Estimating the extremal index through local dependence

Helena Ferreira

Department of Mathematics of University of Beira Interior, Portugal

helena.ferreira@ubi.pt

Marta Ferreira

Center of Mathematics of Minho University

Center for Computational and Stochastic Mathematics of University of Lisbon

Center of Statistics and Applications of University of Lisbon, Portugal

msferreira@math.uminho.pt

Abstract

The extremal index is an important parameter in the characterization of extreme values of a stationary sequence. Our new estimation approach for this parameter is based on the extremal behavior under the local dependence condition $\mathrm{D}^{(k)}(u_n)$. We compare a process satisfying one of this hierarchy of increasingly weaker local mixing conditions with a process of cycles satisfying the $\mathrm{D}^{(2)}(u_n)$ condition. We also analyze local dependence within moving maxima processes and derive a necessary and sufficient condition for $\mathrm{D}^{(k)}(u_n)$. In order to evaluate the performance of the proposed estimators, we apply an empirical diagnostic for local dependence conditions, we conduct a simulation study and compare with existing methods. An application to a financial time series is also presented.

keywords: extreme value theory, stationary sequences, dependence conditions, extremal index

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1 Introduction

Let $\{X_n\}_{n\geq 1}$ be a stationary sequence with marginal distribution $F_{X_n} = F$. Consider $M_{i,j} = \bigvee_{s=i+1}^{j} X_s$, where $x \vee y$ denotes $\max(x,y)$, with $M_{0,j} = M_j$ and $M_{i,j} = -\infty$ for $i \geq j$. The sequence $\{X_n\}_{n\geq 1}$ has extremal index $\theta \in [0,1]$ if, for each $\tau > 0$, there is a sequence of normalized levels $\{u_n \equiv u_n^{(\tau)}\}_{n\geq 1}$, i.e.,

$$n(1-F(u_n)) \to \tau,$$

as $n \to \infty$, such that

$$P(M_n \leq u_n) \to e^{-\theta\tau} \tag{1}$$

(Leadbetter *et al.* [21] 1983). A null θ corresponds to "pathological" cases, not addressed here. When $\theta = 1$ the exceedances of high thresholds u_n, by the variables in $\{X_n\}_{n\geq 1}$, tend to occur isolated as in an independent variables context. However, if $\theta < 1$ we have groups of exceedances in the limit. Clusters of extreme values are linked with incidences and durations of catastrophic phenomena, an important issue in areas like environment, finance, insurance, engineering, among others. The extremal index is a key parameter in this context and its estimation has been greatly addressed in literature. The most popular procedures are the blocks and the runs estimators (e.g., Nandagopalan [24] 1990; Hsing [14] 1993; Smith and Weissman [30] 1994; Weissman and Novak [34] 1998; Robert *et al.* [26] 2009). Both methods require a clustering identification parameter which is a largely arbitrarily task to comply and has some impact in inference. The interexceedance times methods (Ferro and Segers ([10] 2003; Süveges [32] 2007) overcome this issue by avoiding this parameter but presents some threshold sensitivity. Laurini and Tawn ([18] 2003) proposes a two-threshold estimator leading to a more complete cluster identification. The maximum likelihood estimator in Süveges ([32] 2007) demands a local dependence condition to hold. The K-gaps estimator (Süveges and Davison [33] 2010; Fukutome *et al.* [11] 2014) implies the choice of a run-length K. Unlike these methods which depend on the choice of a threshold, the maxima procedures (Gomes [12] 1993; Ancona-Navarrete and Tawn [1] 2000; Northrop [25] 2015) are based on the choice of a block size.

In this paper we propose a new estimation procedure that works under the local dependence condition $\mathrm{D}^{(k)}(u_n)$ of Chernick *et al.* ([3], 1991). This condition requires the dependence condition $\mathrm{D}(u_n)$ of Leadbetter ([20], 1974), which states that for $\alpha_{n,l_n} \to 0$, as $n \to \infty$, for some sequence $l_n = o(n)$, where

$$\alpha_{n,l} = \sup\{|P(M_{i_1,i_1+p} \leq u_n, M_{j_1,j_1+q} \leq u_n) - P(M_{i_1,i_1+p} \leq u_n)\, P(M_{j_1,j_1+q} \leq u_n)|,$$

for any integers $1 \leq i_1 < i_1 + p + l \leq j_1 < j_1 + q \leq n$. We say that condition $\mathrm{D}^{(k)}(u_n)$ holds for

$\{X_n\}_{n\geq 1}$, if for some $\{b_n\}_{n\geq 1}$ such that,

$$b_n \to \infty,\ b_n\alpha_{n,l_n} \to 0,\ b_n l_n/n \to 0, \tag{2}$$

as $n\to\infty$, we have

$$nP\left(X_1 > u_n, M_{1,k} \leq u_n < M_{k,r_n}\right) \underset{n\to\infty}{\longrightarrow} 0,$$

with $\{r_n = [n/b_n]\}_{n\geq 1}$ ($[x]$ denotes the integer part of x). Condition $\mathrm{D}^{(k)}(u_n)$ is implied by

$$n\sum_{j=k+1}^{r_n} P\left(X_1 > u_n, M_{1,k} \leq u_n < X_j\right) \underset{n\to\infty}{\longrightarrow} 0.$$

This corresponds to condition $\mathrm{D}^{'}(u_n)$ of Leadbetter *et al.* ([21], 1983) whenever $k=1$ which locally restricts the occurrence of clusters of exceedances and thus leads to $\theta=1$. If $k=2$ we have condition $\mathrm{D}^{''}(u_n)$ of Leadbetter and Nandagopalan ([22], 1989). This condition locally restricts the occurrence of two or more upcrossings within a cluster, but still allows clustering of exceedances.

In Chernick *et al.* ([3], 1991) it is proved that, under $\mathrm{D}^{(k)}(u_n)$, the extremal index exists and is given by

$$\theta_X = \lim_{n\to\infty} P(M_{1,k} \leq u_n | X_1 > u_n). \tag{3}$$

The runs estimator can be derived from this relation by taking the runs parameter r equal to k. In particular, under condition $\mathrm{D}^{(2)}(u_n)$, Nadagopalan ([24], 1990) found

$$\begin{aligned}\theta_X = & \lim_{n\to\infty} P(X_2 \leq u_n | X_1 > u_n) \\ = & \lim_{n\to\infty} \frac{P(X_1 \leq u_n < X_2)}{P(X_2 > u_n)},\end{aligned}$$

which motivates his estimator based on the ratio between the number of upcrossings (equal to the number of downcrossings) and the number of exceedances. Although the $\mathrm{D}^{(2)}(u_n)$ condition implies $\mathrm{D}^{(k)}(u_n)$ for $k>2$ and we have several representations for θ_X as in (3), under $\mathrm{D}^{(2)}(u_n)$ we have only to be concerned with the count of upcrossings and exceedances, rather than the length r for runs of non-exceedances or intervals between exceedances. It is this easy approach in the Nandagopalan's estimator that we want to take advantage in this paper, by estimating θ_X through the extremal index of an auxiliary sequence satisfying $\mathrm{D}^{(2)}(u_n)$.

The results that motivate our new estimation approach are given in Section 2. In this section we relate the extremal index θ_X of the process satisfying $\mathrm{D}^{(k)}(u_n)$ with the one of a process of cycles satisfying $\mathrm{D}^{(2)}(u_n)$, deriving new representations for θ that motivate the estimators. In this way, we promote the application of the estimation procedures that work under $\mathrm{D}^{(2)}(u_n)$.

Knowledge about $\mathrm{D}^{(k)}(u_n)$ has not only impact on the computation of the extremal index of a process but also informs about the cluster structure of extreme values. In moving maximum processes we directly obtain the extremal index by calculating the limit in (1). It may be the reason why there is no study in literature, as far as we know, concerning local dependence within these processes. In Section 3 we derive a necessary and sufficient condition for $\mathrm{D}^{(k)}(u_n)$ to hold within moving maxima processes.

Section 4 is devoted to inference. We state a diagnostic tool to analyze $\mathrm{D}^{(k)}(u_n)$ since it is the context of our framework. Therefore, we are also moving forward in diminishing the arbitrarity in the declustering scheme of the runs estimator. We analyze the performance of the new estimators through simulation and illustrate with an application to a financial time series. We conclude in Section 5.

2 Extremal index of grouped variables

Let $\{I_0 = 0, I_n\}_{n\geq 1}$ be an increasing sequence of integer random variables (r.v.s) such that $\{S_n = I_n - I_{n-1}\}_{n\geq 1}$ is an i.i.d. sequence satisfying $E(S_n) = p$, with p positive integer. From such a renewal process and a stationary sequence $\{X_n\}_{n\geq 1}$, define

$$Z_n = M_{I_{n-1},I_n},\ n \geq 1. \tag{4}$$

Driven by the strategy used by Rootzén (1988, [27]) in the study of the extremal behavior of the regenerative processes, we will compare M_n with the maximum of the first $[n/p]$ variables in the sequence of cycles $\{Z_n\}_{n\geq 1}$.

Proposition 2.1. *Let $\{X_n\}_{n\geq 1}$ be a stationary sequence and $\{Z_n\}_{n\geq 1}$ defined by (4), for some renewal process $\{S_n\}_{n\geq 1}$ such that $E(S_n) = p$. If $\left\{n \bigvee_{i\geq 1} P\left(M_{I_{i-1},I_i} > u_n\right)\right\}_{n\geq 1}$ is bounded, then*

$$P(M_n \leq u_n) - P\left(\bigcap_{i=1}^{[n/p]} \{M_{I_{i-1},I_i} \leq u_n\}\right) \to 0,\ n \to \infty.$$

Proof. Let $L_n = \sup\{k : I_k \leq n\}$ and $U_n = \inf\{k : I_k > n\}$. By the law of large numbers, $\forall \epsilon > 0$, we have

$$P\left(\left|\frac{L_n}{n} - \frac{1}{p}\right| > \epsilon\right) \underset{n\to\infty}{\to} 0 \text{ and } P\left(\left|\frac{U_n}{n} - \frac{1}{p}\right| > \epsilon\right) \underset{n\to\infty}{\to} 0.$$

Furthermore,

$$\begin{aligned} P(M_n \leq u_n) &\leq P\left(\textstyle\bigcap_{i=1}^{L_n} \{M_{I_{i-1},I_i} \leq u_n\}\right) \\ &= P\left(\textstyle\bigcap_{i=1}^{L_n} \{M_{I_{i-1},I_i} \leq u_n\}, n(1/p - \epsilon) \leq L_n \leq n(1/p + \epsilon)\right) + o(1) \\ &\leq P\left(\textstyle\bigcap_{i=1}^{[n(1/p-\epsilon)]} \{M_{I_{i-1},I_i} \leq u_n\}\right) + o(1). \end{aligned}$$

Similarly, we derive

$$P(M_n \le u_n) \quad \ge P\left(\bigcap_{i=1}^{[n(1/p+\epsilon)]}\{M_{I_{i-1},I_i} \le u_n\}\right) + o(1).$$

Now, just observe that

$$0 \le P\left(\bigcap_{i=1}^{[n/p]}\{M_{I_{i-1},I_i} \le u_n\}\right) - P\left(\bigcap_{i=1}^{[n(1/p+\epsilon)]}\{M_{I_{i-1},I_i} \le u_n\}\right)$$

$$\le n\epsilon \bigvee_{i\ge 1}\{P\left(M_{I_{i-1},I_i} > u_n\right)\} \le \epsilon k,$$

as well as

$$0 \le P\left(\bigcap_{i=1}^{[n(1/p-\epsilon)]}\{M_{I_{i-1},I_i} \le u_n\}\right) - P\left(\bigcap_{i=1}^{[n/p]}\{M_{I_{i-1},I_i} \le u_n\}\right) \le \epsilon k,$$

for some constant k. □

If we assume that $\{Z_n\}_{n\ge1}$ is stationary satisfying D(u_n) and a local dependence condition D$^{(k)}(u_n)$ then, by applying the previous proposition, we can compute the extremal index of $\{X_n\}_{n\ge1}$ from the knowledge of the joint distribution of a finite number of consecutive terms of $\{Z_n\}_{n\ge1}$.

In what concerns the local behavior of the large values of $\{Z_n\}_{n\ge1}$, we are going to consider two ways: in Proposition 2.2 we derive the extremal index by assuming the local independence condition D$^{(1)}(u_n)$ and in Proposition 2.3 by assuming the local dependence condition D$^{(2)}(u_n)$.

Proposition 2.2. *Under the conditions of Proposition 2.1, if $\{Z_n\}_{n\ge1}$ is stationary and satisfies D(u_n) and D$^{(1)}(u_n)$ conditions for u_n such that $u_n \equiv u_n^{(\tau)}$ for $\{X_n\}_{n\ge1}$ and $u_n \equiv u_n^{(\tau^*)}$ for $\{Z_n\}_{n\ge1}$, then $\{X_n\}_{n\ge1}$ has extremal index*

$$\theta_X = \lim_{n\to\infty}\frac{P(M_{I_1} > u_n)}{pP(X_1 > u_n)} = \frac{\tau^*}{p\tau}.$$

Proof. We have that $\{Z_n\}_{n\ge1}$ has extremal index $\theta_Z = 1$ and thus, by applying Proposition 2.1,

$$\lim_{n\to\infty} P(M_n \le u_n) = \lim_{n\to\infty} P\left(\bigvee_{i=1}^{[n/p]} Z_i \le u_n\right) = \lim_{n\to\infty}\left(P(M_{I_1} \le u_n)\right)^{[n/p]} = e^{-\tau^*/p}.$$

Therefore, $\lim P(M_n \le u_n) = e^{-\theta_X \tau}$, as $n \to \infty$, with

$$\theta_X = \frac{\tau^*}{\tau p} = \lim_{n\to\infty}\frac{P(M_{I_1} > u_n)}{P(X_1 > u_n)p}.$$

□

This is what happens in regenerative processes with independent cycles (see expression (4.2) in Rootzén, [27] 1988, obtained directly).

Proposition 2.3. *Under the conditions of Proposition 2.1, if* $\{Z_n\}_{n\geq 1}$ *is stationary and satisfies* $D(u_n)$ *and* $D^{(2)}(u_n)$ *conditions for* u_n *such that* $u_n \equiv u_n^{(\tau)}$ *for* $\{X_n\}_{n\geq 1}$, $u_n \equiv u_n^{(\tau^*)}$ *for* $\{Z_n\}_{n\geq 1}$ *and* $nP\left(M_{I_1} \leq u_n < M_{I_1,I_2}\right) \to \nu^*$, *then* $\{X_n\}_{n\geq 1}$ *has extremal index*

$$\theta_X = \frac{\theta_Z \tau^*}{p\tau} = \lim_{n\to\infty} \frac{P(M_{I_1} \leq u_n < M_{I_1,I_2})}{pP(X_1 > u_n)} = \frac{\nu^*}{p\tau}.$$

Proof. We have that $\{Z_n\}_{n\geq 1}$ has extremal index

$$\theta_Z = \lim_{n\to\infty} \frac{P(M_{I_1} \leq u_n < M_{I_1,I_2})}{P(M_{I_1} > u_n)} = \frac{\nu^*}{\tau^*} \tag{5}$$

and thus, by applying Proposition 2.1,

$$\lim_{n\to\infty} P(M_n \leq u_n) = \lim_{n\to\infty} P\left(\bigvee_{i=1}^{[n/p]} Z_i \leq u_n\right) = e^{-\theta_Z \tau^*/p} = e^{-\tau\theta_X},$$

with

$$\theta_X = \frac{\theta_Z \tau^*}{p\tau} = \lim_{n\to\infty} \frac{P(M_{I_1} \leq u_n < M_{I_1,I_2})}{pP(X_1 > u_n)}. \tag{6}$$

□

This is what happens in regenerative processes with 1−dependent cycles. (see comment after expression (4.2) in Rootzén, [27] 1988).

Since

$$nP\left(Z_1 > u_n, \bigvee_{i=2}^{r_n} Z_i > u_n\right) = nP\left(Z_1 > u_n, Z_2 \leq u_n < \bigvee_{i=3}^{r_n} Z_i\right) + nP\left(Z_1 > u_n, Z_2 > u_n\right)$$

and

$$nP\left(Z_1 > u_n, Z_2 > u_n\right) = nP\left(Z_1 > u_n\right) - nP\left(Z_1 > u_n \geq Z_2\right),$$

we can remark that, for $\{Z_n\}_{n\geq 1}$ satisfying $\mathrm{D}^{(2)}(u_n)$, it holds that $\{Z_n\}_{n\geq 1}$ satisfies $\mathrm{D}^{(1)}(u_n)$ if and only if $\tau^* = \nu^*$, that is, the limiting mean number of exceedances is asymptotically equal to the limiting mean number of upcrossings (or downcrossings). Also, for any $k > 2$, provided that $\{X_n\}_{n\geq 1}$ satisfies $\mathrm{D}^{(k)}(u_n)$, $\mathrm{D}^{(k-1)}(u_n)$ holds if and only if

$$\lim_{n\to\infty} n\Big(P\left(X_1 > u_n, M_{1,k-1} \leq u_n\right) - P\left(X_1 > u_n, M_{1,k} \leq u_n\right)\Big) = 0.$$

This remark will help in the choice of a value for k, in Section 4, dedicated to the estimation of θ_X.

The presented results also point out a way to obtain the limiting law of the maximum term of the first $\sum_{i=1}^{n} S_i$ r.v.s of the sequence $\{X_n\}_{n\geq 1}$. In fact, for $\{X_n\}_{n\geq 1}$ and $\{S_n\}_{n\geq 1}$ as above, it holds that

$$P\left(M_{\sum_{i=1}^{n} S_i} \leq u_n\right) = P\left(\bigvee_{j=1}^{\sum_{i=1}^{n} S_i} X_j \leq u_n\right) = P(Z_1 \leq u_n, \ldots, Z_n \leq u_n)$$

$$\underset{n\to\infty}{\to} \quad e^{-\theta_Z \tau^*} = e^{-p\tau\theta_X} = \exp\left\{-E(S_1) \lim_{n\to\infty} nP(X_1 > u_n, X_2 \leq u_n, \ldots, X_k \leq u_n)\right\},$$

if $\{X_n\}_{n\geq 1}$ satisfies $\mathrm{D}^{(k)}(u_n)$.

We could state a general result analogous to the above Propositions 2.2 and 2.3, by considering $\{Z_n\}_{n\geq 1}$ satisfying $\mathrm{D}^{(k)}(u_n)$ with $k > 2$. However, our final goal is to relate the extremal index of a sequence $\{X_n\}_{n\geq 1}$ satisfying $\mathrm{D}^{(k)}(u_n)$ with $k > 2$, with the extremal index of an auxiliary sequence $\{Z_n\}_{n\geq 1}$ satisfying $\mathrm{D}^{(k)}(u_n)$ with $k \leq 2$. This will enable us to take profit of the estimation of the extremal index under $\mathrm{D}^{(1)}(u_n)$ or $\mathrm{D}^{(2)}(u_n)$, after a suitable transformation of the data. The identification of clusters reduces then to the identification of blocks of consecutive exceedances. The following results discuss relations on long-range and local dependence conditions for $\{X_n\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 1}$, which can be easily obtained in the particular case of a deterministic I_n, $n \geq 0$, considered later for the main proposal of this work.

Proposition 2.4. *Let $\{X_n\}_{n\geq 1}$ be such that $\{X_i\}_{i\in B}$ is independent of $\{S_i\}_{i\in A}$ whenever $A\cap B \neq 0$, for some renewal process $S = \{I_0 = 0, S_n = I_n - I_{n-1}\}_{n\geq 1}$ with $t \leq S_n \leq s$, $n \geq 1$. If, conditionally on S, $\{X_n\}_{n\geq 1}$ satisfies condition $D(u_n)$ with spacer sequence l_n, then $\{Z_n\}_{n\geq 1}$, defined by (4), satisfies $D(u_n)$ with $l_n^* = [2l_n/t]$.*

Proof. Let $I = \{i_1, \ldots, i_p\}$ and $J = \{j_1, \ldots, j_q\}$ with $1 \leq i_1 < \ldots < i_p < i_p + l_n^* < j_1 < \ldots < j_q \leq n$ and $l_n^* = [2l_n/t]$. Then

$$\left|P\left(\bigvee_{i\in I} Z_i \leq u_n, \bigvee_{i\in J} Z_i \leq u_n\right) - P\left(\bigvee_{i\in I} Z_i \leq u_n\right) P\left(\bigvee_{i\in J} Z_i \leq u_n\right)\right|$$

$$= \left|E\left(P\left(\bigvee_{i\in I^*(S_I)} X_i \leq u_n, \bigvee_{i\in J^*(S_J)} X_i \leq u_n\right)|S\right)\right.$$

$$\left. -E\left(P\left(\bigvee_{i\in I^*(S_I)} X_i \leq u_n\right)|S\right) E\left(P\left(\bigvee_{i\in J^*(S_J)} X_i \leq u_n\right)|S\right)\right|,$$

where S_A denotes the vector of r.v.s S_i, $i \in A$, and $I^*(S_I)$ and $J^*(S_J)$ are separated by at least l_n, since we have $I_{i_p} \leq i_p s$, $I_{j_1-1} \geq (j_1 - 1)t$ and thus $(j_1 - 1)t + 1 - i_p s \geq l_n$, for large enough n.

Therefore, and meeting the given assumptions, the previous expression is upper bounded by

$$
\begin{aligned}
& \Big| E\left(P\left(\bigvee_{i\in I^*(S_I)} X_i \le u_n, \bigvee_{i\in J^*(S_J)} X_i \le u_n\right) |S_{I\cup J}\right) \\
& -E\left(P\left(\bigvee_{i\in I^*(S_I)} X_i \le u_n\right) P\left(\bigvee_{i\in J^*(S_J)} X_i \le u_n\right) |S_{I\cup J}\right)\Big| \\
+ \quad & \Big| E\left(P\left(\bigvee_{i\in I^*(S_I)} X_i \le u_n\right) P\left(\bigvee_{i\in J^*(S_J)} X_i \le u_n\right) |S_{I\cup J}\right) \\
& -E\left(P\left(\bigvee_{i\in I^*(S_I)} X_i \le u_n\right) |S_I\right) E\left(P\left(\bigvee_{i\in J^*(S_J)} X_i \le u_n\right) |S_J\right)\Big| \\
\le \quad & \alpha_{n,l_n} .
\end{aligned}
$$

□

Proposition 2.5. *Let $\{X_n\}_{n\ge1}$ be a stationary sequence and $\{Z_n\}_{n\ge1}$ defined by (4), for some renewal process $S=\{I_0=0, S_n=I_n-I_{n-1}\}_{n\ge1}$ with $t\le S_n\le s$, $n\ge1$.*

(a) If $\{Z_n\}_{n\ge1}$ satisfies $D^{(2)}(u_n)$ with $r_n=[n/b_n]$, then $\{X_n\}_{n\ge1}$ satisfies $D^{(2s-t+1)}(u_n)$ with the same r_n.

(b) If $2t>s$ and $\{X_n\}_{n\ge1}$ satisfies $D^{(k)}(u_n)$ for some $k\le 2t-s+1$, with $r_n=[n/b_n]$, then $\{Z_n\}_{n\ge1}$ satisfies $D^{(2)}(u_n)$ with $r_n^*=[r_n/s]$.

Proof. To obtain (a) we take into account the following inequalities:

$$
\begin{aligned}
& nP\left(X_1>u_n, M_{1,2s-t+1}\le u_n < M_{2s-t+1,r_n}\right) \\
\le \quad & nP\left(X_1>u_n, M_{1,2s-t+1}\le u_n < M_{2s-t+1,tr_n-t+1)}\right) \\
= \quad & nP\left(X_t>u_n, M_{t,2s}\le u_n < M_{2s,tr_n}\right) \\
\le \quad & nP\left(M_{I_1}>u_n, M_{I_1,I_2}\le u_n < \bigvee_{i=3}^{r_n} M_{I_{i-1},I_i}\right) \\
= \quad & o(1),\, n\to\infty .
\end{aligned}
$$

For (b) we have:

$$
\begin{aligned}
& nP\left(M_{I_1}>u_n, M_{I_1,I_2}\le u_n < \bigvee_{i=3}^{r_n^*} M_{I_{i-1},I_i}\right) \\
\le \quad & nP\left(M_s>u_n, M_{s,2t}\le u_n < \bigvee_{i=2t+1}^{r_n^* s} X_j\right) \\
\le \quad & n\sum_{j=1}^{s} P\left(X_j>u_n, M_{j,s}\le u_n, M_{s,2t}\le u_n < M_{2t,r_n}\right)
\end{aligned}
$$

and each of the s terms in the sum above tends to zero by the $\mathrm{D}^{(2t-s+1)}(u_n)$ condition for $\{X_n\}_{n\ge1}$.

□

We state now a result on the “clustered" process $\{Z_n\}_{n\ge1}$ that resumes the path to obtain its extremal index θ_Z and to recover θ_X for the "declustered" process $\{X_n\}_{n\geq 1}$, showing that counting the mean number of upcrossings (or downcrossings) for $\{Z_1, \ldots, Z_n\}$ is asymptotically equivalent to counting the mean number of runs $\{X_i > u_n, X_{i+1} \leq u_n, \ldots, X_{i+k-1} \leq u_n\}$, $i \leq np$.

Corollary 2.6. *Let $\{X_n\}_{n\geq 1}$ and S be in the conditions of Proposition 2.4 and that $E(S_n) = p$. Suppose that $u_n \equiv u_n^{(\tau)}$ for $\{X_n\}_{n\geq 1}$ which satisfies $D^{(k)}(u_n)$ for some $k \leq 2t - s + 1$. Then*

(a) $\{Z_n\}_{n\geq 1}$ defined by (4) satisfies $D(u_n)$ and $D^{(2)}(u_n)$ conditions.

(b) If $u_n \equiv u_n^{(\tau^*)}$ for $\{Z_n\}_{n\geq 1}$ and there exists $\lim_{n\to\infty} nP(M_{I_1} \leq u_n < M_{I_1,I_2}) = \nu^*$ then it holds that

$$\theta_Z = \frac{\nu^*}{\tau^*},\ \theta_X = \frac{\theta_Z \tau^*}{p\tau}$$

and

$$\lim_{n\to\infty} nP(M_{I_1} \leq u_n < M_{I_1,I_2}) = p \lim_{n\to\infty} nP(X_1 > u_n, X_2 \leq u_n, \ldots, X_k \leq u_n).$$

We now focus on the particular case of $I_n = n(k-1)$, $n \geq 0$, for some $k > 2$. Therefore, we have $s = t = k - 1$. If is this the case then, from the previous result, condition $\mathrm{D}^{(2)}(u_n)$ for $\{Z_n\}_{n\geq 1}$ implies condition $\mathrm{D}^{(k)}(u_n)$ for $\{X_n\}_{n\geq 1}$ with the same length r_n to model "local behavior". Otherwise, the validity of condition $\mathrm{D}^{(k)}(u_n)$ for $\{X_n\}_{n\geq 1}$ leads to condition $\mathrm{D}^{(2)}(u_n)$ for $\{Z_n\}_{n\geq 1}$ with $r_n^* = [r_n/(k-1)]$. We can then state the following corollary, which can also be proved directly.

Corollary 2.7. *Let $\{X_n\}_{n\geq 1}$ be a stationary sequence and $\{Z_n\}_{n\geq 1}$ defined by (4) with $I_n = n(k-1)$, $n \geq 0$, for some $k > 2$.Then $\{Z_n\}_{n\geq 1}$ satisfies condition $D^{(2)}(u_n)$ if and only if $\{X_n\}_{n\geq 1}$ satisfies condition $D^{(k)}(u_n)$.*

Thus, under conditions of Proposition 2.5 and according to (6), the extremal index of $\{X_n\}_{n\geq 1}$ can be written as

$$\begin{aligned} \theta_X &= \frac{\theta_Z \tau^*}{(k-1)\tau} \\ &= \lim_{n\to\infty} \frac{P(M_{I_1} \leq u_n < M_{I_1,I_2})}{(k-1)P(X_1 > u_n)} = \lim_{n\to\infty} \frac{P(M_{k-1} \leq u_n < M_{k-1,2(k-1)})}{(k-1)P(X_1 > u_n)}. \end{aligned} \tag{7}$$

By using the stationarity, this limit can be rewritten as the one obtained in Chernick *et al.* ([3], 1991) under condition $\mathrm{D}^{(k)}(u_n)$ and given in (3). However, representation (7) allows to estimate θ_X through the estimation of θ_Z for the cycles $Z_i = M_{I_{i-1},I_i}$, $i \geq 1$, for which $\mathrm{D}^{(2)}(u_n)$ holds, as we will present in Section 4.

Here we illustrate the above results with finite moving maxima processes (MM) and in the next section we devote special attention to the local dependence in this kind of processes.

Example 2.1. Consider the moving maximum process, $X_n = \bigvee_{j=0}^{2} \alpha_j Y_{n-j}$, $\alpha_0 = 2/6$, $\alpha_1 = 1/6$ and $\alpha_2 = 3/6$, $n \geq 1$, with sequence $\{Y_n\}_{n\geq -1}$ independent and having standard Fréchet marginal distribution, $F_Y = \exp(-1/x)$, $x > 0$. This stationary sequence satisfies $\mathrm{D}^{(3)}(u_n)$ for levels $u_n = n/\tau$, $\tau > 0$, as will be seen in the next section and $\theta_X = ((2/6) \vee (1/6) \vee (3/6)) = 1/2$ (see Weissman and Cohen [35], 1995). For $Z_n = X_{2n-1} \vee X_{2n}$, we have $nP(Z_1 > u_n) \to (5/3)\tau = \tau^*$, as $n \to \infty$, since

$$P(Z_1 \leq u_n) = F_Y(u_n)F_Y(3u_n/2).$$

Observe also that

$$P(Z_2 \leq u_n, Z_1 \leq u_n) = F_Y(u_n/2)F_Y(3u_n/2)$$

and, provided that $\mathrm{D}^{(2)}(u_n)$ holds for $\{Z_n\}_{n\geq 1}$, we have

$$\theta_Z = \lim_{n\to\infty} P(Z_2 \leq u_n | Z_1 > u_n) = 3/5.$$

By applying (7), we obtain $\theta_X = 1/2$. □

We have seen that, for every stationary sequence $\{X_n\}_{n\geq 1}$ satisfying $\mathrm{D}^{(k)}(u_n)$, $k > 2$, we can build a stationary sequence $\{Z_n\}_{n\geq 1}$ satisfying $\mathrm{D}^{(2)}(u_n)$ by taking the maxima of $k-1$ consecutive variables of sequence $\{X_n\}_{n\geq 1}$. For big values of k, such aggregation can result in reduced accuracy in the estimation of θ_X via the sample based on $\{Z_n\}_{n\geq 1}$, as will be pointed in Section 4. Proposition 2.5 states that we can also consider mixtures of big cycles of several lengths in order to build the sequence $\{Z_n\}_{n\geq 1}$ satisfying $\mathrm{D}^{(2)}(u_n)$, as it is illustrated in the next example.

Example 2.2. Let $S = \{I_0 = 0, S_n = I_n - I_{n-1}\}_{n\geq 1}$ be a sequence of independent r.v.s uniformly distributed on $\{k, k+2\}$, with a fixed $k \geq 6$, and independent of $\{X_n\}_{n\geq 1}$. For $\{X_n\}_{n\geq 1}$ take a stationary sequence satisfying $\mathrm{D}^{(5)}(u_n^{(\tau)})$ and $\mathrm{D}(u_n^{(\tau)})$. Let Z_n be as in (4), which is stationary and also satisfies $\mathrm{D}(u_n)$. Since $2k - (k+2) + 1 \geq 5$, then $\{Z_n\}_{n\geq 1}$ satisfies condition $\mathrm{D}^{(2)}(u_n)$. It holds that

$$\tau^* = \lim_{n\to\infty} \frac{1}{2} \sum_{s\in\{k,k+2\}} nP\left(\bigvee_{i=1}^{s} X_i > u_n\right),$$

$$\nu^* = \lim_{n\to\infty} \frac{1}{4} \sum_{s_1,s_2\in\{k,k+2\}} nP\left(\bigvee_{i=1}^{s_1} X_i \leq u_n < \bigvee_{i=s_1+1}^{s_1+s_2} X_i\right),$$

$\theta_Z = \frac{\nu^*}{\tau^*}$ and $\theta_X = \frac{\nu^*}{(k+1)\tau}$. Un example of a process $\{X_n\}_{n\geq 1}$ satisfying the conditions above is a moving maxima, given in the next section, with signatures $\alpha_{l,j}$ and $u_n^{(\tau)} = n/\tau$. □

3 Condition $\mathrm{D}^{(k)}(u_n)$ for Moving Maxima processes

A moving maxima process (MM) is defined as

$$X_n = \bigvee_{l\geq 1} \bigvee_{-\infty<j<\infty} \alpha_{l,j} Y_{l,n-j},\ n \geq 1, \tag{8}$$

where $\{Y_{l,j}\}_{l\geq 1,\,-\infty<j<\infty}$ is an i.i.d. sequence of r.v.s, usually unit Fréchet and $\{\alpha_{l,j}\}_{l\geq 1,\,-\infty<j<\infty}$ are non negative constants (usually denoted signatures) such that $\sum_{l\geq 1}\sum_{-\infty<j<\infty}\alpha_{l,j}=1$ (Deheuvels [6] 1983, Davis and Resnick [5] 1989, Smith and Weissman [31] 1996, Hall *et al.* [13] 2002, Meinguet [23] 2012).

Under the condition $\sum_{-\infty<j<\infty}\sum_{l\geq 1} l\alpha_{l,j} < \infty$, the MM process is strong-mixing (Meinguet, [23] 2012) and therefore it satisfies $\mathrm{D}(u_n)$.

An interesting feature of these processes is that the transformation of $\{Y_{l,j}\}_{l\geq 1,\,-\infty<j<\infty}$ induces a dependence structure with extremes occurring in temporal clusters. Any stationary process with finite-dimensional marginal distributions of multivariate extreme value type can be approximated by an MM process with marginals of extreme value type (Hall *et al.* [13] 2002). Examples of finite MM processes (i.e., with l and j finite) are not difficult to deal with and are often used to illustrate long range and local dependence conditions within extreme values. The extremal index is directly obtained through $\lim_{n\to\infty} P(M_n \leq n/\tau)$, $\tau > 0$, even for infinite MM processes and thus avoids the validity of some $\mathrm{D}^{(k)}$ condition. In Meinguet ([23] 2012; Theorem 4) it was presented a nice finite-cluster condition which prevents a sequence of extremes occurring in MM from being infinite over time. However, it does not enable a representation for θ from finite marginal distributions of the process. The local dependence conditions brings us enlightenment about the clustering structure of extreme values. Any finite MM is m-dependent for some positive integer m and thus $\mathrm{D}^{(k)}$ holds at least for some $k \geq m$. From simple examples, we know that small changes in the values of coefficients $\alpha_{l,j}$ may lead to large diferences within the clusters structure. Hence, this raises the question of which conditions $\alpha_{l,j}$ must satisfy so that some $\mathrm{D}^{(k)}$ holds for an MM process. The next result presents a necessary and sufficient condition.

Proposition 3.1. *Let $\{X_n\}_{n\geq 1}$ be an MM process as defined in (8), where $\{Y_{l,j}\}_{l\geq 1,\,-\infty<j<\infty}$ is an i.i.d. sequence of unit Fréchet r.v.s. Then*

(a) $\{X_n\}_{n\geq 1}$ satisfies condition $D^{(k)}(u_n)$, $k \geq 2$, if and only if, for all $l \geq 1$ and $-\infty < j < \infty$,

$$\alpha_{l,j} \wedge \left(\bigvee_{s\geq k+1} \alpha_{l,j+s-1} \right) \leq \bigvee_{s=2}^{k} \alpha_{l,j+s-1}, \tag{9}$$

where $x \wedge y$ denotes $\min(x,y)$.

(b) $\{X_n\}_{n\geq 1}$ satisfies condition $D^{(1)}(u_n)$ if and only if, for all $l \geq 1$ there is only one $-\infty < j < \infty$ such that $\alpha_{l,j} > 0$.

Proof. (a) Observe that

$$\begin{aligned} & nP\left(X_1 > u_n, \bigvee_{s=2}^{k} X_s \le u_n, \bigvee_{s=k+1}^{r_n} X_s > u_n\right) \\ = \;& nP\left(X_1 > u_n, \bigvee_{s=k+1}^{r_n} X_s > u_n\right) - nP\left(X_1 > u_n, \bigvee_{s=2}^{k} X_s > u_n, \bigvee_{s=k+1}^{r_n} X_s > u_n\right). \end{aligned} \tag{10}$$

Since $X_s = \bigvee_l \bigvee_j \alpha_{l,j} Y_{l,1-(j-s+1)}$ and $\{Y_{l,j},\, l \ge 1,\, -\infty < j < \infty\}$ is a sequence of independent r.v.s, we have

$$\begin{aligned} & \lim_{n\to\infty} nP\left(X_1 > u_n, \bigvee_{s=k+1}^{r_n} X_s > u_n\right) \\ = \;& \lim_{n\to\infty} n \sum_l \sum_j P\left(\alpha_{l,j} Y_{l,1-j} > u_n, \bigvee_{s=k+1}^{r_n} \alpha_{l,j+s-1} Y_{l,1-j} > u_n\right). \end{aligned}$$

Thus, for levels $u_n = n/\tau$, $\tau > 0$,

$$\begin{aligned} & \lim_{n\to\infty} nP\left(X_1 > n/\tau, \bigvee_{s=k+1}^{r_n} X_s > n/\tau\right) \\ = \;& \lim_{n\to\infty} n \sum_l \sum_j P\left(Y_{l,1-j} > \frac{n/\tau}{\alpha_{l,j}} \vee \frac{n/\tau}{\bigvee_{s=k+1}^{r_n} \alpha_{l,j+s-1}}\right) \\ = \;& \lim_{n\to\infty} n \sum_l \sum_j P\left(Y_{l,1-j} > \frac{n/\tau}{\alpha_{l,j} \wedge \bigvee_{s=k+1}^{r_n} \alpha_{l,j+s-1}}\right). \end{aligned}$$

Using the same reasoning on the second term in (10) and by the theorem of dominated convergence,

$$\begin{aligned} & \lim_{n\to\infty} nP\left(X_1 > u_n, \bigvee_{s=2}^{k} X_s \le u_n, \bigvee_{s=k+1}^{r_n} X_s > u_n\right) \\ = \;& \sum_l \sum_j \lim_{n\to\infty} n \left(F_Y\left(\frac{n/\tau}{\alpha_{l,j} \wedge \bigvee_{s=2}^{k} \alpha_{l,j+s-1} \wedge \bigvee_{s=k+1}^{r_n} \alpha_{l,j+s-1}}\right)\right. \\ & \left. - F_Y\left(\frac{n/\tau}{\alpha_{l,j} \wedge \bigvee_{s=k+1}^{r_n} \alpha_{l,j+s-1}}\right)\right) \\ = \;& \sum_l \sum_j \tau \left(\left(\alpha_{l,j} \wedge \bigvee_{s\ge k+1} \alpha_{l,j+s-1}\right) - \left(\alpha_{l,j} \wedge \bigvee_{s=2}^{k} \alpha_{l,j+s-1} \wedge \bigvee_{s\ge k+1} \alpha_{l,j+s-1}\right)\right), \end{aligned}$$

since $\alpha_{l,j} \to 0$, as $l \to \infty$ and $j \to \infty$. Now just observe that the limit will be null whenever relation (9) holds, for all $l \ge 1$ and $-\infty < j < \infty$.

(b) The conclusion in (b) follows from

$$\lim_{n\to\infty} nP\left(X_1 > u_n, \bigvee_{s=2}^{r_n} X_s > u_n\right) = \sum_l \sum_j \tau \left(\alpha_{l,j} \wedge \bigvee_{s\ge 2} \alpha_{l,j+s-1}\right).$$

Table 1: Verification of conditions $\mathrm{D}^{(3)}(u_n)$ and $\mathrm{D}^{(2)}(u_n)$ for, respectively, the MM processes $\{X_n\}_{n\geq 1}$ and $\{W_n\}_{n\geq 1}$ of Example 3.1, according to relation in (9).

condition $\mathrm{D}^{(3)}(u_n)$		for $\{X_n\}_{n\geq 1}$	condition $\mathrm{D}^{(2)}(u_n)$		for $\{W_n\}_{n\geq 1}$
j	$\alpha_j \wedge \left(\bigvee_{s\geq 4}\alpha_{j+s-1}\right)$	$\alpha_{j+1}\vee\alpha_{j+2}$	j	$\alpha_j \wedge \left(\bigvee_{s\geq 3}\alpha_{j+s-1}\right)$	α_{j+1}
≤ -3	$0\wedge 3/6$	0	≤ -2	$0\wedge 3/6$	0
$-2,-1$	$0\wedge 3/6$	2/6	-1	$0\wedge 3/6$	1/6
0	$2/6\wedge 0$	3/6	0	$1/6\wedge 2/6$	3/6
1	$1/6\wedge 0$	3/6	1	$3/6\wedge 0$	2/6
2	$3/6\wedge 0$	0	2	$2/6\wedge 0$	0
≥ 3	$0\wedge 0$	0	≥ 3	$0\wedge 0$	0

The latter sum is null if and only if, for each l, there is j^* such that $\alpha_{l,j^*}>0$ and $\alpha_{l,j}=0$ for $j\neq j^*$.

□

Example 3.1. Consider the moving maximum processes, $X_n=\bigvee_{j=0}^{2}\alpha_j Y_{n-j}$, $\alpha_0=2/6$, $\alpha_1=1/6$ and $\alpha_2=3/6$, $n\geq 1$, given in Example 2.1, and $W_n=\bigvee_{j=0}^{2}\alpha_j Y_{n-j}$, $\alpha_0=1/6$, $\alpha_1=3/6$ and $\alpha_2=2/6$, $n\geq 1$, with sequence $\{Y_n\}_{n\geq -1}$ independent and having standard Fréchet marginal distribution. We will see that $\{X_n\}_{n\geq 1}$ satisfies $\mathrm{D}^{(3)}(u_n)$ (and not $\mathrm{D}^{(2)}(u_n)$) and that $\{W_n\}_{n\geq 1}$ satisfies $\mathrm{D}^{(2)}(u_n)$, for levels $u_n=n/\tau$, $\tau>0$, by applying relation (9). The calculations are summarized in Table 1. Observe that $\mathrm{D}^{(2)}(u_n)$ does not hold for $\{X_n\}_{n\geq 1}$ since, if $j=0$ then $\alpha_0\wedge\left(\bigvee_{s\geq 3}\alpha_{s-1}\right)=2/6\wedge 3/6>1/6=\alpha_1$. Presumably in $\{X_n\}_{n\geq 1}$ persists longer than $\{W_n\}_{n\geq 1}$ because the largest signature is the most recent one. □

Inference within MM processes has been addressed in literature (see Zhang and Smith [36], 2010). Therefore, as an alternative to the empirical method of Süveges ([32] 2007), we can check the validity of $\mathrm{D}^{(k)}(u_n)$ within these processes by estimating coefficients $\alpha_{l,j}$ and applying (9).

The MM processes are stationary max-stable processes for which, under $\mathrm{D}^{(k)}(u_n)$, we can derive the extremal index from a tail dependence coefficient. Suppose that the stationary process $\{X_n\}_{n\geq 1}$ has unit Fréchet marginals $F(x)=\exp(-1/x)$, $x>0$. If $\mathrm{D}^{(k)}(u_n^{(\tau)})$ holds for $\{X_n\}_{n\geq 1}$, then

$$\begin{aligned}\theta_X &= \lim_{n\to\infty} P\left(M_{1,k}\leq n/\tau | X_1>n/\tau\right)\\ &= 1-\lim_{n\to\infty} P\left(M_{1,k}> n/\tau | X_1>n/\tau\right)\\ &= 1-\Lambda_U^{(I_1|I_2)}(1,1),\end{aligned}\tag{11}$$

provided the limit exists, where $I_1=\{2,\ldots,k\}$, $I_2=\{1\}$ and $\Lambda_U^{(I_1|I_2)}(1,1)$ is the upper tail dependence coefficient considered in Ferreira and Ferreira ([9], 2012b). In the case of max-stable processes or, more generally, processes satisfying the max-domain of attraction condition, the limit in (11) is always defined. By applying the propositions 2.1 and 3.1 in Ferreira and Ferreira ([9], 2012b), we conclude that

$$\theta_X = \frac{E\left(e^{-M_k^{-1}}\right)}{1-E\left(e^{-M_k^{-1}}\right)} - \frac{E\left(e^{-M_{1,k}^{-1}}\right)}{1-E\left(e^{-M_{1,k}^{-1}}\right)} \tag{12}$$

and, in particular for $k=2$, it holds that

$$\theta_X = \frac{1}{1-E\left(e^{-(X_1\vee X_2)^{-1}}\right)} - 2. \tag{13}$$

This representation for θ_X motivates its estimation from moment estimators for $E\left(e^{-M_k^{-1}}\right) = E\left(\bigvee_{i=1}^k F(X_i)\right)$, as considered in Ferreira and Ferreira ([9], 2012b).

We apply now the results of the previous section in order to compute θ_X from θ_Z of the process $\{Z_n = \bigvee_{i=(n-1)(k-1)+1}^{n(k-1)} X_i\}_{n\geq 1}$ under $\text{D}^{(2)}(u_n)$. The estimation of θ_Z is considerably simpler as suggested by (13).

Proposition 3.2. *Let $\{X_n\}_{n\geq 1}$ be a stationary max-stable process with unit Fréchet marginals F and $u_n^{(\tau)} = n/\tau$, $\tau > 0$. Then*

(a) $\{Z_n\}_{n\geq 1}$ is stationary and max-stable with marginal distribution $F_Z(x) = F^{\epsilon_{k-1}}(x)$, where $\epsilon_{k-1} = -\log F_{(X_1,\ldots,X_{k-1})}(1,\ldots,1) \in [1,k-1]$ is the $(k-1)$-th extremal coefficient of $\{X_n\}_{n\geq 1}$.

(b) If $\{X_n\}_{n\geq 1}$ satisfies $D(u_n)$ and $D^{(k)}(u_n)$, $k>2$, then $\{Z_n\}_{n\geq 1}$ satisfies $D(u_n)$ and $D^{(2)}(u_n)$,

$$\theta_Z = \frac{1}{1-E\left(F_Z(Z_1)\vee F_Z(Z_2)\right)} - 2 \tag{14}$$

and

$$\theta_X = \theta_Z \frac{-\log F_Z(1)}{k-1}. \tag{15}$$

Proof. Part (a) follows trivially and we only justify (b).

We first consider the sequence of cycles $\{Z_n^* = \bigvee_{i=(n-1)(k-1)+1}^{n(k-1)} X_i/\epsilon_{k-1}\}_{n\geq 1}$ which satisfies the same local and long-range dependence conditions as $\{Z_n\}_{n\geq 1}$. For this stationary and max-stable sequence with unit Fréchet marginals, by applying (13), we obtain

$$\theta_{Z^*} = \frac{1}{1-E\left(e^{-(Z_1^*\vee Z_2^*)^{-1}}\right)} - 2.$$

Then

$$\theta_Z = \theta_{Z^*} = \frac{1}{1-E\left(e^{-(Z_1\vee Z_2)^{-1}\epsilon_{k-1}}\right)} - 2 = \frac{1}{1-E\left(F_Z(Z_1)\vee F_Z(Z_2)\right)} - 2.$$

To obtain the relation (15) we apply Proposition 2.3 with

$$\tau^* = \lim_{n\to\infty} nP(Z > n/\tau) = -\log F_Z(1)\tau.$$

□

This result suggests the estimation of θ_X via the estimation of $-\log F_Z(1)$ and $E\left(F_Z(Z_1) \vee F_Z(Z_2)\right)$.

4 Estimation

Our new estimation proposal consists in first, to state the sequence of cycles, $Z_n = \bigvee_{s=(n-1)(k-1)+1}^{n(k-1)} X_s$, $n \geq 1$, and then estimate θ based on $\{Z_n\}_{n\geq 1}$. Observe that, from Proposition 2.3, we can write

$$\theta_X = \frac{n/(k-1)P(Z_1 \leq u_n < Z_2)}{nP(X_1 > u_n)} = \theta_Z \frac{n/(k-1)P(Z_1 > u_n)}{nP(X_1 > u_n)},$$

and thus define the estimator

$$\widehat{\theta}_X = \frac{U_n^Z(u_n)}{N_n^X(u_n)}, \tag{16}$$

as well as, the estimator

$$\widehat{\theta}_X = \frac{\widehat{\theta}_Z N_n^Z(u_n)}{N_n^X(u_n)}, \tag{17}$$

where $U_n^Z(u_n)$ and $N_n^Z(u_n)$ are, respectively, the number of upcrossings of u_n and the number of exceedances of u_n within $\{Z_1, \ldots, Z_{[n/(k-1)]}\}$ and $N_n^X(u_n)$ is the number of exceedances of u_n within $\{X_1, \ldots, X_n\}$. Since $\mathrm{D}^{(2)}(u_n)$ holds for $\{Z_n\}_{n\geq 1}$, estimators under this condition can be used to calculate $\widehat{\theta}_Z$, e.g., the maximum likelihood estimator in Süveges ([32], 2007) and the upcrossings estimator in Nandagopalan ([24], 1990).

Now observe that, based on (11), we can write θ_Z as

$$\theta_Z = 1 - \lim_{n\to\infty} \frac{P(M_{I_1,I_2} > u_n | M_{I_1} > u_n)}{P(M_{I_1} > u_n)} = 1 - \lambda_Z,$$

where λ_Z is the so called "tail dependence coefficient" (see Joe [16] 1997 p. 33, Coles *et al.* [4] 1999, Schmidt and Stadmüller [28] 2006 and references therein; see also Ferreira and Ferreira [8] 2012a Proposition 4). Hence, we can derive

$$\theta_X = \frac{(1-\lambda_Z)\tau^*}{(k-1)\tau}, \tag{18}$$

and thus also state the estimator

$$\widehat{\theta}_X = \frac{(1-\widehat{\lambda}_Z) N_n^Z(u_n)}{N_n^X(u_n)}. \tag{19}$$

We can estimate the tail dependence coefficient by applying a non-parametric procedure, e.g., the one in Schmidt and Stadmüller ([28], 2006). This estimator will be denoted $\widehat{\theta}^{SS}$. For the particular case of max-stable processes, by representation (14), we can apply the estimator $\widehat{\theta}_Z$ in Ferreira and Ferreira ([9], 2012b) for θ_Z and again $\widehat{\theta}_X$ as in (17). In the following, this method is denoted $\widehat{\theta}^{FF}$. A similar procedure based on (15) leads to a second estimator for max-stable processes, namely

$$\widehat{\theta}_X = \widehat{\theta}_Z \frac{-\log \widehat{F}_Z(1)}{k-1},$$

where $\widehat{F}_Z(1)$ is the empirical distribution function. The notation for this latter is $\widehat{\theta}^{FF^*}$.

In the next section we analyze our new proposal through simulation. For $\widehat{\theta}_Z$ in expression (17), we consider the upcrossings estimator of Nandagopalan ([24], 1990), the estimator of Ferro and Segers ([10], 2003) also known as intervals estimator and the maximum likelihood estimator of Süveges ([32], 2007), and denote our extremal index estimators, respectively, $\widehat{\theta}^U$, $\widehat{\theta}^I$ and $\widehat{\theta}^{ML}$. We also compare with the intervals and runs estimators applied directly on $\{X_n\}_{n\geq 1}$. For these estimators we use notation $\widetilde{\theta}^I$ and $\widetilde{\theta}^R$, respectively.

In order to analyze $\mathrm{D}^{(k)}(u_n)$ and construct the cycles $\{Z_n\}_{n\geq 1}$, we can extend the methodology in Süveges ([32], 2007) considered to check $\mathrm{D}^{(2)}(u_n)$. More precisely, we compute the proportion of anti-$\mathrm{D}^{(k)}(u_n)$ events by

$$p_k(u_n, r_n) = \frac{\sum_{j=1}^{n-r_n+1} \mathbb{1}_{\{X_j>u_n, X_{j+1}\leq u_n, \ldots, X_{j+k-1}\leq u_n, M_{j+k-1, r_n+j-1}>u_n\}}}{\sum_{j=1}^{n} \mathbb{1}_{\{X_j>u_n\}}},$$

for normalized levels u_n approximated by the empirical quantiles $1-\tau/n$, for some fixed positive τ and some sequence $\{r_n\}_{n\geq 1}$ satisfying the conditions of Proposition (2.2), where $\mathbb{1}_{\{\cdot\}}$ is the indicator function. We take the proportions $p_k(u_m, r_m)$ for sequences $\{X_1, \ldots, X_m\}$, with increasing length $m \leq n$.

On what concerns the choice of $\{b_n\}_{n\geq 1}$, we can choose, for instance, the family of sequences of integers $b_n^{(s)} = [(\log n)^s]$, $s>0$. Thus, for each τ and s, we can plot the points $(m, p_k(u_m, r_m^{(s)}))$, which must converge to zero, for some s, as m increases if $\mathrm{D}^{(k)}(u_n)$ holds with $b_n^{(s)}$. This is a slightly different approach of the one in Süveges ([32], 2007), but closer to the definition of $\mathrm{D}^{(k)}(u_n)$, since this condition states a limiting behavior as $n\to\infty$, and $u_n \approx F^{-1}(1-\tau/n)$, $r_n = [n/b_n]$ and $p_k(u_n, r_n)$ are functions of n. To avoid three-dimensional plots that arise from the joint variation of τ, s and m, we can separately analyze the evolution of the proportions for different choices of r_n.

For the particular case of $D^{(1)}(u_n)$ condition, we have the proportions

$$p_1(u_n, r_n) = \frac{\sum_{j=1}^{n-r_n+1} \mathbb{1}_{\{X_j > u_n, M_{j,r_n+j-1} > u_n\}}}{\sum_{j=1}^{n} \mathbb{1}_{\{X_j > u_n\}}}.$$

Once accepted the condition $D^{(k)}(u_n)$ for some k, that means we consider that the process satisfies $D^{(s)}(u_n)$ for all $s \geq k$ and does not satisfy $D^{(s)}(u_n)$ for $s < k$. The decision to exclude values less than k may be based on the analysis of $(m, p_{k-1}(u_m, r_m))$ or, from the remark after Proposition 2.3, by comparing $d_{k-1}(u_n, r_n)$ with $d_k(u_n, r_n)$, where

$$d_k(u_n, r_n) = \sum_{j=1}^{n-r_n+1} \mathbb{1}_{\{X_j > u_n, M_{j,j+k-1} \leq u_n\}}.$$

A good choice of k is enhanced by away trajectories for $(m, d_{k-1}(u_m, r_m))$ and $(m, d_k(u_m, r_m))$ and close trajectories for $(m, d_k(u_m, r_m))$ and $(m, d_{k+1}(u_m, r_m))$. In the following we analyze the respective plots based on this purely empirical method. We realize that an intensive Monte Carlo study concerning this technique, out of the scope of this paper, may help us in finding useful guidelines.

An illustration is given in Figure 1, where it was considered a simulated sample of size 10000 from a GARCH(1,1) process with Gaussian innovations, autoregressive parameter $\lambda = 0.25$ and variance parameter $\beta = 0.7$ (Laurini and Tawn, [19] 2012). More precisely, in the first two panels are plotted the proportions of anti-$D^{(3)}(u_n)$ by choosing $b_n = [(\log n)^3]$ and $b_n = [(\log n)^{3.3}]$, respectively, and values $\tau = 50, 100$. We can see that the choice $b_n = [(\log n)^{3.3}]$ may be better within this case. Observe that from Proposition 2.5 we can also analyze $D^{(k)}(u_n)$ by evaluating $D^{(2)}(u_n)$. The last two plots correspond to the proportions of anti-$D^{(2)}(u_n)$ within cycles $\{Z_n\}_{n\geq 1}$, respectively with $k = 4$ and $k = 5$ and $b_n = [(\log n)^3]$ and values $\tau = 15, 20$. The plots also suggest that condition $D^{(3)}(u_n)$ is unlikely to hold for the considered GARCH(1,1) model. A more prominent decrease is observed within the proportions of anti-$D^{(4)}(u_n)$ and anti-$D^{(5)}(u_n)$. It will be seen in the simulation study that these proportions lead to a quite acceptable choice of values of k in the new estimation procedure. A similar exercise (not reported) was implemented with the following models: a first order autoregressive process with Cauchy marginals and autoregressive parameter $\rho = -0.6$ of Chernick ([2], 1978), a negatively correlated uniform AR(1) process of Chernick *et al.* ([3], 1991) with $r = 2$, respectively denoted ARCauchy and ARUnif, an MM process with coefficients $\alpha_0 = 2/6$, $\alpha_1 = 1/6$, $\alpha_2 = 3/6$ as given in Example 2.1, a first order MAR process with standard Fréchet marginals and autoregressive parameter $\phi = 0.5$ of Davis and Resnick ([5], 1989) and a Markov chain with standard Gumbel marginals and logistic joint distribution with dependence parameter $\alpha = 0.5$. In all the cases we considered $b_n = [(\log n)^3]$, and values $\tau = 50, 100$. The MAR process satisfy $D^{(2)}(u_n)$ and so $D^{(3)}(u_n)$ holds, thus leading to small proportions of anti-$D^{(3)}(u_n)$. The same scenario is noticed in the first three models, all satisfying condition $D^{(3)}(u_n)$. There are slightly upper curves within the Markov chain but still comprising small values. A little decrease occurs in the proportions of anti-$D^{(4)}(u_n)$ for the Markov process.

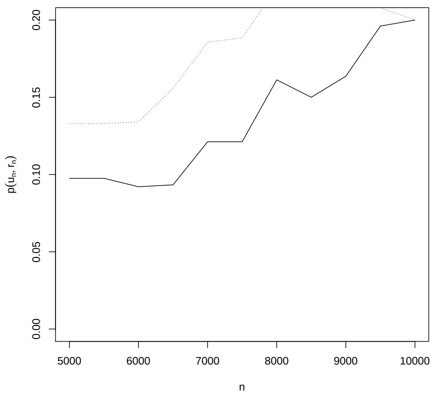
These plots can give us some clue about $D^{(k)}(u_n)$ but they do not allow us to make a definite decision. We can always opt for higher values of k since, if $D^{(k)}(u_n)$ holds then $D^{(s)}(u_n)$ holds for all $s > k$. However, a too large k for the cycles may diminish the precision of the new estimators, as will be pointed in the next section.

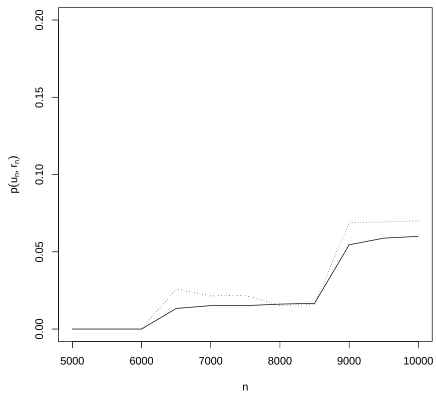
4.1 Simulations

In our study we consider 1000 replicates of simulated samples of size 1000 of each of the models referred previously: ARCauchy ($\rho = -0.6$), ARUnif ($r = 2$), MM ($\alpha_0 = 2/6, \alpha_1 = 1/6, \alpha_2 = 3/6$), MAR ($\phi = 0.5$), Markov chain ($\alpha = 0.5$) and GARCH(1,1) ($\lambda = 0.25$, $\beta = 0.7$). We have calculated the values of the new estimator $\widehat{\theta}$ given in (16), as well as, the values of estimators $\widehat{\theta}^I$, $\widehat{\theta}^{ML}$ and $\widehat{\theta}^U$ based on the new indirect approach in (17), and estimator $\widehat{\theta}^{SS}$ based on (19). Although $\widehat{\theta}^{FF}$ and $\widehat{\theta}^{FF^*}$ are derived under a max-stable premise, we still apply them since, in practice, we are taking cycles $\{Z_n\}_{n\geq 1}$ of maximums which, albeit crudely, can approach a max-stable behavior. We denote all these estimators as indirect. For comparison, we also consider the runs estimator ($\widetilde{\theta}^R$) and the intervals estimator ($\widetilde{\theta}^I$) directly for θ_X. In opposition to indirect estimators, we denote $\widetilde{\theta}^R$ and $\widetilde{\theta}^I$ as direct. The root mean squared errors (rmse) and the absolute mean biases (abias) are given in Table 2, for levels u_n corresponding to the empirical quantile 0.95. Observe that $\widehat{\theta}^{FF^*}$ does not depend on u_n. The bold entries correspond to good performances (the best one is marked with a plus) and the italic entry denotes the worst result. For models MM, ARUnif, ARCauchy and MAR, which satisfy condition $D^{(3)}(u_n)$, all the new estimators were based on the construction of cycles $\{Z_n\}_{n\geq 1}$ by taking $k = 3$. The direct runs estimator $\widetilde{\theta}^R$ was also computed for run $r = k$ (see Section 1). The results for the Markov chain and GARCH(1,1) are given by considering that $D^{(k)}(u_n)$ holds with $k = 4$ in the first case and $k = 5$ in the second model. See Figure 1 and the respective comments on this topic in Section 4 above. In what concerns the direct runs estimator $\widetilde{\theta}^R$, we choose a run r equal to $k = 4$ in the Markov chain model and run r equal to $k = 5$ in the GARCH(1,1) model. Ancona-Navarrete and Tawn ([1], 2000) considered $r = 10$ for the runs estimator $\widetilde{\theta}^R$ in the Markov chain model. Indeed, if we take $r = 10$ in our simulations for this model, we obtain slightly lower rmse's for this estimator. We have also considered $r = 10$ in the GARCH(1,1) model which led to an overall decreasing of 0.1 in the rmse's for estimator $\widetilde{\theta}^R$. The presented choice of the values k for the indirect estimators leads to the best results among other values of k also tried in simulations. Figure 2 illustrates this. Indeed, if the models satisfy condition $D^{(k)}(u_n)$, the results by taking $k+1$ are quite close but if we continue to increase k, they get worst for both bias and rmse. Observe that a too much large k means larger cycles $\{Z_n\}_{n\geq 1}$ and thus some loss of information. On the other hand, choosing k too small also raises bias and rmse (see the GARCH plots in Figure 2).

GARCH

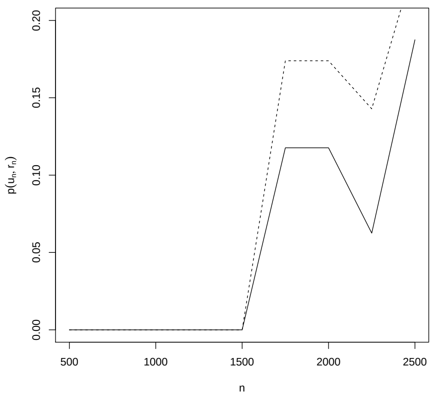
$(k = 3;\, b_n = [(\log n)^3])$ $(k = 3;\, b_n = [(\log n)^{3.3}])$





GARCH: Cycles $\{Z_n\}_{n\geq 1}$

$(k = 4;\, b_n = [(\log n)^3])$ $(k = 5;\, b_n = [(\log n)^3])$



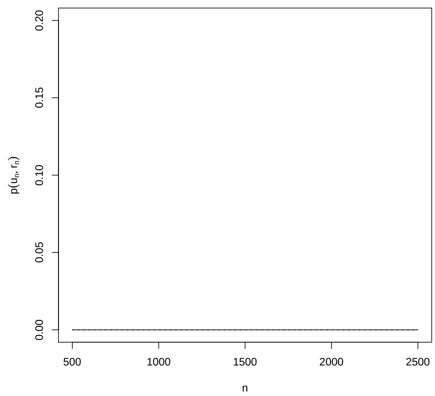


Figure 1: From left to right and top to bottom, proportions of anti-$\text{D}^{(3)}(u_n)$ with $b_n = [(\log n)^3]$ and anti-$\text{D}^{(3)}(u_n)$ with $b_n = [(\log n)^{3.3}]$ for GARCH(1,1), for $\tau = 50$ (full line) and $\tau = 100$ (dotted line), and proportions of anti-$\text{D}^{(2)}(u_n)$ of cycles $\{Z_n\}_{n\geq 1}$ for GARCH(1,1) with $k = 4$ and $k = 5$ and $b_n = [(\log n)^3]$, for $\tau = 15$ (full line) and $\tau = 20$ (dotted line).

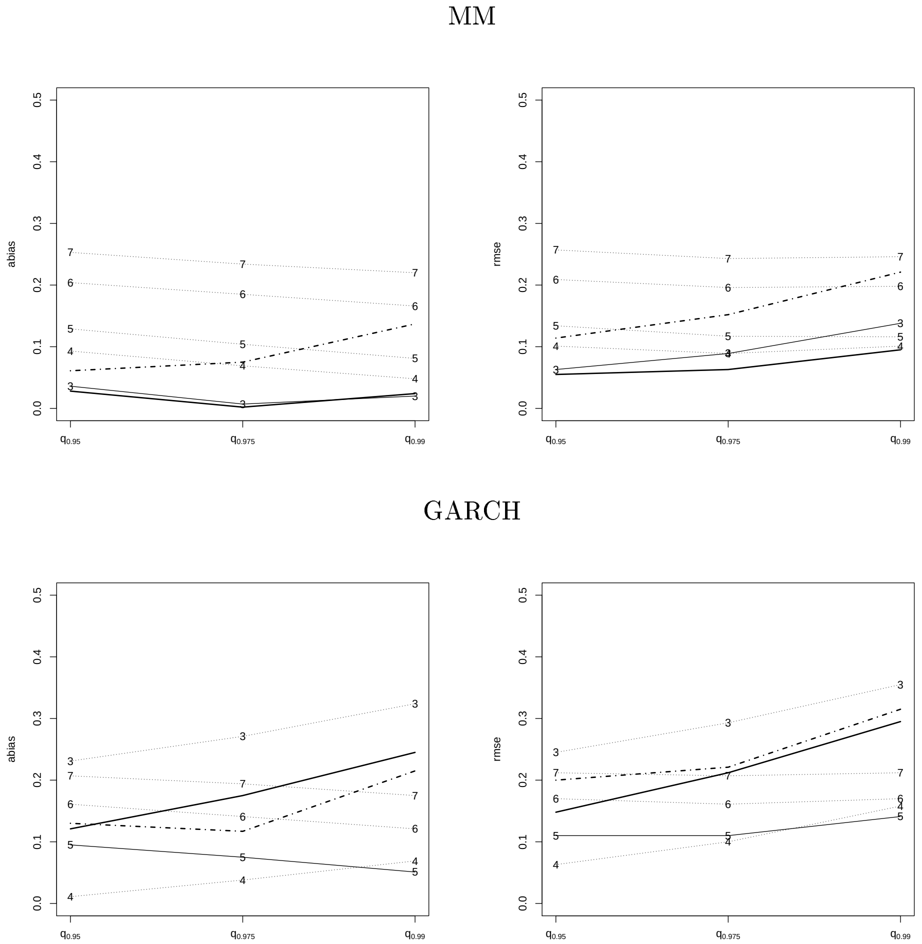


Figure 2: The thicker full and dash-dot lines correspond, respectively, to the direct runs and intervals estimators. The dotted lines correspond to estimator $\widehat{\theta}$, where the labels indicate the value of k and the full line corresponds to the "true" k. The left panels represent the absolute bias and the right panels the root mean squared error obtained for quantiles 0.95, 0.975 and 0.99, respectively denoted, $q_{0.95}$, $q_{0.975}$ and $q_{0.99}$, of models MM and GARCH.

The new approach presents good results, particularly with estimators $\widehat{\theta}$ and $\widehat{\theta}^U$. As expected, the upcrossings estimator is a competitor within our framework. The estimator $\widehat{\theta}^{FF}$ has also a good performance, except for the ARUnif model. In this case the results are better if we take $k=4$, leading to a rmse ranging from 0.084 to 0.158. One reason is that the cycles $\{Z_n\}_{n\geq 1}$ with $k=4$ for this model may be closer to max-stable behavior. We observe a similar situation with estimator $\widehat{\theta}^{FF^*}$. It performs well except in model ARUnif where, for $k=4$, we obtain a rmse of 0.077, as well as in model ARCauchy where $k=4$ leads to a rmse of 0.063. The intervals estimator yields the largest errors and behaves better if applied indirectly in the case of the Markov chain and the GARCH(1,1). The indirect estimators $\widehat{\theta}^{ML}$ and $\widehat{\theta}^{SS}$ have a similar performance.

Table 2: The root mean squared error (rmse) and the absolute mean bias (abias) obtained by considering the empirical quantile 0.95. The direct runs estimator $\widetilde{\theta}^R$ is based on run $r = 3$ for models MM, ARUnif, ARCauchy and MAR, and for models Markov chain (MC) and GARCH(1,1), on run $r = 4$ for model MC and $r = 5$ for model GARCH. The results in bold correspond to the best performances (the plus signal indicates the least value) and the italic denotes the worst performance.

rmse	MM	ARUnif	ARCauchy	MAR	MC	GARCH(1,1)
$\widetilde{\theta}^R$	**0.055**	**0.063**$^+$	**0.077**$^+$	0.071	0.084	0.148
$\widetilde{\theta}^I$	0.114	0.200	0.158	0.134	0.141	0.200
$\widehat{\theta}$	**0.057**	**0.063**$^+$	**0.084**	0.071	**0.071**	0.110
$\widehat{\theta}^U$	**0.055**	**0.089**	0.095	0.077	**0.071**	0.105
$\widehat{\theta}^I$	0.141	0.182	0.179	0.145	0.118	0.134
$\widehat{\theta}^{ML}$	0.063	**0.089**	0.095	0.077	0.084	0.073
$\widehat{\theta}^{SS}$	**0.055**	**0.089**	0.089	0.077	0.077	**0.071**
$\widehat{\theta}^{FF}$	**0.032**$^+$	0.335	**0.084**	**0.045**	**0.071**	**0.063**
$\widehat{\theta}^{FF^*}$	**0.032**$^+$	*0.875*	*0.602*	**0.032**$^+$	**0.055**$^+$	**0.045**$^+$
abias	**MM**	**ARUnif**	**ARCauchy**	**MAR**	**MC**	**GARCH(1,1)**
$\widetilde{\theta}^R$	0.028	**0.005**	0.041	0.005	0.024	0.121
$\widetilde{\theta}^I$	0.061	0.179	0.095	0.067	0.082	0.130
$\widehat{\theta}$	0.036	**0.003**$^+$	0.051	0.026	0.036	0.095
$\widehat{\theta}^U$	0.013	0.011	0.018	0.009	**0.022**	0.076
$\widehat{\theta}^I$	0.071	0.130	0.088	0.075	0.032	0.085
$\widehat{\theta}^{ML}$	0.009	0.015	**0.014**	0.005	**0.021**	**0.010**$^+$
$\widehat{\theta}^{SS}$	**0.000**$^+$	0.020	**0.003**$^+$	**0.002**$^+$	**0.014**$^+$	**0.010**$^+$
$\widehat{\theta}^{FF}$	**0.003**	0.331	0.072	**0.003**	0.053	0.020
$\widehat{\theta}^{FF^*}$	**0.003**	*0.861*	*0.595*	**0.006**	0.050	**0.010**$^+$

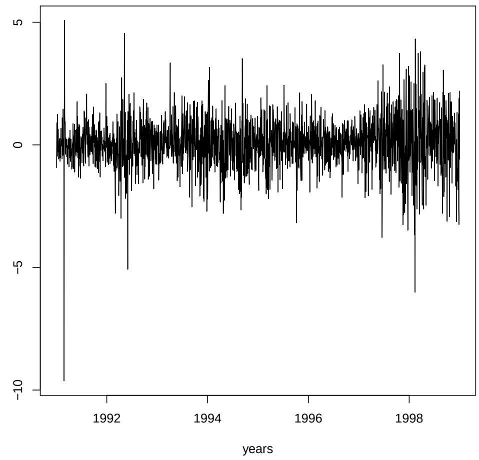


Figure 3: Daily log-returns of DAX, from 1991 to 1998, with 1786 observations (successive equal prices excluded).

4.2 Application to financial data

Log-returns of a financial time series usually present high volatility and clustering of large values. Klar *et al.* ([17], 2012) have analyzed DAX German stock market index time series and concluded that GARCH(1,1) is a good model to describe these data. In particular they considered the series of log-returns of DAX closing prices from 1991 to 1998 (see Figure 3) and fitted a GARCH(1,1) model with autoregressive parameter $\lambda \simeq 0.08$, variance parameter $\beta = 0.87$ and innovations t_7 (after removing null log-returns). By the tabulated values of the extremal index of GARCH(1,1) models in Laurini and Tawn ([19], 2012), the true value is around 0.3. In Table 3 we report the estimates, derived according to the conclusions of the simulations concerning the GARCH(1,1) model (the anti-$\mathrm{D}^{(k)}(u_n)$ plots were considered with $b_n = [(\log n)^{2.5}]$, for $\tau = 15$ and $\tau = 20$ and the anti-$\mathrm{D}^{(2)}(u_n)$ plots of the respective cycles $\{Z_n\}_{n\geq 1}$ were considered with $b_n = [(\log n)^2]$, for $\tau = 5$ and $\tau = 10$). Thus the direct runs estimator $\widetilde{\theta}^R$ was computed with run 5 and the indirect estimators ($\widehat{\theta}$, $\widehat{\theta}^U$, $\widehat{\theta}^I$, $\widehat{\theta}^{ML}$, $\widehat{\theta}^{SS}$, $\widehat{\theta}^{FF}$ and $\widehat{\theta}^{FF^*}$) were calculated by considering cycles $\{Z_n\}_{n\geq 1}$ with $k = 5$. The estimates were obtained based on the quantile 0.95. We have also tried other values for k and found that $k = 6$ leads to $\widehat{\theta} = 0.34$ and the other estimators have values of approximately 0.39, except for the intervals and the direct runs estimator where the estimates were 0.12 and 0.68. If we consider the direct runs estimator $\widetilde{\theta}^R$ with run 10 (see Section 4) we obtain the estimate 0.48.

Table 3: Estimates of the extremal index of the DAX series at quantile 0.95. The direct runs estimator was derived with run 5. The indirect estimators ($\widehat{\theta}$, $\widehat{\theta}^U$, $\widehat{\theta}^I$, $\widehat{\theta}^{ML}$, $\widehat{\theta}^{SS}$, $\widehat{\theta}^{FF}$ and $\widehat{\theta}^{FF^*}$) were obtained based on cycles $\{Z_n\}_{n\geq 1}$ with $k = 5$.

$\widetilde{\theta}^R$	$\widetilde{\theta}^I$	$\widehat{\theta}$	$\widehat{\theta}^U$	$\widehat{\theta}^I$	$\widehat{\theta}^{ML}$	$\widehat{\theta}^{SS}$	$\widehat{\theta}^{FF}$	$\widehat{\theta}^{FF^*}$
0.72	0.50	0.40	0.36	0.37	0.48	0.50	0.47	0.49

5 Conclusions

In this work we consider the estimation of the extremal index, an important dependence parameter within extreme values of stationary sequences. The new approach requires the validity of the local dependence condition $\mathrm{D}^{(k)}(u_n)$ of Chernick *et al.* ([3], 1991). The results are promising under a suitable choice for k and an empirical procedure was proposed for this evaluation. We also find that it is a useful tool for the well-known runs estimator, by guiding a first choice for the run. Since it is a crucial issue within our framework, further strength in diagnostic tools to identify the proper k of $\mathrm{D}^{(k)}(u_n)$ will be addressed in a future work.

Our aim, within the estimation of the extremal index, is to provide a new approach, being, however, aware that it does not solve the open problem of the best choice of k. We recognize that different estimators available in the literature have strengths and also vulnerabilities that led to our contribution, and we hope that future works can test it.

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