*(3-
8*v+3*pow(v,2)))+pow(H3,2)*(pow(H2,6)*pow(v-1,4)+(pow(H3,2)-16*pow(v-
1,2))*pow(v,4)*pow(-4+pow(H3,2)+4*v,2)+pow(H2,4)*pow(v-1,2)*v*(-8*pow(v-
1,2)*(2*v+1)+pow(H3,2)*(3*v-2))+pow(H2,2)*(v-1)*pow(v,2)*(pow(H3,4)*(3*v-
1)+16*pow(v-1,3)*(1+8*v)+pow(H3,2)*(17-49*v+64*pow(v,2)-
32*pow(v,3))))+pow(H1,2)*(pow(H2,6)*pow(v-1,4)+pow(H2,4)*pow(v-1,2)*(-1*pow(v-
1,2)*(1+8*v)+2*pow(H3,2)*(4-9*v+4*pow(v,2)))+pow(H2,2)*(v-1)*v*(8*pow(v-
1,3)*(1+2*v)+pow(H3,4)*(14-33*v+13*pow(v,2))+pow(H3,2)*(-4+38*v-
98*pow(v,2)+64*pow(v,3)))+2*pow(v,2)*(-8*pow(v-1,4)+pow(H3,6)*(3-
8*v+3*pow(v,2))+4*pow(H3,2)*pow(v-1,2)*(7-63*v+52*pow(v,2))+pow(H3,4)*(-23+134*v-
187*pow(v,2)+76*pow(v,3))))-pow(H1,4)*(pow(H2,4)*pow(v-1,2)*(2*v-3)+pow(H2,2)*(v-
1)*(-2+4*v-19*pow(v,2)+17*pow(v,3)+pow(H3,2)*(13-33*v+14*pow(v,2)))+v*(-8*pow(v-
1,2)*(3*v-1)+pow(H3,4)*(16-37*v+16*pow(v,2))+pow(H3,2)*(-32+209*v-
298*pow(v,2)+121*pow(v,3)))));
    p[5] = 32*H1*pow(H2,2)*H3*v*(-2*pow(H2,6)*pow(v-1,5)+2*pow(H1,6)*(pow(v,2)-v+1)-
pow(H2,4)*pow(v-1,2)*v*(pow(v-1,2)*(11+4*v)+pow(H3,2)*(5-10*v+6*pow(v,2)))-
2*pow(v,3)*(40*pow(v-1,4)*(4*v-1)+pow(H3,6)*(pow(v,2)-v+1)+2*pow(H3,2)*pow(v-
1,2)*(27-79*v+56*pow(v,2))+pow(H3,4)*(-15+46*v-53*pow(v,2)+22*pow(v,3)))-
pow(H2,2)*(v-1)*pow(v,2)*(-8*pow(v-1,3)*(16*v-1)+pow(H3,4)*(6*pow(v,2)-
8*v+5)+pow(H3,2)*(-33+131*v-146*pow(v,2)+48*pow(v,3)))+pow(H1,4)*(pow(H2,2)*(6-
14*v+13*pow(v,2)-5*pow(v,3))+v*(14-61*v+92*pow(v,2)-45*pow(v,3)-2*pow(H3,2)*(6-
7*v+6*pow(v,2))))+pow(H1,2)*(pow(H2,4)*pow(v-1,2)*(6-
10*v+5*pow(v,2))+pow(H2,2)*(v-1)*v*(-3+41*v-56*pow(v,2)+18*pow(v,3)+pow(H3,2)*(17-
26*v+17*pow(v,2)))+2*pow(v,2)*(pow(H3,4)*(6*pow(v,2)-7*v+6)+2*pow(v-
1,2)*(57*pow(v,2)-64*v+11)+pow(H3,2)*(67*pow(v,3)-155*pow(v,2)+125*v-37))));
    p[6] = -16*pow(H2,2)*(pow(H2,8)*pow(v-1,6)-2*pow(H2,6)*(pow(H3,2)*(1-
2*v)+8*pow(v-1,2))*v*pow(v-1,4)+pow(H1,8)*(pow(v,2)+1)-2*pow(H2,2)*(v-
1)*pow(v,3)*(-84*pow(H3,2)*pow(v-1,3)+128*pow(v-1,5)+17*pow(H3,4)*(v-
1)*v+pow(H3,6)*(-1+v-2*pow(v,2)))+pow(v,4)*(pow(-4+pow(H3,2)+4*v,2))*(-
17*pow(H3,2)*pow(v-1,2)+16*pow(v-1,4)+pow(H3,4)*(1+pow(v,2)))+pow(H2,4)*pow(v-
1,2)*pow(v,2)*(96*pow(v-1,4)-3*pow(H3,2)*pow(v-1,2)*(3+8*v)+pow(H3,4)*(2-
6*v+6*pow(v,2)))+pow(H1,6)*(-2*pow(H2,2)*(-2+3*v-2*pow(v,2)+pow(v,3))+v*(8-
25*v+42*pow(v,2)-25*pow(v,3)-4*pow(H3,2)*(4-3*v+4*pow(v,2))))-
pow(H1,2)*(2*pow(H2,6)*(v-2)*pow(v-1,4)+pow(H2,4)*pow(v-1,2)*v*(3*pow(v-
1,2)*(8+3*v)+pow(H3,2)*(20-34*v+20*pow(v,2)))+pow(H2,2)*(v-1)*pow(v,2)*(-168*pow(v-
1,3)*v+pow(H3,4)*(30-40*v+34*pow(v,2))+17*pow(H3,2)*(-7+27*v-
27*pow(v,2)+7*pow(v,3)))+2*pow(v,3)*(8*pow(v-1,4)*(25*v-8)+pow(H3,6)*(8-
6*v+8*pow(v,2))+4*pow(H3,2)*pow(v-1,2)*(67-122*v+67*pow(v,2))+pow(H3,4)*(-
104+247*v-230*pow(v,2)+87*pow(v,3))))+2*pow(H1,4)*(pow(H2,4)*pow(v-1,2)*(3-
3*v+pow(v,2))+pow(H2,2)*(v-1)*v*(-17*v*(v-1)+pow(H3,2)*(17-
```

```
20*v+15*pow(v,2)))+pow(v,2)*(2*pow(H3,4)*(9-8*v+9*pow(v,2))+4*pow(v-1,2)*(4-
21*v+21*pow(v,2))+pow(H3,2)*(-87+230*v-247*pow(v,2)+104*pow(v,3)))));
   p[7] = -32*H1*pow(H2,2)*H3*(-2*pow(H2,6)*pow(v-1,5)+2*pow(H1,6)*(pow(v,2)-v+1)-
pow(H2,4)*pow(v-1,2)*v*(-1*pow(v-1,2)*(4+11*v)+pow(H3,2)*(5-10*v+6*pow(v,2)))-
pow(H2,2)*(v-1)*pow(v,2)*(-8*(v-16)*pow(v-1,3)+pow(H3,4)*(5-
8*v+6*pow(v,2))+pow(H3,2)*(-18+56*v-41*pow(v,2)+3*pow(v,3)))-pow(v,3)*(80*(v-
4)*pow(v-1,4)+2*pow(H3,6)*(pow(v,2)-v+1)+4*pow(H3,2)*pow(v-1,2)*(57-
64*v+11*pow(v,2))+pow(H3,4)*(-45+92*v-
61*pow(v,2)+14*pow(v,3)))+pow(H1,4)*(pow(H2,2)*(6-14*v+13*pow(v,2)-5*pow(v,3))-
2*v*(-22+53*v-46*pow(v,2)+15*pow(v,3)+pow(H3,2)*(6-
7*v+6*pow(v,2))))+pow(H1,2)*(pow(H2,4)*pow(v-1,2)*(6-
10*v+5*pow(v,2))+pow(H2,2)*(v-1)*v*(-48+146*v-
131*pow(v,2)+33*pow(v,3)+pow(H3,2)*(17-
26*v+17*pow(v,2)))+2*pow(v,2)*(pow(H3,4)*(6-7*v+6*pow(v,2))+2*pow(v-1,2)*(56-
79*v+27*pow(v,2))+pow(H3,2)*(-67+155*v-125*pow(v,2)+37*pow(v,3)))));
   p[8] = -16*pow(H2,2)*(pow(H1,8)+pow(H1,6)*(pow(H2,2)*(3-
4*v+pow(v,2))+2*(pow(H3,2)*(3-8*v+3*pow(v,2))-4*(2-
5*v+3*pow(v,2))))+pow(H3,2)*(pow(H2,6)*pow(v-1,4)+pow(v,4)*(pow(H3,2)-1*pow(v-
1,2))*pow((-4+pow(H3,2)+4*v),2)-pow(H2,4)*pow(v-1,2)*v*(pow(H3,2)*(2-3*v)+pow(v-
1,2)*(8+v))+pow(H2,2)*(v-1)*pow(v,2)*(8*pow(v-1,3)*(2+v)+pow(H3,4)*(3*v-
1)+pow(H3,2)*(17-19*v+4*pow(v,2)-2*pow(v,3))))-pow(H1,4)*(pow(H2,4)*pow(v-
1,2)*(2*v-3)+pow(H2,2)*(v-1)*(-32+64*v-49*pow(v,2)+17*pow(v,3)+pow(H3,2)*(13-
33*v+14*pow(v,2)))+v*(-16*pow(v-1,2)*(9*v-8)+pow(H3,4)*(16-
37*v+16*pow(v,2))+2*pow(H3,2)*(-76+187*v-
134*pow(v,2)+23*pow(v,3))))+pow(H1,2)*(pow(H2,6)*pow(v-1,4)+2*pow(H2,4)*pow(v-
1,2)*(pow(H3,2)*(4-9*v+4*pow(v,2))-4*(2-3*v+pow(v,3)))+pow(H2,2)*(v-1)*v*(16*pow(v-
1,3)*(8+v)+pow(H3,4)*(14-33*v+13*pow(v,2))+pow(H3,2)*(-64+98*v-
38*pow(v,2)+4*pow(v,3)))+pow(v,2)*(-256*pow(v-1,4)+2*pow(H3,6)*(3-
8*v+3*pow(v,2))+8*pow(H3,2)*pow(v-1,2)*(52-63*v+7*pow(v,2))+pow(H3,4)*(-121+298*v-
209*pow(v,2)+32*pow(v,3)))));
   p[9] = -32*H1*pow(H2,2)*H3*(2*pow(H1,6)+pow(H3,2)*(-1*pow(H2,4)*pow(v-1,2)*(2*v-
1)+pow(H2,2)*(v-1)*v*(-16+pow(H3,2)*(1-4*v)+16*v-pow(v,2)+pow(v,3))+pow(v,3)*(-
2*pow(H3,4)+4*(v-5)*pow(v-1,2)+pow(H3,2)*(13-
14*v+pow(v,2))))+pow(H1,4)*(pow(H2,2)*(4-5*v+pow(v,2))+2*(-8+22*v-
14*pow(v,2)+pow(H3,2)*(1-6*v+pow(v,2))))-pow(H1,2)*(pow(H2,4)*(v-2)*pow(v-
1,2)+pow(H2,2)*(v-1)*(-16+16*v-pow(v,2)+pow(v,3)+pow(H3,2)*(3-
13*v+3*pow(v,2)))+2*v*(-8*pow(v-1,2)*(5*v-4)+pow(H3,4)*(1-6*v+pow(v,2))+pow(H3,2)*(-
18+53*v-38*pow(v,2)+3*pow(v,3)))));
   p[10] = -16*pow(H1,2)*pow(H2,2)*pow(H3,2)*(6*pow(H1,4)+pow(H2,4)*pow(v-1,2)-
pow(H2,2)*(v-1)*((pow(v,3)-pow(v,2)+16*v-16)+pow(H3,2)*(1-
7*v))+pow(v,2)*(6*pow(H3,4)+16*pow(v-1,2)-pow(H3,2)*(pow(v,2)-
26*v+25))+pow(H1,2)*(pow(H3,2)*(pow(v,2)-16*v+1)+pow(H2,2)*(pow(v,2)-8*v+7)-
8*(5*pow(v,2)-7*v+2)));
   p[11] = -32*pow(H1,3)*pow(H2,2)*pow(H3,3)*(2*pow(H1,2)-pow(H2,2)*(v-1)-
2*v*(pow(H3,2)+2*v-2));
   p[12] = -16*pow(H1,4)*pow(H2,2)*pow(H3,4);

   return (0);
}

/* */
SA_int SA_Polynomial_Root(LaVectorDouble &Real_Part, LaVectorDouble &Img_Part,
                SA_double h1, SA_double h2, SA_double h3, SA_double v)
{

   SA_double p[13];
```

```
   /* Polynomial Coefficients */
   SA_Polynomial(p,  h1,  h2,  h3,  v);

   /* Matrix Construction Based on Vieta's Formulas */
   double A[]={0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -p[12]/p[0],
            1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -p[11]/p[0],
            0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -p[10]/p[0],
            0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -p[9]/p[0],
            0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, -p[8]/p[0],
            0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -p[7]/p[0],
            0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -p[6]/p[0],
            0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -p[5]/p[0],
            0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -p[4]/p[0],
            0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -p[3]/p[0],
            0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -p[2]/p[0],
            0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -p[1]/p[0]};

   /* Lapackpp Assignment */
   LaGenMatDouble Aug_Matrix(A,12,12,true);


   /* Eigenvalues Computation */
   LaEigSolve(Aug_Matrix,Real_Part,Img_Part,Aug_Matrix);

   return (0);
}


SA_int SA_eq_quad(SA_double Out_Vect[2], SA_float a, SA_float b, SA_float c)
{
   SA_float d = b*b - 4*a*c;

   if ((fabs((double)a) > SA_EPS_10) && (d > SA_EPS_10))
   {
      Out_Vect[0] = (-b + (SA_float)sqrt((double)d) )/(2.0*a);
      Out_Vect[1] = (-b - (SA_float)sqrt((double)d) )/(2.0*a);

      return (SA_TWO);
   }
   else if ((fabs((double)a) > SA_EPS_10) && ((d <SA_EPS_10) && (d >SA_EPS_10)))
   {
      Out_Vect[0] = (-b + (SA_float)sqrt((double)d) )/(2.0*a);
      Out_Vect[1] = (-b - (SA_float)sqrt((double)d) )/(2.0*a);
      return (SA_ONE);
   }
   else
   {
      return (SA_ZERO);
   }

}

SA_int SA_Analyse_Root(LaVectorDouble &Real_Part, LaVectorDouble &Img_Part,
SA_double y[], SA_int y_status[],
                 SA_double h1, SA_double h2, SA_double h3, SA_double v)
{
```

```
   SA_double a[3];
   SA_double b[5];
   SA_double Root[2];



   for (int i=0; i<12; i++)
   {

      SA_Coef(a, b, h1, h2, h3, v, Real_Part(i));
      SA_eq_quad(Root, a[0], a[1], a[2]);

      SA_double Eq_2 = fabs(b[0]*pow(Root[0],4) + b[1]*pow(Root[0],3) +
b[2]*pow(Root[0],2) + b[3]*Root[0] + b[4]);

      if( Eq_2 < .00001)
      {
         y[i] = Root[0];
      }
      else
      {
         y[i] = Root[1];
      }

      if (fabs((double) Img_Part(i)) > SA_EPS_6)
      {
         y_status[i] = SA_ZERO;
      }
      else
      {
         y_status[i] = SA_ONE;
      }


   }


   return (0);
}

SA_int SA_Stability_Analysis(SA_double Angles[], SA_double AA[], SA_double Cond[],
                       SA_double A[][3], SA_double h1, SA_double h2, SA_double h3,
SA_double v)
{

   SA_double varphi = atan(A[2][0]/A[2][1]);
   if(varphi < 0)
      varphi+=M_PI;
   SA_double theta  = (acos(A[2][2]));

   SA_double Psi    = asin(A[0][2]/sin(theta));

   SA_double A_Psi_Psi       = v*((pow(A[0][0],2) - pow(A[1][0],2))+(pow(A[0][2],2) -
pow(A[1][2],2))) + h1*A[1][0] + h2*A[1][1] + h3*A[1][2];
   SA_double A_theta_theta   = (3 + pow(cos(Psi),2))*(1 -
v*pow(sin(varphi),2))*cos(2*theta) - 1/4*v*sin(2*Psi)*cos(theta)*sin(2*varphi)
+(h1*sin(varphi) + h2*cos(varphi))*cos(Psi)*cos(theta) + h3*A[1][2];
```

```
   SA_double A_varphi_varphi = v*((pow(A[1][1],2)-pow(A[1][0],2))-3*(pow(A[2][1],2)-
pow(A[2][0],2)))+h1*A[1][0]+h2*A[1][1];
   SA_double A_Psi_theta     = -0.5*sin(2*Psi)*sin(2*theta) + v*(A[0][0]*A[1][2] +
A[0][2]*A[1][0]) - sin(Psi)*(h1*A[2][0] + h2*A[2][1] + h3*A[2][2]);
   SA_double A_Psi_varphi    = v*(A[0][0]*A[1][1]+A[0][1]*A[1][0])-h1*A[0][1]+h2*A[0][0];
   SA_double A_theta_varphi  = -3/2*v*sin(2*theta)*sin(2*varphi) + v*(A[1][0]*cos(varphi)
+ A[1][1])*A[1][2] - (h1*cos(varphi) -h2*sin(varphi))*A[1][2];

   SA_double Cond_I   = A_Psi_Psi;
   SA_double Cond_II  = A_Psi_Psi*A_theta_theta  - A_Psi_theta*A_Psi_theta;
   SA_double Cond_III = A_Psi_Psi*A_theta_theta*A_varphi_varphi +
2*A_Psi_theta*A_Psi_varphi*A_theta_varphi - A_Psi_Psi*pow(A_theta_varphi,2) -
A_theta_theta*pow(A_Psi_varphi,2) - A_varphi_varphi*pow(A_Psi_theta,2);

   Angles[0] = Psi;
   Angles[1] = theta;
   Angles[2] = varphi;

   AA[0] = A_Psi_Psi;
   AA[1] = A_theta_theta;
   AA[2] = A_varphi_varphi;
   AA[3] = A_Psi_theta;
   AA[4] = A_Psi_varphi;
   AA[5] = A_theta_varphi;

   Cond[0] = Cond_I;
   Cond[1] = Cond_II;
   Cond[2] = Cond_III;

   if ((Cond_I > 0.0) && (Cond_II > 0.0) && (Cond_III > 0.0))
   {
      return (SA_ONE);
   }
   else
   {
      return (SA_ZERO);
   }



}
```

A.4 – NUMBER OF EQUILIBRIA POSITIONS AND CORRESPONDENT STABILITY CONFIRMATION (MATHEMATICA)

```
a0 = H2*(H1 - v*H3*x);
a1 = H1*H3 + (4*v*(1 - v) + H1^2 - (1 - v)*H2^2 - v*H3^2)*x -
   v*H1*H3*x^2;
a2 = -(1 - v)*H2*(H1*x + H3)*x;
b0 = H2^2;
b1 = 2*H2*(H1*x + H3);
b2 = (H2^2 + H3^2 - 16) + 2*H1*H3*x + (H1^2 + H2^2 - 16*v^2)*x^2;
b3 = 2*H2*(H1*x + H3)*(1 + x^2);
b4 = (H1*x + H3)^2*(1 + x^2) - 16*(1 - v)^2*x^2;
\Solve[{z1 == 0, z2 == 0}, {x, y}]
Rx = {{a0, a1, a2, 0, 0, 0}, {0, a0, a1, a2, 0, 0}, {0, 0, a0, a1, a2,
     0}, {0, 0, 0, a0, a1, a2}, {b0, b1, b2, b3, b4, 0}, {0, b0, b1,
    b2, b3, b4}};
eq = Det[Rx];
AA = Collect[eq, x, Simplify];
Clear[x]
Clear[v]
Clear[H3]
// x = 1;
v = 0.2;
H1 = 1.6;
H2 = 0.1;
H3 = 0.4;
\AA;
Solve[AA == 0, x];
CountRoots[Collect[eq, x, Simplify], x];
yy1 = a0*y^2 + a1*y + a2;
Solve[yy1 == 0, y]
yr = -0.07368374734103442`
yy2 = b0*yr^4 + b1*yr^3 + b2*yr^2 + b3*yr + b4
-8.60423*10^-16
fi = ArcTan[x/yr]
```

A.5 – AXIALLY SYMMETRIC CALCULATIONS (MATHEMATICA)

```
plot1 = Plot[(4^(2/3) - x^(2/3))^(3/2), {x, -4, 4},
  PlotRange -> {-4, 4}, AxesOrigin -> {0, 0}, AxesLabel -> {n, m},
  PlotStyle -> {Black}];
plot2 = Plot[-(4^(2/3) - x^(2/3))^(3/2), {x, -4, 4},
  PlotRange -> {-4, 4}, AxesOrigin -> {0, 0}, PlotStyle -> {Black}];
plot3 = Show[plot1,
  plot1 /.
   L_Line :> {Black,
    GeometricTransformation[L, ReflectionTransform[{1, 1}]]},
  PlotRange -> All];
plot4 = Show[plot2,
  plot2 /.
   L_Line :> {Black,
    GeometricTransformation[L, ReflectionTransform[{-1, 1}]]},
  PlotRange -> All];
plot5 = Plot[(1^(2/3) - x^(2/3))^(3/2), {x, -4, 4},
  PlotRange -> {-4, 4}, AxesOrigin -> {0, 0}, PlotStyle -> {Black}];
plot6 = Plot[-(1^(2/3) - x^(2/3))^(3/2), {x, -4, 4},
  PlotRange -> {-4, 4}, AxesOrigin -> {0, 0}, PlotStyle -> {Black}];
plot7 = Show[plot5,
  plot5 /.
   L_Line :> {Black,
    GeometricTransformation[L, ReflectionTransform[{1, 1}]]},
  PlotRange -> All];
plot8 = Show[plot6,
  plot6 /.
   L_Line :> {Black,
    GeometricTransformation[L, ReflectionTransform[{1, -1}]]},
  PlotRange -> All];
Show[plot1, plot2, plot3, plot4, plot5, plot6, plot7, plot8]
m = 0.4;
n = 0.4;
plot9 = Plot[(1 - x^2)^(1/2), {x, -1.2, 1.2},
  PlotRange -> {-1.2, 1.2}, AxesOrigin -> {0, 0},
  AxesLabel -> {a21, a11}, PlotStyle -> {Black}];
plot10 =
 Plot[-(1 - x^2)^(1/2), {x, -1.2, 1.2}, PlotRange -> {-1.2, 1.2},
  AxesOrigin -> {0, 0}, AxesLabel -> {a21, a11},
  PlotStyle -> {Black}];
plot11 =
 Plot[(n*x)/(x + m), {x, -1.2, 1.2}, PlotRange -> {-1.2, 1.2},
  AxesOrigin -> {0, 0}, AxesLabel -> {a21, a11},
  PlotStyle -> {Black}];
plot12 =
 Plot[-(n*x)/(x + m), {x, -1.2, 1.2}, PlotRange -> {-1.2, 1.2},
  AxesOrigin -> {0, 0}, AxesLabel -> {a21, a11},
  PlotStyle -> {Black}];
Show[plot9, plot10, plot11, plot12]
plot13 =
 Plot[-4*x, {x, -1, 1}, PlotRange -> {-6, 6}, AxesOrigin -> {0, 0},
  AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
  PlotStyle -> {Black}];
plot14 = Show[Plot[0, {x, 0, 1}], Graphics[Line[{{1, -6}, {1, 6}}]]];
```

```
plot15 =
  Show[Plot[0, {x, 0, 1}], Graphics[Line[{{-1, -6}, {-1, 6}}]]];
plot16 =
  Plot[-4 x^3, {x, -1, 0}, PlotRange -> {-6, 6}, AxesOrigin -> {0, 0},
    AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
    PlotStyle -> {Black}, Filling -> Top];
plot17 =
  Plot[-4*x, {x, 0, 1}, PlotRange -> {-6, 6}, AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Bottom];
plot18 =
  Plot[-4 x^3, {x, 0, 1}, PlotRange -> {-6, 6}, AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Bottom];
plot19 =
  Plot[-4*x, {x, -1, 0}, PlotRange -> {-6, 6}, AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Top];
Show[plot13, plot14, plot15, plot16, plot17, plot18, plot19]



plot20 =
  Plot[-x, {x, -1, 1}, PlotRange -> {-2, 2}, AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}];
plot21 = Show[Plot[0, {x, 0, 1}], Graphics[Line[{{1, -6}, {1, 6}}]]];
plot22 =
  Show[Plot[0, {x, 0, 1}], Graphics[Line[{{-1, -6}, {-1, 6}}]]];
plot23 =
  Plot[-x^3, {x, -1, 0}, PlotRange -> {-2, 2}, AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Bottom];
plot24 =
  Plot[-x, {x, 0, 1}, PlotRange -> {-2, 2}, AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Top];
plot25 =
  Plot[-x^3, {x, 0, 1}, PlotRange -> {-2, 2}, AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Top];
plot26 =
  Plot[-x, {x, -1, 0}, PlotRange -> {-2, 2}, AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Bottom];
Show[plot20, plot21, plot22, plot23, plot24, plot25, plot26]


plot29 =
  Plot[-4 x^3, {x, -1, 0}, PlotRange -> {{-1, 1}, {-5, 5}},
   AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Bottom];
plot27 = Show[Plot[0, {x, -1, 1}], Graphics[Line[{{1, -5}, {1, 5}}]]];
plot28 =
  Show[Plot[0, {x, -1, 1}], Graphics[Line[{{-1, -5}, {-1, 5}}]]];
plot30 =
```

```
  Plot[-4*x, {x, 0, 1}, PlotRange -> {{-1, 1}, {-5, 5}},
   AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Bottom];
plot31 =
  Plot[-x, {x, -1, 0}, PlotRange -> {{-1, 1}, {-5, 5}},
   AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Top];
plot32 =
  Plot[-x^3, {x, 0, 1}, PlotRange -> {{-1, 1}, {-5, 5}},
   AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Top];
Show[plot27, plot28, plot29, plot30, plot31, plot32]


plot29 =
  Plot[-x^3, {x, -1, 0}, PlotRange -> {{-1, 1}, {-5, 5}},
   AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Bottom];
plot30 =
  Plot[-4*x, {x, 0, 1}, PlotRange -> {{-1, 1}, {-5, 5}},
   AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Bottom];
plot31 =
  Plot[-4 x, {x, -1, 0}, PlotRange -> {{-1, 1}, {-5, 5}},
   AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Top];
plot32 =
  Plot[-x^3, {x, 0, 1}, PlotRange -> {{-1, 1}, {-5, 5}},
   AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}, Filling -> Top];
plot27 = Show[Plot[0, {x, 0, 1}], Graphics[Line[{{1, -5}, {1, 5}}]]];
Show[plot29, plot30, plot31, plot32, plot27]


plot33 =
  Plot[-x^3, {x, -1, 1}, PlotRange -> {{-1, 1}, {-2, 2}},
   AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}];
plot34 =
  Plot[-x, {x, -1, 1}, PlotRange -> {{-1, 1}, {-2, 2}},
   AxesOrigin -> {0, 0},
   AxesLabel -> {"\!\(\*SubscriptBox[\"sin\[Beta]\", \"0\"]\)", "m"},
   PlotStyle -> {Black}];
Show[plot33, plot34]
```

A.6 – MERGE OF ALL EQUILIBRIA CALCULATION FILES (MSDOS)

```
type v*.txt > salida.txt
```

Note: v is the common first letter of name of all files that are needed to be merged.

A.7 – CONVERT INTO A READABLE FORMAT (MATLAB)

```
clear
clc
dat=load('c:\Home\u37244\MATLAB\salida.txt','-ascii'); %load file into ascii forma tinto the variable a

save 'c:\Home\u37244\MATLAB\dat.mat' dat -mat %save the variable a into .mat estension

clear %

load 'c:\Home\u37244\MATLAB\dat.mat' %load variable a.mat loaded in the file
```

Note: Load into a Matlab readable format the merged txt file.

A.8 – EQUILIBRIA 3D GRAPHICAL CALCULATIONS - SCATTER (MATLAB)

```
clear
clc
load -mat datos.m; %for loading the variable dat

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

roots=12;
res=1;% 1 is the complete resolution

H1=dat(dat(1:res:end,5)==roots,1);
H2=dat(dat(1:res:end,5)==roots,2);
H3=dat(dat(1:res:end,5)==roots,3);
%%%%%root=dat(dat(:,5)==roots,5);double check
hold on
plot3(H1,H2,H3,'.','MarkerSize',.1,'MarkerEdgeColor','k')

%[1 1 0] y yellow
%[1 0 1] m magenta
%[0 1 1] c cyan
%[1 0 0] r red
%[0 1 0] g green
%[0 0 1] b blue
%[1 1 1] w white
%[0 0 0] k black

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

roots=10;
res=1;% 1 is the complete resolution

H1=dat(dat(1:res:end,5)==roots,1);
H2=dat(dat(1:res:end,5)==roots,2);
H3=dat(dat(1:res:end,5)==roots,3);
%%%%%root=dat(dat(:,5)==roots,5);double check
hold on
plot3(H1,H2,H3,'.','MarkerSize',.1,'MarkerEdgeColor','b')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

roots=8;
res=1;% 1 is the complete resolution

H1=dat(dat(1:res:end,5)==roots,1);
H2=dat(dat(1:res:end,5)==roots,2);
H3=dat(dat(1:res:end,5)==roots,3);
%%%%root=dat(dat(:,5)==roots,5);double check
hold on
plot3(H1,H2,H3,'.','MarkerSize',.1,'MarkerEdgeColor','g')
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

roots=6;
res=1;% 1 is the complete resolution

H1=dat(dat(1:res:end,5)==roots,1);
H2=dat(dat(1:res:end,5)==roots,2);
H3=dat(dat(1:res:end,5)==roots,3);
%%%%root=dat(dat(:,5)==roots,5);double check
hold on
plot3(H1,H2,H3,'.','MarkerSize',.1,'MarkerEdgeColor','r')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

roots=4;
res=1;% 1 is the complete resolution

H1=dat(dat(1:res:end,5)==roots,1);
H2=dat(dat(1:res:end,5)==roots,2);
H3=dat(dat(1:res:end,5)==roots,3);
%root=dat(dat(:,5)==roots,5);double check
hold on
plot3(H1,H2,H3,'.','MarkerSize',.1,'MarkerEdgeColor','m')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

legend('Black - 12','Blue - 10','Green - 8','Red - 6','Magenta - 4')
title({'υ=0.2';'0.01<H3<1.5';'Step H3=0.01'})
xlabel('H1')
ylabel('H2')
zlabel('H3')
grid on
box on
```

A.9 – EQUILIBRIA 3D GRAPHICAL CALCULATIONS - SURFACE (MATLAB)

```
clear
clc
load -mat datos.m; %load data into the variable dat

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

roots=12;
res=1;% 1 is the complete resolution

H1=dat(dat(1:res:end,5)==roots,1);
H2=dat(dat(1:res:end,5)==roots,2);
H3=dat(dat(1:res:end,5)==roots,3);


%root=dat(dat(:,5)==roots,5);double check
%hold on
%plot3(H1,H2,H3,'.','MarkerSize',.1,'MarkerEdgeColor','k')

%[1 1 0] y yellow
%[1 0 1] m magenta
%[0 1 1] c cyan
%[1 0 0] r red
%[0 1 0] g green
%[0 0 1] b blue
%[1 1 1] w white
%[0 0 0] k black


%%%surf command can lso be
used%%%%http://www.mathworks.com/matlabcentral/newsreader/view_thread/21181

%tri = delaunay(H1,H2);
%[r,c] = size(tri);
%disp(r);
%h = trisurf(tri, H1, H2, H3,'FaceColor','cyan','EdgeColor','none');

xgrid = linspace(min(H1),max(H1),500);
ygrid = linspace(min(H2),max(H2),500);
[X,Y] = meshgrid(xgrid,ygrid);
Z = griddata(H1,H2,H3, X, Y);
figure;
surf(X,Y,Z,'FaceColor','cyan','EdgeColor','none');
hold on
%surf(-X,-Y,-Z,'FaceColor','cyan','EdgeColor','none');
%hold on
%surf(-X,-Y,Z,'FaceColor','cyan','EdgeColor','none');
%hold on
%surf(-X,Y,-Z,'FaceColor','cyan','EdgeColor','none');
%hold on
%surf(X,-Y,-Z,'FaceColor','cyan','EdgeColor','none');
%hold on
```

```
%surf(-X,Y,Z,'FaceColor','cyan','EdgeColor','none');
%hold on
%surf(X,-Y,Z,'FaceColor','cyan','EdgeColor','none');
%hold on
%surf(X,Y,-Z,'FaceColor','cyan','EdgeColor','none');
%hold on


camlight left;
lighting phong;
hold on

%legend('Black - 12','Blue - 10','Green - 8','Red - 6','Magenta - 4')
title({'υ=0.2';'0.01<H3<1.5';'Step H3=0.01';'12s Region'})
xlabel('H1')
ylabel('H2')
zlabel('H3')
grid on
box off
```

APPENDIX B – NOMENCLATURE

$\dot{\varphi}_i$: Constant angular velocity from the i-rotor relatively to the gyrostat body.

$\vec{\omega}$: Total angular velocity of the gyrostat satellite on the principal axis of inertia.

$\vec{\omega_S}$: Total angular velocity of the gyrostat satellite in relation with the principal axis of inertia.

$\vec{\omega_0}$: Total angular velocity of the gyrostat satellite in relation with the central attracting body.

$\alpha, \beta, \gamma, \psi, \theta, \varphi$ are the direction cosine between the principal axis and the gyrostat axis.

ω_{0x}: Angular velocity in the coordinate x relatively to the inertial central attracting body.

ω_{0y}: Angular velocity in the coordinate y relatively to the inertial central attracting body.

ω_{0z}: Angular velocity in the coordinate z relatively to the inertial central attracting body.

h_i: Projection of the Gyrostatic Motion Vector into axis Ox, Oy and Oz.

i: The gyrostat number of rotors.

m: Mass of the Satellite

p: Angular velocity in relation with the principal axis x.

q: Angular velocity in relation with the principal axis y.

r: Angular velocity in relation with the principal axis z.

t: time

A, B and C are the principal moments of inertia in relation to the gyrostat centre of inertia.

G: Universal Gravitic Constant - 6.67259×10^{-11} Nm^2/kg^2

H: Hamiltonian

J_i: Moment of inertia from the i-rotor.

M: Mass of the central attracting body

O = Gyrostat Centre of Mass

OXYZ: Orbital Reference Frame

OX: Axis aligned with the orbital plane, with the positive direction in the direction of speed.

OY: Axis normal to the orbital plane.

OZ: Axis that connects the centre of mass of the planet with the centre of mass of the gyrostat.

Oxyz: Gyrostat Coordinate System

T: Kinetic Energy

U: Force Function