

Conformal minimal immersions of constant curvature of Riemann surfaces into symmetric spaces and flag manifolds

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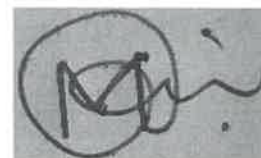
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Abstract

Following the seminal result by Eugenio Calabi establishing the local classification of complex submanifolds with constant holomorphic sectional curvature in complex space forms, several researchers have investigated minimal immersions with constant curvature of Riemann surfaces into symmetric spaces. For isometric immersions, recall that minimality is equivalent to harmonicity, hence the rich theory of harmonic maps has played here an important role.

There exists a well-established theory on twistorial constructions of harmonic maps from Riemann surfaces into symmetric spaces. An important class of twistor lifts is that of *primitive maps* into k -symmetric spaces. In this thesis, we investigate primitive immersions of constant curvature from Riemann surfaces into flag manifolds equipped with invariant metrics and their canonical structure of k -symmetric spaces. First we consider the case of primitive lifts associated to pseudoholomorphic maps from surfaces into complex Grassmannians. We establish that any such primitive lift from the two-sphere S^2 into a flag manifold has constant curvature with respect to all invariant metrics, provided that it has constant curvature with respect to *at least* one such invariant metric. This lead us to conclude as a corollary that any primitive immersion of constant curvature from S^2 into the full flag manifold is unitarily equivalent to the primitive lift of a Veronese map. We prove a partial generalization of this result to the case where the domain is a general simply connected Riemann surface. On the way, we consider the problem of finding the invariant metric on the flag manifold, under a certain normalization condition, that maximizes the induced area of the two-sphere by a given primitive immersion. Finally, we explicitly classify all the primitive immersions of constant curvature from S^2 into certain low dimensional flag manifolds, namely $F_{2,1,1}$ and $F_{2,2,1}$.

Keywords

primitive immersion; harmonic map; Riemann surface; flag manifold; Veronese sequence; Grassmannian manifold; constant curvature; two-sphere.

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Resumo Alargado

Eugenio Calabi estabeleceu, na década de 50 do século passado, a classificação local de subvariedades complexas com curvatura seccional holomorfa constante em espaços forma complexos. Na esteira deste resultado seminal, vários matemáticos têm investigado imersões mínimas com curvatura constante de superfícies de Riemann em espaços simétricos. Para imersões isométricas, recorde-se que minimalidade é equivalente a harmonicidade; deste modo, a teoria das aplicações harmónicas tem desempenhado aqui um papel fundamental.

Existe uma teoria muito desenvolvida sobre construções twistoriais de aplicações harmónicas de superfícies de Riemann em espaços simétricos. Uma classe importante de levantamentos twistor é aquela formada pelas *aplicações primitivas* sobre espaços k -simétricos. Nesta tese, investigamos imersões primitivas de curvatura constante de superfícies de Riemann numa variedade bandeira equipada com métricas invariantes e a sua estrutura canónica de espaço k -simétrico. Em primeiro lugar, consideramos o caso de aplicações primitivas associadas a aplicações pseudoholomorfas de superfícies de Riemann em variedades Grassmannianas complexas. Provamos que qualquer imersão primitiva da superfície esférica S^2 sobre uma variedade bandeira tem curvatura constante em relação a todas as métricas invariantes, desde que tenha curvatura constante em relação a pelo menos uma dessas métricas invariantes. Como corolário, concluímos que qualquer imersão primitiva de curvatura constante de S^2 na variedade bandeira completa é unitariamente equivalente ao levantamento primitivo de uma aplicação de Veronese. Provamos uma generalização parcial desse resultado para o caso em que o domínio é uma superfície de Riemann simplesmente conexa, não necessariamente fechada. De seguida, consideramos o problema de encontrar a métrica invariante na variedade bandeira, sob uma certa condição de normalização, que maximiza a área induzida em S^2 por uma dada imersão primitiva. Finalmente, classificamos explicitamente todas as imersões primitivas de curvatura constante de S^2 em variedades bandeira de baixa dimensão, a saber, $F_{2,1,1}$ e $F_{2,2,1}$.

Palavras-chave

imersão primitiva; aplicação harmónica; superfície de Riemann; espaço simétrico, variedade bandeira; sequência de Veronese; variedade de Grassmannian; curvatura constante.

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Initialisms

CMA	Centro de Matemática e Aplicações
FCT	Fundação para a Ciência e a Tecnologia
UBI	Universidade da Beira Interior

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List of Symbols

\mathbb{N}	Set of natural numbers $0, 1, \dots$
\mathbb{R}	Set of real numbers
\mathbb{C}	Set of complex numbers
$\mathbb{C}P^n$	Complex projective space
M	Riemann surface
S^2	Two-sphere
$G_k(\mathbb{C}^n)$	Grassmannian manifold
$U(n)$	Unitary group
F_{k_0, \dots, k_p}	Flag manifold
V_i^n	i^{th} element of the n -Veronese sequence $V^n : S^2 \rightarrow \mathbb{C}P^n$

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Chapter 1

Introduction

1.1 Introduction

It is a classical problem in Riemannian geometry to determine whether a Riemannian manifold can be isometrically embedded into a Euclidean space of sufficiently high dimension or, more generally, into a space of constant curvature. In 1953, Eugenio Calabi [12] considered this problem in the special case where both the Riemannian manifold and the ambient space possess a complex structure and the embedding is required to be holomorphic. He investigated which complex Hermitian manifolds can be holomorphically and isometrically embedded into a complex space form. His findings had significant implications, including the local classification of complex submanifolds with constant holomorphic sectional curvature of a complex space form. In particular, E. Calabi showed that any such submanifold of the complex projective space $\mathbb{C}P^n$ is locally congruent to a piece of the *round Veronese embedding*. Later in 1988, J. Bolton et al. [7] extended Calabi's result, by showing that any conformal minimal immersion of the two-sphere S^2 in the complex projective space $\mathbb{C}P^n$ belongs to some *Veronese sequence*, up to holomorphic isometries. Then in 1989, Q.-S. Chi and Y. Zheng [13, Theorem 2] further generalized to the case where the domain is a simply connected Riemann surface, not necessarily closed.

Since the complex projective space $\mathbb{C}P^n$ can be viewed as a complex Grassmannian $G_1(\mathbb{C}^{n+1})$, it is natural to consider the same problem for general complex Grassmannians $G_k(\mathbb{C}^n)$, with $k > 1$. In 1989, Q.-S. Chi and Y. Zheng [13, Theorem 3] showed that there exist two families of holomorphic, noncongruent immersions of constant curvature 2 from S^2 into $G_2(\mathbb{C}^4)$. This implies that any holomorphic immersion from S^2 into $G_2(\mathbb{C}^4)$ with constant curvature 2 is unitarily equivalent to exactly one of these two. This led to the broader problem of classifying minimal two-spheres in complex Grassmannians. Several researchers have made progress in this direction. In 1999, Zhenqi Li and Zhu-Huan Yu [41] provided a complete classification of minimal two-spheres of constant curvature in $G_2(\mathbb{C}^4)$. They showed that any holomorphic immersion of constant curvature from S^2 into $G_2(\mathbb{C}^4)$ falls exactly into one of four mutually noncongruent families. Furthermore, they also classified minimal immersions of constant curvature that are neither holomorphic nor antiholomorphic from S^2 into $G_2(\mathbb{C}^4)$. Later, X. Jiao and his collaborators classified various types of constantly curved minimal two-

sphere in $G_2(\mathbb{C}^5)$, including nonsingular holomorphic curves, totally unramified holomorphic curves and those which are neither holomorphic nor antiholomorphic curves [27, 33, 34, 35]. More recently, in [16] the authors constructed explicit examples of constantly curved holomorphic two-spheres of degree 6 in $G_2(\mathbb{C}^5)$, but it seems like a complete classification is still missing.

Recall that, for isometric immersions, minimality is equivalent to harmonicity [18, 51], hence the rich theory of harmonic maps comes into play. The study of harmonic maps from Riemann surfaces into complex Grassmannians through their harmonic sequences was initiated by S. S. Chern and J. G. Wolfson [14, 15, 53] and further developed by J. C. Wood and his collaborators in a sequence of papers [4, 5, 10, 19, 20]. Due to the fact that there does not exist any nonzero holomorphic differentials on S^2 , there are strong orthogonality conditions on harmonic sequences and diagrams associated to harmonic maps from S^2 into complex Grassmannians. In recent years the method of harmonic sequences has been intensively used to classify minimal immersions of constant curvature of S^2 into different Riemannian symmetric spaces [26, 27, 28, 29, 30, 31, 32, 35, 36, 40, 45].

There exists a well-established theory on twistorial constructions of harmonic maps from Riemann surfaces into symmetric spaces [8, 9]. An important class of twistor lifts is that of *primitive maps* into k -symmetric spaces [8, 24]. By definition, a primitive map satisfies certain first-order equations of Cauchy-Riemann type arising from the geometry of the k -symmetric space G/K . For $k > 2$, primitive maps are harmonic with respect to all G -invariant metrics. Thus, a natural problem arising from these observations is that of classifying primitive immersions of constant curvature of Riemann surfaces into k -symmetric spaces, when these are equipped with G -invariant metrics. In this thesis, we address this problem. To the best of our knowledge, this is the first approach to this problem in the literature. We will consider the case of primitive immersions of Riemann surfaces into the complex flag manifold F_{k_0, \dots, k_p} of all $(p+1)$ -tuples (ψ_0, \dots, ψ_p) , where the ψ_i 's are mutually orthogonal complex vector subspaces of \mathbb{C}^n , and the dimension of each ψ_i is k_i . As a homogeneous space:

$$F_{k_0, \dots, k_p} \cong \frac{U(n)}{U(k_0) \times \dots \times U(k_p)},$$

with $k_0 + \dots + k_p = n$. Note that, while the complex Grassmannian admits only one (up to positive scalar multiplication) $U(n)$ -invariant metric, F_{k_0, \dots, k_p} admits infinite $U(n)$ -invariant metrics for $p > 1$.

Next we describe the organization of this thesis and the main results we have obtained.

Chapter 2: This chapter provides necessary preliminaries. Firstly we revisit the key concept of *absolute value type function* [21, 35, 41] on Riemann surfaces, and we introduce the notion of *generalized absolute value type function*. This generalization will play a crucial role in studying primitive immersions of constant curvature into flag manifolds. Following [10, 15, 53], we then review the construction of harmonic sequences and harmonic diagrams associated with harmonic maps from Riemann surfaces into complex Grassmannians. In the final section of this chapter, we describe the geometry of the flag manifolds F_{k_0, \dots, k_p} and review the concept of *primitive map*.

Chapter 3: We consider the important case of primitive lifts of *pseudoholomorphic* maps from S^2 into complex Grassmannians. By definition, any pseudoholomorphic map into a complex Grassmannian $G_r(\mathbb{C}^n)$ belongs to the harmonic sequence ψ_0, \dots, ψ_p associated to some holomorphic map ψ_0 (which we will assume to be linearly full) into a complex Grassmannian $G_{r_0}(\mathbb{C}^n)$, with $r_0 \geq r$; the corresponding *primitive lift* $\Psi = (\psi_0, \dots, \psi_p)$ takes values in the flag manifold F_{r_0, \dots, r_p} , endowed with its natural structure of k -symmetric space, with $k = p + 1$. In Theorem 3.1.1 we will prove that *if any such primitive lift $\Psi = (\psi_0, \dots, \psi_p)$ from S^2 has constant curvature with respect to at least one invariant metric, then it has constant curvature with respect to all invariant metrics; moreover, each $\psi_j : S^2 \rightarrow G_{k_j}(\mathbb{C}^n)$ is an immersion of constant curvature with constant Kähler angle*. Since all harmonic maps from S^2 into $\mathbb{C}P^{n-1}$ are pseudoholomorphic [19], we will conclude (see Corollary 3.1.2) that *any full primitive immersion from S^2 into the full flag manifold which has constant curvature with respect to at least one invariant metric is unitarily equivalent to the primitive lift of a Veronese map*. In the final section, we prove a partial generalization of this result to the case where the domain is a general simply connected Riemann surface. The technique introduced by Q.-S. Chi and Y. Zheng [13] will play an important role in this generalization. On the way, we consider the problem of finding the invariant metric on the flag manifold, under a certain normalization condition, that maximizes the induced area of S^2 by a given primitive immersion.

Chapter 4: We classify primitive immersions of constant curvature from the two-sphere S^2 into the low dimensional flag manifolds $F_{2,1,1}$ and $F_{2,2,1}$. Our methods involve harmonic sequences and harmonic diagrams, and the aforementioned notion of generalized absolute value type. We start by describing some constructions of new primitive immersions of constant curvature from existing ones. Such constructions will enable us to reduce our problem to the classification of primitive immersions of constant curvature which are lifts of full harmonic maps from S^2 into the Grassmannians $G_2(\mathbb{C}^4)$ and $G_2(\mathbb{C}^5)$. The outcome of our main results (Theorem 4.2.1, Theorem 4.3.1, and Theorem 4.3.4) is that all primitive immersions of constant curvature from the two-sphere S^2 into the low dimensional flag manifolds $F_{2,1,1}$ and $F_{2,2,1}$ arise by operating

algebraically on certain Veronese sequences.

Chapter 5: In this final chapter, we summarize the achieved goals throughout the thesis and outline possible future research directions and perspectives for further work.

We would like to conclude this introduction by mentioning that the main results of this thesis are included in the preprints [42, 43].

Chapter 2

Preliminaries

2.1 Riemann surfaces

Let M be a Riemann surface, i.e., a complex one-dimensional manifold. In a local complex chart $(U, z = x + iy)$, any conformal Riemannian metric can be written as

$$ds^2 = \lambda^2(z) dz d\bar{z},$$

where $\lambda : U \rightarrow \mathbb{R}$ is smooth and satisfies $\lambda(z) > 0$ for all $z \in U$; we call λ the *conformal factor* of the local chart $(U, z = x + iy)$. The corresponding element of area is given by

$$dA = \lambda^2(z) \frac{i}{2} dz \wedge d\bar{z}.$$

We denote $\partial_z := \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\partial_{\bar{z}} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. The Laplace-Beltrami operator with respect to the metric ds^2 is locally given by

$$\Delta_M = \frac{4}{\lambda^2(z)} \partial_z \partial_{\bar{z}};$$

and the (Gaussian) curvature of M with respect to ds^2 is given by

$$K = -\frac{1}{2} \Delta_M \log \lambda^2(z).$$

A very important example of a Riemann surface is the Riemann sphere. We can interpret the Riemann sphere as the one-point compactification of the complex plane \mathbb{C} , i.e. $S^2 = \mathbb{C} \cup \{\infty\}$. Set $U_0 = S^2 \setminus \{\infty\} = \mathbb{C}$ and $U_\infty = S^2 \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}$. We define the *canonical charts* (U_0, z) and (U_∞, w) by

$$z(x) = x, \quad w(x) = \frac{1}{x}.$$

With respect to the canonical chart (U_0, z) of S^2 , the round metric of (constant) curvature K is given by

$$ds^2 = \frac{4}{K(1 + z\bar{z})^2} dz d\bar{z}.$$

2.1.1 Absolute value type functions

In this subsection, following [21, 35, 41], we will recall the definition and some important results on *absolute value type functions*, and we will generalize them for our specific purpose. For completeness, all proofs will be given.

A nonnegative smooth (real) function f on a Riemann surface M is said to be of *absolute value type*, [21, 41] if either f is identically zero or, for any point $x \in M$, there is a local complex chart (U, z) centered at x (i.e. $z(x) = 0$) such that $f(z) = (z\bar{z})^p f_1(z)$, where $p \geq 0$ is an integer and f_1 is a nonvanishing smooth function on U . In [35], the authors admit p to be any positive or negative integer. The following generalization will play an important role in our study of primitive immersions of constant curvature into flag manifolds.

Definition 2.1.1. *A continuous function $f : M \rightarrow [0, \infty]$ on a Riemann surface M is said to be of generalized absolute value type if either f is identically zero or, for any point $x \in M$, there is a local complex chart (U, z) centered at x (i.e. $z(x) = 0$), such that*

$$f(z) = (z\bar{z})^p f_1(z), \tag{2.1.1}$$

for some $p \in \mathbb{R}$ and positive smooth function $f_1 : U \rightarrow (0, \infty)$. If $p > 0$, we say that x is a zero of order p ; if $p < 0$, we say that x is a pole of order $|p|$. We denote $p = \text{order}_x(f)$.

Remark 2.1.2. 1. From the definition, it follows that the zeros and poles of a generalized absolute value type function f are isolated. Moreover, if M is compact, the set S of zeros and poles is finite; in this case, we set

$$\text{order}(f) = \sum_{x \in S} \text{order}_x(f).$$

Observe that $\text{order}_x(f)$ does not depend on the local complex chart we choose.

2. Observe that $f : M \setminus S \rightarrow (0, \infty)$ is smooth.
3. Given a meromorphic function g on M and $p \in \mathbb{R}$, then $|g|^p$ is a generalized absolute value type function. Moreover, if $h : M \rightarrow \mathbb{R}$ is a positive smooth function, then $|g|^p h$ is also a generalized absolute value type function.
4. Let f and g be generalized absolute value type functions, then fg and $\frac{f}{g}$ (if g is not identically zero) are also generalized absolute value type functions.

Example 2.1.3. Let $\sigma : \mathbb{C} \rightarrow \mathbb{C}^{n+1}$ be a nonvanishing polynomial map of degree m :

$$\sigma(z) = \sum_{i=0}^m a_i z^i, \quad \text{where } a_i \in \mathbb{C}^{n+1}.$$

Consider the continuous function $f : S^2 \rightarrow [0, \infty]$ defined by $f(z) = \|\sigma(z)\|^{2\lambda}$ on \mathbb{C} and $f(\infty) = \infty$, with $\lambda > 0$. Clearly, f does not have zeros in \mathbb{C} . Since

$$f(z) = \|\sigma(z)\|^{2\lambda} = \left\| \sum_{i=0}^m a_i z^i \right\|^{2\lambda} = (z\bar{z})^{m\lambda} \left\| \sum_{i=0}^m a_i z^{i-m} \right\|^{2\lambda},$$

then, in terms of the local canonical chart (U_∞, w) centered at $x = \infty$, we have

$$f(w) = (w\bar{w})^{-m\lambda} f_1(w), \quad \text{with } f_1(w) = \left\| \sum_{i=0}^m a_i w^{m-i} \right\|^{2\lambda}.$$

Observe that $f_1(0) = \|a_m\|^{2\lambda} \neq 0$. Hence f_1 is a positive smooth function on U_∞ , and we conclude that f is a generalized absolute value type function with a pole of order $m\lambda$. This example will play an important role later on.

Lemma 2.1.4. [35] *Let (U, z) be a local chart of a Riemann surface M , where U is an open connected subset of M . Let $f : U \rightarrow \mathbb{R}$ be a smooth function which is not identically equal to zero, and consider the metric on M given locally by $ds^2 = \omega\bar{\omega}$, where $\omega = \lambda dz$ (with $\lambda : U \rightarrow \mathbb{C}$ smooth). If θ is a purely imaginary smooth 1-form on U such that*

$$(df + f\theta) \wedge \omega = 0 \tag{2.1.2}$$

on U , then $|f|$ is an absolute value type function on U and it satisfies

$$\frac{1}{2} \Delta_M \log \|f\| \bar{\omega} \wedge \omega = d\theta.$$

Proof. Since θ is pure imaginary, we can write

$$\theta = a dz - \bar{a} d\bar{z} = 2i\mathfrak{I}m(adz), \tag{2.1.3}$$

for some smooth complex valued function a on U . Combining (2.1.2) and (2.1.3), we obtain

$$((\partial_z f + af)dz + (\partial_{\bar{z}} f - \bar{a}f)d\bar{z}) \wedge dz = 0$$

hence

$$\partial_{\bar{z}}f = \bar{a}f. \quad (2.1.4)$$

Let u be any solution of the inhomogeneous Cauchy-Riemann equation

$$\partial_{\bar{z}}u = \bar{a}.$$

By the $\partial_{\bar{z}}$ -Poincaré Lemma in one variable (see [22, pp.5]), such solutions always exist locally. Then $f_1 = e^u$ is a nonvanishing function satisfying $\partial_{\bar{z}}f_1 = \bar{a}f_1$. Together with (2.1.4), this gives

$$\partial_{\bar{z}}\left(\frac{f}{f_1}\right) = \frac{(\partial_{\bar{z}}f)f_1 - f(\partial_{\bar{z}}f_1)}{f_1^2} = 0.$$

This means that $\frac{f}{f_1}$ is holomorphic, hence there exists a holomorphic function f_0 on U such that $f = f_0f_1$, which implies that $|f|$ is an absolute value type function (see Remark 2.1.2).

From equation (2.1.4), we have

$$\bar{a} = \frac{\partial_{\bar{z}}f}{f} = \partial_{\bar{z}}\log f.$$

Now, combining with (2.1.3), this gives

$$\begin{aligned} d\theta &= da \wedge dz - d\bar{a} \wedge d\bar{z} = (\partial_{\bar{z}}a + \partial_z\bar{a})d\bar{z} \wedge dz \\ &= (\partial_{\bar{z}}\partial_z \log \bar{f} + \partial_z\partial_{\bar{z}} \log f)d\bar{z} \wedge dz \\ &= \partial_z\partial_{\bar{z}}(\log |f|^2)d\bar{z} \wedge dz \\ &= \frac{1}{2}\Delta_M \log |f| \bar{\omega} \wedge \omega. \end{aligned}$$

□

In [21, Lemma 4.1], Eschenburg et al. established that for an absolute value type function f on a compact Riemann surface M , the following holds:

$$\int_M \Delta_M \log(f) dA = -4\pi \text{order}(f). \quad (2.1.5)$$

Next we extend this result for generalized absolute value type functions.

Lemma 2.1.5. *Let f be a generalized absolute value type function on a compact Riemann surface M , equipped with a conformal Riemannian metric. Then (2.1.5) holds.*

Proof. The proof is exactly the same as the proof of [21, Lemma 4.1]. Let S be the set

of zeros and poles of f . Since M is compact, the set S is finite. For each $x_0 \in S$, we consider a local complex chart $(U(x_0), z)$ centered at x_0 . For sufficiently small ε , the open disks

$$U_\varepsilon(x_0) = \{x \in U(x_0) \subset M : |z(x)| < \varepsilon\}, \quad x_0 \in S$$

are disjoint.

Now define the compact subset

$$M_\varepsilon := M - \bigcup_{x_0 \in S} U_\varepsilon(x_0).$$

By the divergence theorem,

$$\int_{M_\varepsilon} \Delta_M \log(f) dA = \sum_{x_0 \in S} \int_{\partial U_\varepsilon(x_0)} \langle \nabla \log(f), v \rangle ds, \quad (2.1.6)$$

where dA is the area element, ds is the line element of the boundary $\partial U_\varepsilon(x_0)$, $\langle \cdot, \cdot \rangle$ stands for the metric and v is the unit normal vector of $\partial U_\varepsilon(x_0)$ pointing inside $U_\varepsilon(x_0)$.

Fix $x_0 \in S$ and a local complex chart $(U(x_0), z)$ centered at x_0 , with conformal factor λ . Set $r(x) = |z(x)|$ for each $x \in U(x_0)$. If the function f has order $p \in \mathbb{R}$ at the point x_0 , then we have $f = r^{2p} a_1$ on an open neighborhood of x_0 , for some smooth positive function a_1 . Substituting this in equation (2.1.6), we obtain the following

$$\int_{\partial U_\varepsilon(x_0)} \langle \nabla \log(f), v \rangle ds = 2p \int_{\partial U_\varepsilon(x_0)} \langle \nabla \log r, v \rangle ds + \int_{\partial U_\varepsilon(x_0)} \langle \nabla \log a_1, v \rangle ds \quad (2.1.7)$$

Since a_1 is a smooth positive function, $\langle \nabla \log a_1, v \rangle$ is bounded, and so

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial U_\varepsilon(x_0)} \langle \nabla \log a_1, v \rangle ds = 0.$$

Concerning the first term on the right-hand side of (2.1.7), observe first that $v = -\frac{z}{\lambda r}$. We have

$$2p \langle \nabla \log r, v \rangle = -\frac{2p}{\lambda r}.$$

Hence, we have

$$2p \int_{\partial U_\varepsilon(x_0)} \langle \nabla \log r, v \rangle ds = \int_{r=\varepsilon} -\frac{2p}{\lambda r} ds = -\frac{2p}{\varepsilon} \int_{r=\varepsilon} |dz| = -4\pi p.$$

Since $p = \text{order}_{x_0}(f)$, taking the sum over $x_0 \in S$ we obtain the result. \square

The following lemma is a generalization of [41, Lemma 2.3] and [35, Lemma 2.8].

Lemma 2.1.6. *Consider on S^2 a conformal Riemannian metric ds^2 of constant cur-*

vature K . Let f be a generalized absolute value type function on S^2 . Assume there exists a constant c such that

$$\frac{1}{4}\Delta_{S^2} \log f = c, \quad (2.1.8)$$

off the set of zeros and poles. Then the following holds:

1. the constant c is given by

$$c = -\frac{K}{4}\text{order}(f); \quad (2.1.9)$$

2. in the canonical chart (U_0, z) , with $U_0 = \mathbb{C}$, we have

$$f(z) = a \frac{|z - z_0|^{2p_0} \dots |z - z_m|^{2p_m}}{|z - w_0|^{2q_0} \dots |z - w_l|^{2q_l}} (1 + z\bar{z})^{-\text{order}(f)},$$

where: z_0, \dots, z_m are the zeros f contained in \mathbb{C} , and p_0, \dots, p_m their real orders; w_0, \dots, w_l are the poles of f contained in \mathbb{C} , and q_0, \dots, q_l their real orders; a is a positive real constant.

Proof. Firstly, we will prove (2.1.9). Since, by assumption, the curvature is constant, the Gauss-Bonnet theorem gives

$$\int_{S^2} dA = \frac{4\pi}{K}, \quad (2.1.10)$$

where dA is the area element of S^2 . On the other hand, by combining (2.1.5) and (2.1.8), we obtain

$$\int_{S^2} dA = -\frac{\text{order}(f)\pi}{c}. \quad (2.1.11)$$

From (2.1.10) and (2.1.11), we have the required conclusion $c = -\frac{K}{4}\text{order}(f)$.

Let z_0, \dots, z_m and w_0, \dots, w_l be the zeros and poles of f contained in $U_0 = \mathbb{C}$, and let p_0, \dots, p_m and q_0, \dots, q_l , respectively, be their real orders. We have

$$f(z) = \frac{|z - z_0|^{2p_0} \dots |z - z_m|^{2p_m}}{|z - w_0|^{2q_0} \dots |z - w_l|^{2q_l}} g(z), \quad (2.1.12)$$

where $g(z)$ is a smooth positive function on \mathbb{C} . Recall that

$$\Delta_{S^2} = K(1 + z\bar{z})^2 \partial_z \partial_{\bar{z}}. \quad (2.1.13)$$

Applying " $\partial_z \partial_{\bar{z}} \log$ " on both sides of the equation (2.1.12) and using the fact that

$$\partial_z \partial_{\bar{z}} \log |H(z)|^2 = 0$$

for any holomorphic function $H(z)$, we have

$$\partial_z \partial_{\bar{z}} \log f(z) = \partial_z \partial_{\bar{z}} \log g(z) \quad (2.1.14)$$

Combining (2.1.8), (2.1.9), (2.1.13), and (2.1.14), we obtain

$$\partial_z \partial_{\bar{z}} \log g(z) + \frac{\text{order}(f)}{(1 + z\bar{z})^2} = 0,$$

which can be rewritten as

$$\partial_z \partial_{\bar{z}} \log [g(z)(1 + z\bar{z})^{\text{order}(f)}] = 0, \quad \forall z \in \mathbb{C}.$$

This implies that $\log [g(z)(1 + z\bar{z})^{\text{order}(f)}]$ is a harmonic function on \mathbb{C} . Since \mathbb{C} is simply connected, this means there exist a holomorphic function \tilde{h} such that

$$\log [g(z)(1 + z\bar{z})^{\text{order}(f)}] = \tilde{h}(z) + \overline{\tilde{h}(z)}.$$

Taking exponential, and setting $h(z) = e^{\tilde{h}(z)}$, we obtain

$$g(z)(1 + z\bar{z})^{\text{order}(f)} = |h(z)|^2. \quad (2.1.15)$$

Hence, in view of (2.1.12),

$$f(z) = |h(z)|^2 \frac{|z - z_0|^{2p_0} \dots |z - z_m|^{2p_m}}{|z - w_0|^{2q_0} \dots |z - w_l|^{2q_l}} (1 + z\bar{z})^{-\text{order}(f)}. \quad (2.1.16)$$

It remains to prove that $|h(z)|^2$ is constant. Set $p = \sum_i p_i$ and $q = \sum_i q_i$. From (2.1.16), we see, by continuity of f , that

$$\begin{aligned} & \lim_{|z| \rightarrow \infty} \frac{|h(z)|^2}{(z\bar{z})^{\text{order}(f) + q - p}} \\ &= \lim_{|z| \rightarrow \infty} \frac{(1 + z\bar{z})^{\text{order}(f)}}{(z\bar{z})^{\text{order}(f)}} \frac{|z|^{2p}}{|z - z_0|^{2p_0} \dots |z - z_m|^{2p_m}} \frac{|z - w_0|^{2q_0} \dots |z - w_m|^{2q_l}}{|z|^{2q}} f(z) \\ &= f(\infty). \end{aligned} \quad (2.1.17)$$

Suppose $f(\infty)$ is finite. In this case, in view of the definition of $\text{order}(f)$, we have $\text{order}(f) + q - p = \text{order}_\infty(f) \geq 0$. Choose $\varepsilon > 0$ such that $N := \text{order}(f) + q - p + \varepsilon$

is a positive integer. Then, (2.1.17) gives

$$\lim_{|z| \rightarrow \infty} \left| \frac{h(z)}{z^N} \right| = 0.$$

From this we see that the entire function h is a polynomial. Since $h = e^{\tilde{h}}$ is nonvanishing, h must be constant.

Suppose $f(\infty)$ is infinite. In this case, $\text{order}(f) + q - p = \text{order}_\infty(f) < 0$. Choose $\varepsilon > 0$ such that $N := \text{order}(f) + q - p - \varepsilon$ is a negative integer. From (2.1.17), we have

$$\lim_{|z| \rightarrow \infty} \left| z^{|N|} h(z) \right| = \infty.$$

This means that the entire function $z \mapsto z^{|N|} h(z)$ has a pole (of finite order) at ∞ , which implies that h is a polynomial. Again, since $h = e^{\tilde{h}}$ is nonvanishing, h must be constant.

□

2.2 Complexification of a real analytic function

The notion of complexification of a real analytic function will appear along some proofs of our main results, so let us recall what this notion means exactly.

Lemma 2.2.1. *Suppose $U \subset \mathbb{C}$ is an open set and $f : U \rightarrow \mathbb{C}$ is real analytic. Then there exists an open set $V \subset \mathbb{C} \times \mathbb{C}$ and a unique holomorphic function $F : V \rightarrow \mathbb{C}$ such that*

$$\{(z, w) : w = \bar{z}, z \in U\} \subset V,$$

and $F(z, \bar{z}) = f(z)$ for all $z \in U$. (F is called the complexification of f .)

Proof. Being real analytic, the function $f(z)$ is represented around each point $z_0 \in U$ by a power series of the form

$$f(z) = \sum_{n,m} c_{n,m} (z - z_0)^n (\bar{z} - \bar{z}_0)^m;$$

this power series converges absolutely on a disk $D_R(z_0) = \{z : |z - z_0| < R\} \subset U$. Hence the series

$$\sum_{n,m} c_{n,m} (z - z_0)^n (w - \bar{z}_0)^m$$

converges absolutely on the polydisc $D_R(z_0) \times D_R(\bar{z}_0)$, so it represents a holomorphic function $F(z, w)$ defined on $D_R(z_0) \times D_R(\bar{z}_0)$. For $w = \bar{z}$, we have $F(z, \bar{z}) = f(z)$.

Let $G(z, w)$ be another holomorphic function on $D_R(z_0) \times D_R(\bar{z}_0)$ satisfying $G(z, \bar{z}) = f(z)$. We need to prove that $G(z, w) = F(z, w)$ on $D_R(z_0) \times D_R(\bar{z}_0)$. We assume without loss of generality that $f(z) = 0$ on $D_R(z_0)$ and $G(z, w) = 0$ on $D_R(z_0) \times D_R(\bar{z}_0)$. Under these assumptions, $F(z, \bar{z}) = 0$ for all $z \in D_R(z_0)$. Applying the operators ∂_z and $\partial_{\bar{z}}$ together with the chain rule, we obtain

$$0 = \partial_z [F(z, \bar{z})] = \frac{\partial F}{\partial z}(z, \bar{z}), \quad 0 = \partial_{\bar{z}} [F(z, \bar{z})] = \frac{\partial F}{\partial w}(z, \bar{z}).$$

Then, the holomorphic functions $\frac{\partial F}{\partial z}, \frac{\partial F}{\partial w}$ defined on $D_R(z_0) \times D_R(\bar{z}_0)$ are zero when $w = \bar{z}$. By induction, we see that

$$\frac{\partial^{n+m} F}{\partial^n z \partial^m w}(z, \bar{z}) = 0$$

for all $z \in D_R(z_0)$ and all integers $n, m \geq 0$. The uniqueness of the power series representation yields $F(z, w) = 0$ on $D_R(z_0) \times D_R(\bar{z}_0)$.

For each $z_0 \in U$, we have constructed a holomorphic function $F(z, w)$ on a polydisk $D_R(z_0) \times D_R(\bar{z}_0)$, which is the unique holomorphic function satisfying $F(z, \bar{z}) = f(z)$ for all $z \in D_R(z_0)$. Finally, we can extend this construction to obtain a holomorphic function $F(z, w)$ on $V = \bigcup_{z_0 \in U} D_R(z_0) \times D_R(\bar{z}_0)$, satisfying $F(z, \bar{z}) = f(z)$ for all $z \in U$; by the identity theorem [22, pp. 7], the function $F(z, w)$ is well defined on V and it is the unique holomorphic function on V satisfying $F(z, \bar{z}) = f(z)$. \square

2.3 Harmonic maps and minimal immersions

2.3.1 Harmonic maps

The notion of harmonic maps between two Riemannian manifolds was introduced by J. Eells and J. Sampson [18] in 1964.

Definition 2.3.1. *A smooth map $\psi : (M, g) \rightarrow (N, h)$ between two Riemannian Manifolds is said to be harmonic if it is a critical point of the energy functional:*

$$E(\psi) = \frac{1}{2} \int |d\psi|^2 v_g,$$

on every compact subdomain of M , where v_g is the volume element on M .

The Euler-Lagrange equation associated to this variational problem is given by

$$\tau_\psi := \text{tr } \tilde{\nabla} d\psi = 0,$$

where $\tilde{\nabla}$ is the connection on $T^*M \otimes \psi^{-1}TN$ induced by the Levi-civita connections on M and N . The smooth vector field τ_ψ along ψ is called the *tension field* of ψ .

Recall that an isometric immersion $\psi : (M, g) \rightarrow (N, h)$ is *minimal* if its mean curvature vector field vanishes identically. For any such isometric immersion, the mean curvature vector field equals the tension field, hence we have:

Theorem 2.3.2. [17, 18, 51]: *If $\psi : (M, g) \rightarrow (N, h)$ is an isometric immersion, then ψ is harmonic if and only if ψ is minimal.*

Throughout this thesis, the domain M will be two-dimensional. In this particular case, the Euler-Lagrange equation is conformally invariant for the domain metric, thus the notion of harmonic map from a Riemann surface to a Riemannian manifold makes sense. Moreover, if z is a local complex coordinate on the Riemann surface M , then $\psi : M \rightarrow (N, h)$ is harmonic if and only if

$$\psi^{-1}\nabla_{\bar{Z}}^N d\psi(Z) = 0,$$

where $Z := \frac{\partial}{\partial z}$ and ∇^N is the Levi-Civita connection on N . This equation says that $d\psi(Z)$ is a holomorphic section of $\psi^{-1}TN$ with respect to the *Koszul-Malgrange holomorphic structure* on the bundle $\psi^{-1}TN$ over M .

Theorem 2.3.3 (Koszul-Malgrange [38]). *Let E be a complex vector bundle over a Riemann surface M with connection ∇ . Then there is a unique holomorphic structure on E with respect to which a local section σ of E is holomorphic if and only if $\nabla_{\bar{Z}}\sigma = 0$ for all local coordinate vector field $Z := \frac{\partial}{\partial z}$.*

2.3.2 Harmonic sequences and harmonic diagrams

We recall from [7, 10, 32, 53] the definition and properties of harmonic sequences and harmonic diagrams associated to harmonic maps from Riemann surfaces into complex Grassmannians.

2.3.2.1 Harmonic maps into complex Grassmannians

We consider on \mathbb{C}^n the standard Hermitian inner product

$$\langle v, w \rangle = v_1\bar{w}_1 + \dots + v_n\bar{w}_n,$$

with $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{C}^n$. The Grassmannian $G_k(\mathbb{C}^n)$ of all k -dimensional complex subspaces of \mathbb{C}^n is a Hermitian symmetric $U(n)$ -space, with stabilizers conjugate to $U(k) \times U(n-k)$. Given $L \in G_k(\mathbb{C}^n)$, the complex structure at

the corresponding tangent space is given by

$$T_L^{\mathbb{C}}G_k(\mathbb{C}^n) = T_L^{1,0}G_k(\mathbb{C}^n) \oplus T_L^{0,1}G_k(\mathbb{C}^n) \cong \text{Hom}(L, L^\perp) \oplus \text{Hom}(L^\perp, L), \quad (2.3.1)$$

while the compatible Riemannian metric $h = \text{Re } h_{\mathbb{C}}$, where $h_{\mathbb{C}}$ is the Hermitian metric, is given by

$$h(\xi, \eta) = \frac{1}{2} \text{tr}(\xi\eta^*), \quad (2.3.2)$$

for $\xi, \eta \in T_L G_k(\mathbb{C}^n)$. When $k = 1$, this is the *Fubini-Study metric* of constant holomorphic sectional curvature equal to 4 on $\mathbb{C}P^{n-1}$. Sometimes we will use the notation $h = ds_{G_k(\mathbb{C}^n)}^2$.

We identify a smooth map $\psi : M \rightarrow G_k(\mathbb{C}^n)$ from a Riemann surface M with a (complex) rank- k vector subbundle, of the trivial vector bundle $M \times \mathbb{C}^n$, with fibre at $z \in M$ given by the k -dimensional vector subspace $\psi(z)$. The subbundle ψ is a holomorphic vector subbundle of $M \times \mathbb{C}^n$ if and only if the smooth map $\psi : M \rightarrow G_k(\mathbb{C}^n)$ is holomorphic and it is said to be *harmonic* if the corresponding smooth map is harmonic.

Any subbundle ψ of $M \times \mathbb{C}^n$ inherits a metric from the standard Hermitian inner product of \mathbb{C}^n . The corresponding Levi-Civita connection is given by

$$\nabla_X^\psi s = \pi_\psi(ds(X)),$$

where s is a smooth section of ψ , and $\pi_\psi : M \times \mathbb{C}^n \rightarrow \psi$ denotes the orthogonal (Hermitian) projection. Henceforth, we will always consider on any such subbundle ψ of $M \times \mathbb{C}^n$ the Koszul-Malgrange holomorphic structure induced by ∇^ψ .

Remark 2.3.4. If ψ is a holomorphic subbundle of $M \times \mathbb{C}^n$, then the induced holomorphic structure on ψ as a holomorphic subbundle of $M \times \mathbb{C}^n$ coincides with the Koszul-Malgrange holomorphic structure.

Let ϕ and ψ be two complex vector subbundles of the trivial vector bundle $M \times \mathbb{C}^n$, equipped with connections ∇^ϕ and ∇^ψ , respectively. On the complex vector bundle $\text{Hom}(\phi, \psi)$, we consider the connection induced from ∇^ϕ and ∇^ψ , together with the corresponding Koszul-Malgrange holomorphic structure. Holomorphic sections of $\text{Hom}(\phi, \psi)$ are characterized as follows: a section A of $\text{Hom}(\phi, \psi)$ maps holomorphic sections of ϕ to holomorphic sections of ψ if and only if A is a holomorphic section of $\text{Hom}(\phi, \psi)$.

Given a holomorphic section A of $\text{Hom}(\phi, \psi)$, the rank of $\text{Im } A_x$, $x \in M$ is maximum except at isolated points, where the rank drops. By *filling out zeros* (see [10, Proposi-

tion 2.2]) we can extend $\text{Im } A$ to a well globally defined holomorphic vector subbundle of ψ , which we denote by $\underline{\text{Im}}A$. Similarly, we can define $\underline{\text{ker}}A$.

Remark 2.3.5. Let $s_0, s_1, \dots, s_n : M \rightarrow \psi$ be nonzero holomorphic sections of ψ . The rank of $\text{span}\{s_0(x), \dots, s_n(x)\}$, $x \in M$ is maximum except at isolated points, where the rank drops. However, by [10, Proposition 2.2], it is possible to extend it, by filling out the zeros, in order to obtain a globally defined vector subbundle of ψ , which we will denote by $\underline{\text{span}}\{s_0, \dots, s_n\}$.

For any local complex chart (U, z) of M , the *second fundamental forms* $A'_\psi, A''_\psi : \psi|_U \rightarrow \psi^\perp|_U$ are defined by

$$A'_\psi(s) = \pi_{\psi^\perp} \circ \partial_z s, \quad A''_\psi(s) = \pi_{\psi^\perp} \circ \partial_{\bar{z}} s,$$

where s is a section of $\psi|_U$ and π_{ψ^\perp} is the orthogonal projection onto ψ^\perp . We clearly have

$$A''_\psi = -(A'_{\psi^\perp})^*. \quad (2.3.3)$$

The second fundamental forms can be turned into global objects by considering global sections

$$\mathcal{A}'_\psi \in \Gamma((T^{1,0}M)^* \otimes \text{Hom}(\psi, \psi^\perp)), \quad \mathcal{A}''_\psi \in \Gamma((T^{0,1}M)^* \otimes \text{Hom}(\psi, \psi^\perp))$$

given locally by

$$\mathcal{A}'_\psi = dz \otimes A'_\psi, \quad \mathcal{A}''_\psi = d\bar{z} \otimes A''_\psi.$$

Proposition 2.3.6. [10, Lemma 1.3] *A smooth map $\psi : M \rightarrow (G_k(\mathbb{C}^n), h)$ is harmonic if and only if \mathcal{A}'_ψ is holomorphic, and this holds if and only if \mathcal{A}''_ψ is antiholomorphic.*

In view of (2.3.3), this last proposition implies that ψ is harmonic if and only if ψ^\perp is harmonic.

Proposition 2.3.7. *A smooth map $\psi : M \rightarrow (G_k(\mathbb{C}^n), h)$ is weakly conformal (i.e., conformal except at the points where the differential vanishes) if and only if, on any local complex chart (U, z) ,*

$$\text{tr } A'_\psi \circ A'_{\psi^\perp} = 0.$$

Proof. The component of $d\psi$ along $(T^{1,0}M)^* \otimes T^{\mathbb{C}}G_k(\mathbb{C}^n)$ is given by

$$d\psi\left(\frac{\partial}{\partial z}\right) = A'_\psi - A'_{\psi^\perp}.$$

Hence, since $A'_\psi \circ A'_\psi = A'_{\psi^\perp} \circ A'_{\psi^\perp} = 0$, we have

$$\psi^* h \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right) = \frac{1}{2} \operatorname{tr} (A'_\psi - A'_{\psi^\perp}) \circ (A'_\psi - A'_{\psi^\perp}) = -\operatorname{tr} A'_\psi \circ A'_{\psi^\perp},$$

and the result follows. \square

2.3.2.2 Harmonic sequences

Given a harmonic map $\psi : M \rightarrow G_k(\mathbb{C}^n)$, the holomorphicity of

$$\mathcal{A}'_\psi \in \Gamma \left((T^{1,0}M)^* \otimes \operatorname{Hom}(\psi, \psi^\perp) \right) = \Gamma \left(\operatorname{Hom}(T^{1,0}M \otimes \psi, \psi^\perp) \right)$$

is used to define the first ∂' -Gauss bundle

$$\psi_1 = \underline{\operatorname{Im}} \mathcal{A}'_\psi.$$

If $\psi_1 = \{0\}$, then ψ is an antiholomorphic subbundle of $M \times \mathbb{C}^n$; otherwise ψ_1 is a holomorphic subbundle of ψ^\perp of rank $k' \leq k$. Similarly, the antiholomorphicity of \mathcal{A}''_ψ is used to define the first ∂'' -Gauss bundle

$$\psi_{-1} = \underline{\operatorname{Im}} \mathcal{A}''_\psi$$

of ψ . If $\psi_{-1} = \{0\}$, then ψ is a holomorphic subbundle of $M \times \mathbb{C}^n$; otherwise ψ_{-1} is an antiholomorphic subbundle of ψ^\perp of rank $k'' \leq k$.

It is well known [10, Proposition 2.3] that, *if ψ is harmonic, then both its Gauss bundles ψ_{-1}, ψ_1 are harmonic.*

Proceeding recursively, we can associate to any harmonic map $\psi : M \rightarrow G_k(\mathbb{C}^n)$ a sequence $\{\psi_j\}_{j \in \mathbb{Z}}$ of harmonic subbundles, with $\psi = \psi_0$, which is called the associated *harmonic sequence*.

2.3.2.3 Harmonic diagrams

Let ψ_1, \dots, ψ_s be a collection of mutually orthogonal vector subbundles of $M \times \mathbb{C}^n$ whose direct sum is \mathbb{C}^n . The *second fundamental forms* $A'_{i,j} : \psi_i \rightarrow \psi_j$ are defined by

$$A'_{i,j}(s) = \pi_{\psi_j} \circ \partial_z s,$$

where s is a local section of ψ_i and π_{ψ_j} is the orthogonal projection onto ψ_j . The collection of these vector subbundles, along with their second fundamental forms, is

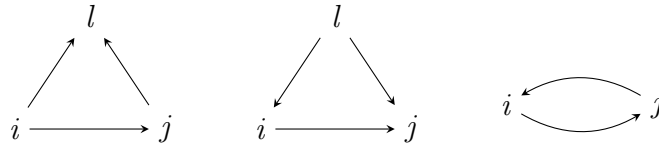
referred as a *diagram*. This diagram is represented by a directed graph, where the vector subbundles ψ_1, \dots, ψ_s are the vertices, and the arrows between them correspond to the maps $A'_{i,j}$, where the arrow from ψ_i to ψ_j is not shown whenever $A'_{i,j}$ is known to be zero.

Proposition 2.3.8. [10, Proposition 1.4] *Let $\{\psi_i, A'_{i,j}\}$ be a diagram. The following statements hold:*

1. ψ_i is holomorphic if and only if there are no edges entering ψ_i , i.e., $A'_{j,i} = 0$ for all $i \neq j$.
2. ψ_i is antiholomorphic if and only if there are no edges leaving ψ_i , i.e., $A'_{i,j} = 0$ for all $i \neq j$.

The next proposition provide sufficient criteria to ensure the holomorphicity of the second fundamental forms.

Proposition 2.3.9. [10, Proposition 1.5] *Let $\{\psi_i, A'_{i,j}\}$ be a diagram. Then, $A'_{i,j}$ is holomorphic if there are no configurations of the following forms:*



The following result provides sufficient criteria for a vertex to be harmonic.

Proposition 2.3.10. [10, Proposition 1.6]¹ *Let $\{\psi_i, A'_{i,j}\}$ be a diagram. Then ψ_i is harmonic if there are no configurations of the following forms:*



Let $\psi : M \rightarrow G_k(\mathbb{C}^n)$ be a harmonic map and $\{\psi_j\}_{j \in \mathbb{Z}}$ be its harmonic sequence. Then ψ is said to have an *isotropic order* r if $\psi \perp \psi_i$ for $1 \leq i \leq r$, but $\psi \not\perp \psi_{r+1}$. If $r = \infty$, then ψ is said to be *strongly isotropic* or *pseudoholomorphic*. By [10, Lemma 3.1], if ψ has isotropy order r , then ψ_j has isotropy order $r_j \geq r$, for all $j > 0$.

¹The correction of [10, Proposition 1.6] can be consulted in <https://people.bath.ac.uk/feb/papers/bw-corr/paper.pdf>

We have the following *fundamental harmonic diagram* associated to a harmonic map $\psi : M \rightarrow G_k(\mathbb{C}^n)$ of isotropy order r :

$$\begin{array}{ccccccc}
 \psi & \xleftarrow{\hspace{1.5cm}} & \psi_1 & \longrightarrow & \cdots & \longrightarrow & \psi_{r-1} & \longrightarrow & \psi_r & \longrightarrow & R \\
 & \searrow & & & & & & & & \nearrow & \\
 & & & & & & & & & &
 \end{array}
 \tag{2.3.4}$$

where $R = \left(\bigoplus_{j=0}^r \psi_j\right)^\perp$. By definition, a *harmonic diagram* is any refinement of a fundamental harmonic diagram. If $\psi : M \rightarrow G_k(\mathbb{C}^n)$ is holomorphic (hence harmonic), then $\psi_{p+1} = 0$ for some p , and the associated fundamental diagram simplifies to

$$\psi \longrightarrow \psi_1 \longrightarrow \cdots \longrightarrow \psi_{p-1} \longrightarrow \psi_p \qquad R \tag{2.3.5}$$

Any vertex belonging to the fundamental diagram associated with a holomorphic map $\psi : M \rightarrow G_k(\mathbb{C}^n)$ is pseudoholomorphic; the converse also holds, i.e., any pseudoholomorphic map is a vertex of the fundamental diagram associated with some holomorphic map.

Remark 2.3.11.

1. Using the terminology of loop group theory for harmonic maps from Riemann surfaces into symmetric spaces (first introduced by Uhlenbeck [50]), all pseudoholomorphic maps have *finite uniton number*. General criteria for finiteness of the uniton number for harmonic maps into complex Grassmannians in terms of the corresponding harmonic sequences and diagrams have been recently developed in [1, 44].
2. The nonexistence of nonzero holomorphic differentials on the two-sphere S^2 ensures that all harmonic maps from S^2 into $\mathbb{C}P^{n-1}$ are pseudoholomorphic [19].
3. An example of a harmonic map into a complex projective space with infinite (both to the left and to the right) harmonic sequence is provided by the *Clifford solution* $\psi : \mathbb{C} \rightarrow \mathbb{C}P^3$, which is defined in homogeneous coordinates by

$$\psi(z) = [e^{2i(z+\bar{z})}, ie^{-2i(z+\bar{z})}, e^{2(\bar{z}-z)}, e^{2(z-\bar{z})}].$$

This has not finite uniton number [1, Example 4.15].

Let $\psi : M \rightarrow (G_k(\mathbb{C}^n), h)$ be a harmonic map and $\{\psi_j\}_{j \in \mathbb{Z}}$ be its harmonic sequence. Take a local complex chart (U, z) of M . If ψ has isotropy order $r \geq 2$, then it is clear

that $A'_{\psi_j^\perp} \circ A'_{\psi_j} = 0$ for all j . Hence, by Proposition 2.3.7, each ψ_j is weakly conformal. Set

$$\gamma_j = \operatorname{tr} A'_{\psi_j} (A'_{\psi_j})^*. \quad (2.3.6)$$

The metric induced from (2.3.2) by ψ_j on M is locally given by (see also [26, 32])

$$\psi_j^* h = (\gamma_{j-1} + \gamma_j) dz d\bar{z}. \quad (2.3.7)$$

Then the Laplacian Δ_j of the metric $\psi_j^* h$ is

$$\Delta_j = \frac{4}{\gamma_{j-1} + \gamma_j} \partial_z \partial_{\bar{z}}$$

and the area form

$$dA_j = (\gamma_{j-1} + \gamma_j) \frac{i}{2} dz \wedge d\bar{z}.$$

The corresponding curvature is given by

$$K(\psi_j) = -\frac{2}{\gamma_{j-1} + \gamma_j} \partial_z \partial_{\bar{z}} \log(\gamma_{j-1} + \gamma_j). \quad (2.3.8)$$

Denoting by θ_j the *Kähler angle* of ψ_j , we have

$$(\tan(\theta_j/2))^2 = \frac{\gamma_{j-1}}{\gamma_j}. \quad (2.3.9)$$

2.3.3 Minimal immersions in the complex projective space

Let $\psi_0 : M \rightarrow \mathbb{C}P^n$ be a *linearly full* (i.e., the image of ψ_0 is not contained in any hyperplane of $\mathbb{C}P^n$) holomorphic map, generating the harmonic sequence

$$\psi_0 \longrightarrow \psi_1 \longrightarrow \cdots \longrightarrow \psi_{n-1} \longrightarrow \psi_n \quad (2.3.10)$$

Let $f_0 : U \subset M \rightarrow \mathbb{C}^{n+1}$ be a local holomorphic section of ψ_0 . Define recursively local holomorphic sections f_1, \dots, f_n of ψ_1, \dots, ψ_n by $f_{j+1} = A'_{\psi_j}(f_j)$. On the complement of a set of isolated points, we have

$$f_{j+1} = \partial_z f_j - \frac{\langle \partial_z f_j, f_j \rangle}{\|f_j\|^2} f_j, \quad \text{for } 0 \leq j \leq n. \quad (2.3.11)$$

In this setting (see also [7]),

$$\gamma_j = \frac{\|f_{j+1}\|^2}{\|f_j\|^2}.$$

Set

$$\gamma_{-1} = 0 = \gamma_n.$$

Once again in the complement of a set of isolated points, the following equations hold (see [7] for more details):

$$\begin{aligned} \partial_z \partial_{\bar{z}} \log \|f_j\|^2 &= \gamma_j - \gamma_{j-1}; \\ \partial_z \partial_{\bar{z}} \log \gamma_j &= \gamma_{j+1} - 2\gamma_j + \gamma_{j-1}; \\ \partial_{\bar{z}} f_j &= -\gamma_{j-1} f_{j-1}. \end{aligned} \tag{2.3.12}$$

The j -th *osculating curve* of ψ_0 is the holomorphic map

$$\sigma_j : U \rightarrow \mathbb{C}P^{\binom{n+1}{j+1}-1}$$

locally spanned by

$$\hat{\sigma}_j = f_0 \wedge \partial_z f_0 \wedge \dots \wedge \partial_z^j f_0, \quad \text{for } 0 \leq j \leq n.$$

The section $\hat{\sigma}_j$ can have isolated zeros. On a sufficiently small open neighborhood of one of these zeros, set

$$\hat{\sigma}_j = \Phi(z) \tilde{\sigma}_j,$$

where $\Phi(z)$ is the greatest common divisor of the $\binom{n+1}{j+1}$ components of $\hat{\sigma}_j$. Then $\tilde{\sigma}_j$ is nowhere zero $\mathbb{C}^{\binom{n+1}{j+1}}$ -valued holomorphic map. If we set $\beta_j = \|\tilde{\sigma}_j\|^2$, then

$$|\Phi(z)|^2 \beta_j = \|f_0\|^2 \dots \|f_j\|^2.$$

From (2.3.12), it follows that

$$\partial_z \partial_{\bar{z}} \log \beta_j = \gamma_j, \quad \text{for } 0 \leq j \leq n-1. \tag{2.3.13}$$

The degree δ_j of σ_j (for the general definition of the degree of a line bundle, see [22, Chap. 1]) is then given by

$$\delta_j = \frac{1}{2\pi i} \int_{S^2} \gamma_j d\bar{z} \wedge dz. \tag{2.3.14}$$

Remark 2.3.12. Consider the *Plücker embedding*

$$\iota : G_k(\mathbb{C}^n) \hookrightarrow \mathbb{C}P^{\binom{n}{k}-1},$$

which is defined as follows (see [22, pp. 209]): if $V = \text{span}\{e_1, \dots, e_k\} \in G_k(\mathbb{C}^n)$, then

$\iota(V)$ is the span of the multivector $e_1 \wedge \dots \wedge e_k \in \bigwedge^k \mathbb{C}^n$,

$$\iota(V) = \text{span}\{e_1 \wedge \dots \wedge e_k\}.$$

This is a holomorphic isometry. Hence, if $\psi : M \rightarrow G_k(\mathbb{C}^n)$ is a holomorphic immersion, then the metric on M induced by ψ is given by

$$\psi^* ds_{G_k(\mathbb{C}^n)}^2 = (\iota \circ \psi)^* ds_{\mathbb{C}P^{\binom{n}{k}-1}}^2, \quad (2.3.15)$$

where $ds_{G_k(\mathbb{C}^n)}^2$ and $ds_{\mathbb{C}P^{\binom{n}{k}-1}}^2$ are the metrics on $G_k(\mathbb{C}^n)$ and $\mathbb{C}P^{\binom{n}{k}-1}$, respectively, generally defined by (2.3.2); moreover, if s is a nonvanishing holomorphic section of $\iota \circ \psi$, then, putting together (2.3.7), (2.3.13) and (2.3.15), we obtain the following useful formula

$$\psi^* ds_{G_k(\mathbb{C}^n)}^2 = \partial_z \partial_{\bar{z}} \log \|s\|^2 dz d\bar{z}. \quad (2.3.16)$$

2.3.3.1 The Veronese sequence

The n -Veronese map $V^n : S^2 \rightarrow \mathbb{C}P^n$ is defined by

$$V^n(z) = \left[1, \sqrt{\binom{n}{1}} z, \dots, \sqrt{\binom{n}{r}} z^r, \dots, z^n \right], \quad (2.3.17)$$

in terms of the canonical chart (U_0, z) . This is a linearly full holomorphic immersion of constant curvature. Moreover, as shown by E. Calabi (see [39] and the references therein), the n -Veronese map is the unique such immersion, up to unitary congruence. The harmonic sequence V_0^n, \dots, V_n^n associated to $V^n = V_0^n$ is called the n -Veronese sequence.

Theorem 2.3.13. [7, Theorem 5.2] *The Veronese sequence $V_0^n, \dots, V_n^n : S^2 \rightarrow \mathbb{C}P^n$ is given by $V_j^n = [\hat{V}_j^n]$, with $\hat{V}_j^n = (f_{j,0}, \dots, f_{j,n})$, where*

$$f_{j,r}(z) = \frac{j!}{(1+z\bar{z})^j} \sqrt{\binom{n}{r}} z^{r-j} \sum_k (-1)^k \binom{r}{j-k} \binom{n-r}{k} (z\bar{z})^k. \quad (2.3.18)$$

Moreover, each V_j^n is a minimal immersion with induced metric

$$V_j^{n*} h = (\gamma_{j-1}^n + \gamma_j^n) dz d\bar{z} = \frac{n+2j(n-j)}{(1+z\bar{z})^2} dz d\bar{z}, \quad \gamma_j^n = \frac{(j+1)(n-j)}{(1+z\bar{z})^2}, \quad (2.3.19)$$

and hence it has constant curvature $K(V_j^n) = \frac{4}{n+2j(n-j)}$. Finally, if $T : S^2 \rightarrow S^2$ is the antipodal map (i.e., $T(z) = -\frac{1}{\bar{z}}$ on (U_0, z)), then

$$V_{n-j}^n = V_j^n \circ T.$$

From (2.3.18), one obtains

$$\|\hat{V}_j^n\|^2 = \frac{n!j!}{(n-j)!} (1+z\bar{z})^{n-2j};$$

hence

$$\hat{V}_j^n = \partial_z \hat{V}_{j-1}^n - \frac{\langle \partial_z \hat{V}_{j-1}^n, \hat{V}_{j-1}^n \rangle}{\|\hat{V}_{j-1}^n\|^2} \hat{V}_{j-1}^n = \partial_z \hat{V}_{j-1}^n - \frac{(n-2j)\bar{z}}{(1+z\bar{z})} \hat{V}_{j-1}^n, \quad \text{for } 0 \leq j \leq n. \quad (2.3.20)$$

2.3.4 The moving frame approach

In this section, we will fix some notation concerning the structure equations for moving frames adapted to maps into the complex Grassmannian. We mainly follow [14, 15, 35].

Let us again consider on \mathbb{C}^n the standard Hermitian inner product i.e.

$$\langle v, w \rangle = v_1 \bar{w}_1 + \dots + v_n \bar{w}_n,$$

for $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{C}^n$. We will use the following ranges of indices:

$$1 \leq A, B \dots \leq n, \quad 1 \leq i, j \dots \leq k, \quad k+1 \leq \alpha, \beta \dots \leq n.$$

A frame on \mathbb{C}^n is a set of n linearly independent vectors e_1, e_2, \dots, e_n , hence

$$e_1 \wedge e_2 \wedge \dots \wedge e_n \neq 0.$$

This frame is unitary if

$$\langle e_A, e_B \rangle = \delta_{AB}.$$

Let $U(n)$ be the unitary group, with Lie algebra $\mathfrak{u}(n)$. We can identify $U(n)$ with the space of unitary frames by putting the vectors e_1, e_2, \dots, e_n side by side to form a $n \times n$ matrix. The $\mathfrak{u}(n)$ -valued one form $\omega = (\omega_A^B)$ defined by

$$de_A = \sum_{B=1}^n \omega_A^B e_B$$

is the *Maurer-Cartan form* of $U(n)$. The components ω_A^B satisfy the *Maurer-Cartan structure equations*

$$d\omega_A^B = \sum_{C=1}^n \omega_A^C \wedge \omega_C^B, \quad \omega_A^B = -\overline{\omega_B^A}. \quad (2.3.21)$$

As a homogeneous space,

$$G_k(\mathbb{C}^n) = \frac{U(n)}{U(k) \times U(n-k)}.$$

From this point of view, the Hermitian metric on $G_k(\mathbb{C}^n)$ is given by

$$h_{\mathbb{C}} = \sum_{j=1}^k \sum_{\beta=k+1}^n \omega_j^\beta \overline{\omega_j^\beta}. \quad (2.3.22)$$

Let us consider a smooth conformal map $\psi : M \rightarrow G_k(\mathbb{C}^n)$, In a local complex chart (U, z) of M , the induced metric $\psi^* h_{\mathbb{C}}$ is given as follows;

$$\psi^* h_{\mathbb{C}} = \eta^2 dz d\bar{z} = \phi \bar{\phi}, \quad \text{where } \phi = \eta dz. \quad (2.3.23)$$

The pull-back of each ω_j^β can be written as follows:

$$\psi^* \omega_j^\beta = a_j^\beta \phi + b_j^\beta \bar{\phi}. \quad (2.3.24)$$

Then, from equation (2.3.22), we obtain

$$\psi^* h_{\mathbb{C}} = \sum_{j=1}^k \sum_{\beta=k+1}^n \left(|a_j^\beta|^2 + |b_j^\beta|^2 \right) \phi \bar{\phi} + \sum_{j=1}^k \sum_{\beta=k+1}^n \left(a_j^\beta \overline{b_j^\beta} \right) \phi^2 + \sum_{j=1}^k \sum_{\beta=k+1}^n \left(\overline{a_j^\beta} b_j^\beta \right) \bar{\phi}^2. \quad (2.3.25)$$

In view of (2.3.23), this implies

$$\sum_{j=1}^k \sum_{\beta=k+1}^n \left(\overline{a_j^\beta} b_j^\beta \right) = \sum_{j=1}^k \sum_{\beta=k+1}^n \left(a_j^\beta \overline{b_j^\beta} \right) = 0,$$

together with the normalization

$$\sum_{j=1}^k \sum_{\beta=k+1}^n \left(|a_j^\beta|^2 + |b_j^\beta|^2 \right) = 1.$$

2.4 Geometry of flag manifolds and primitive immersions

The complex flag manifold F_{k_0, \dots, k_p} is the set of all $(p+1)$ -tuples (ψ_0, \dots, ψ_p) , where ψ_0, \dots, ψ_p are mutually orthogonal complex subspaces of \mathbb{C}^n , and the dimension of each ψ_i is k_i . The manifold structure on the flag arises by the fact that $U(n)$ act transitively on it:

$$F_{k_0, \dots, k_p} \cong \frac{U(n)}{U(k_0) \times \dots \times U(k_p)}$$

The tangent space of the flag manifold at $\Psi = (\psi_0, \dots, \psi_p)$ is

$$T_{\Psi}^{\mathbb{C}} F_{k_0, \dots, k_p} \cong \sum_{i \neq j} \text{Hom}(\psi_i, \psi_j). \quad (2.4.1)$$

In the above equation, to make the notation consistent, we use the symbol of ordinary sum, although it actually denotes an orthogonal direct sum with respect to $U(n)$ -invariant metrics.

We distinguish the subbundle T^1 of $T^{\mathbb{C}} F_{k_0, \dots, k_p}$ whose fiber at $\Psi = (\psi_0, \dots, \psi_p)$ is given by

$$T_{\Psi}^1 = \sum_{j \in \mathbb{Z}_{p+1}} \text{Hom}(\psi_j, \psi_{j+1}).$$

Definition 2.4.1. *Let M be a Riemann surface. A smooth map*

$$\Psi = (\psi_0, \dots, \psi_p) : M \rightarrow F_{k_0, \dots, k_p}$$

*is said to be primitive if $d\Psi\left(\frac{\partial}{\partial z}\right)$ is a local section of Ψ^*T^1 , for all local complex chart (U, z) of M .*

Remark 2.4.2.

1. The term ‘‘primitive’’ was first introduced by F. Burstall [8] in the more general setting of maps from surfaces into k -symmetric spaces. (The flag manifold F_{k_0, \dots, k_p} carries a structure of k -symmetric space, with $k = p+1$.) See also [24, Ch. 21].
2. For $p > 1$, if $\Psi : M \rightarrow F_{k_0, \dots, k_p}$ is primitive, then each projection $\psi_j : M \rightarrow G_{k_j}(\mathbb{C}^n)$ is harmonic [8]. (For $p = 1$, all maps are primitive.)
3. Let $\psi : M \rightarrow G_k(\mathbb{C}^n)$ be a harmonic map of isotropy order $\geq r$. Let $\{\psi_j\}_{j \in \mathbb{Z}}$ be the harmonic sequence of ψ , holding the fundamental harmonic diagram (2.3.4). Then the smooth map

$$\Psi = (\psi_0, \dots, \psi_{r-1}, R) : M \rightarrow F_{k_0, \dots, k_{r-1}, k_R}$$

with

$$R = \left(\bigoplus_{j=0}^{r-1} \psi_j \right)^\perp, \quad k_j = \text{rank } \psi_j, \quad k_R = \text{rank } R,$$

is primitive. We say that Ψ is a *primitive lift* of ψ .

When $p = 1$, the flag F_{k_0, \dots, k_p} is a Grassmannian and all smooth maps into F_{k_0, \dots, k_j} are primitive. It is known [6] that, for $p > 1$, any primitive map into F_{k_0, \dots, k_p} is harmonic with respect to all $U(n)$ -invariant metrics of F_{k_0, \dots, k_p} . Any such metric g has the following form [2, Proposition 7.4]: given $\xi, \eta \in T_\Psi F_{k_0, \dots, k_p}$, and writing $\xi = \sum \xi_{ij}$, $\eta = \sum \eta_{ij}$ according to (2.4.1), then

$$g(\xi, \eta) = \frac{1}{2} \sum_{i \neq j} \lambda_{ij} \text{tr } \xi_{ij} \eta_{ij}^*, \quad (2.4.2)$$

where η_{ij}^* is the conjugate transpose of the matrix η_{ij} , for some positive constants λ_{ij} satisfying $\lambda_{ij} = \lambda_{ji}$. Hence, while the Grassmannians admit a unique (up to multiplication by a positive constant) $U(n)$ -invariant metric, the flag F_{k_0, \dots, k_p} admits infinite nonequivalent $U(n)$ -invariant metrics.

Given a primitive immersion $\Psi = (\psi_0, \dots, \psi_p) : M \rightarrow F_{k_0, \dots, k_p}$ and a $U(n)$ -invariant metric g on F_{k_0, \dots, k_p} of the form (2.4.2), then the metric on M induced by Ψ is given, in terms of a local complex chart (U, z) , by

$$\Psi^* g = \sum_{j \in \mathbb{Z}_{p+1}} \lambda_j \gamma_j dz d\bar{z}, \quad (2.4.3)$$

where, for each $j \in \mathbb{Z}_{p+1}$, we are denoting $\lambda_j = \lambda_{j, j+1}$; and γ_j is given by (2.3.6). The corresponding curvature is given by

$$K(\Psi) = - \frac{2}{\sum_{j \in \mathbb{Z}_{p+1}} \lambda_j \gamma_j} \partial_z \partial_{\bar{z}} \log \sum_{j \in \mathbb{Z}_{p+1}} \lambda_j \gamma_j. \quad (2.4.4)$$

Example 2.4.3. We consider the n -Veronese sequence V_0^n, \dots, V_n^n and the corresponding primitive immersion $\mathcal{V} = (V_0^n, \dots, V_n^n) : S^2 \rightarrow F_{1, \dots, 1}$, which will be called the *n -Veronese primitive map*. In view of Theorem 2.3.13, given a $U(n)$ -invariant metric on $F_{1, \dots, 1}$ of the form (2.4.2), the metric induced by \mathcal{V} on S^2 is given by

$$\mathcal{V}^* g = \sum_{j=0}^{n-1} \lambda_j \frac{(j+1)(n-j)}{(1+z\bar{z})^2} dz d\bar{z},$$

which has constant curvature

$$K(\mathcal{V}) = \frac{4}{\sum_{j=0}^{n-1} \lambda_j (j+1)(n-j)}.$$

Hence, the n -Veronese primitive map has constant curvature with respect to all invariant metrics of $F_{1,\dots,1}$.

2.4.1 Einstein metrics on flag manifolds

Let (M, g) be a Riemannian manifold. Recall that g is said to be an *Einstein metric* if its Ricci curvature satisfies the following:

$$\text{Ric}(g) = cg,$$

for some constant c . Einstein metrics are considered as privileged metrics on a given Riemannian manifold. The following result classifies $U(n)$ -invariant Einstein metrics on the flag manifold F_{k_0, k_1, k_2} .

Theorem 2.4.4. [2, Theorem 8.4] *Consider the flag manifold*

$$F_{k_0, k_1, k_2} \cong \frac{U(n)}{U(k_0) \times U(k_1) \times U(k_2)},$$

where $k_0 + k_1 + k_2 = n$. Up to scaling, a $U(n)$ -invariant metric

$$g(\xi, \eta) = \frac{1}{2} \sum_{\substack{i \neq j \\ i, j \in \{0, 1, 2\}}} \lambda_{ij} \text{tr } \xi_{ij} \eta_{ij}^*$$

on F_{k_0, k_1, k_2} is an Einstein metric if and only if the coefficients $\lambda_{ij} = \lambda_{ji}$ take one of the following forms:

1. $\lambda_{01} = k_0 + k_1, \quad \lambda_{12} = k_1 + k_2, \quad \lambda_{20} = k_0 + 2k_1 + k_2$
2. $\lambda_{01} = k_0 + k_1 + 2k_2, \quad \lambda_{12} = k_1 + k_2, \quad \lambda_{20} = k_0 + k_2$
3. $\lambda_{01} = k_0 + k_1, \quad \lambda_{12} = 2k_0 + k_1 + k_2, \quad \lambda_{20} = k_0 + k_2$
4. $\lambda_{01} = k_0 + k_1, \quad \lambda_{12} = k_1 + k_2, \quad \lambda_{20} = k_0 + k_2.$

It follows that F_{k_0, k_1, k_2} admits exactly four $U(n)$ -invariant Einstein metrics (up to scaling).

Example 2.4.5. Let $\Psi = (\psi_0, \psi_1, \psi_2) : M \rightarrow F_{1,1,1}$, be an immersion, and $\gamma_0, \gamma_1, \gamma_2$ defined as in (2.3.6). The metrics induced by Ψ from Einstein metrics $\Psi^* ds_{F_{1,1,1}}^2$ on $F_{1,1,1}$ are locally the following (up to scaling):

1. $\Psi^* ds_{F_{1,1,1}}^2 = (\gamma_0 + \gamma_1 + 2\gamma_2) dz d\bar{z}$

$$2. \Psi^* ds_{F_{1,1,1}}^2 = (2\gamma_0 + \gamma_1 + \gamma_2) dz d\bar{z}$$

$$3. \Psi^* ds_{F_{1,1,1}}^2 = (\gamma_0 + 2\gamma_1 + \gamma_2) dz d\bar{z}$$

$$4. \Psi^* ds_{F_{1,1,1}}^2 = (\gamma_0 + \gamma_1 + \gamma_2) dz d\bar{z}.$$

Chapter 3

Primitive immersions of constant curvature of Riemann surfaces into flag manifolds

A natural problem is that of characterizing the primitive immersions with constant curvature of a Riemann surface into a k -symmetric space G/K (equipped with G -invariant metrics). In the present chapter we address this question in the important case of primitive lifts of *pseudoholomorphic* maps from S^2 into complex Grassmannians. In Theorem 3.1.1 we will prove that *if any such primitive lift $\Psi = (\psi_0, \dots, \psi_p)$ from S^2 has constant curvature with respect to at least one invariant metric, then it has constant curvature with respect to all invariant metrics; moreover, each $\psi_j : S^2 \rightarrow G_{k_j}(\mathbb{C}^n)$ is an immersion of constant curvature with constant Kähler angle.* Since all harmonic maps from S^2 into $\mathbb{C}P^{n-1}$ are pseudoholomorphic [19], we will conclude (see Corollary 3.1.2) that *any full primitive immersion from S^2 into the full flag manifold which has constant curvature with respect to at least one invariant metric is unitarily equivalent to the primitive lift of a Veronese map.* In the final section, we prove a partial generalization of this result to the case where the domain is a general simply connected Riemann surface. The technique introduced by Q.-S. Chi and Y. Zheng [13] will play an important role in this generalization. On the way, we consider the problem of finding the invariant metric on the flag manifold, under a certain normalization condition, that maximizes the induced area of S^2 by a given primitive immersion.

3.1 Primitive immersions of constant curvature of S^2

Theorem 3.1.1. *Let $\psi_0 : S^2 \rightarrow G_{k_0}(\mathbb{C}^n)$ be a linearly full holomorphic map, with harmonic sequence ψ_0, \dots, ψ_p . If the primitive lift $\Psi = (\psi_0, \dots, \psi_p) : S^2 \rightarrow F_{k_0, \dots, k_p}$ is an immersion and there exists at least one $U(n)$ -invariant metric on F_{k_0, \dots, k_p} with respect to which Ψ has constant curvature, then:*

1. Ψ has constant curvature with respect to all $U(n)$ -invariant metrics on F_{k_0, \dots, k_p} .
2. Each $\psi_j : S^2 \rightarrow G_{k_j}(\mathbb{C}^n)$ is a minimal immersion of constant curvature and constant Kähler angle.

Proof. Let g be a $U(n)$ -invariant metric on F_{k_0, \dots, k_p} with respect to which Ψ has constant

curvature K . Consider the canonical chart $(U_0 = \mathbb{C}, z)$ on S^2 . By (2.4.3),

$$\Psi^*g = \sum_{j=0}^{p-1} \lambda_j \gamma_j dz d\bar{z}$$

for some positive constants $\lambda_0, \dots, \lambda_{p-1}$. The area form induced by the immersion Ψ is given by

$$dA = \sum_{j=0}^{p-1} \lambda_j \gamma_j \frac{d\bar{z} \wedge dz}{2i}. \quad (3.1.1)$$

By Gauss-Bonnet theorem, together with (2.3.14) and (3.1.1), we have

$$\begin{aligned} 4\pi &= 2\pi\chi(S^2) = \int_{S^2} K dA \\ &= K \sum_{j=0}^{p-1} \lambda_j \frac{1}{2i} \int_{S^2} \gamma_j d\bar{z} \wedge dz = K\pi \sum_{j=0}^{p-1} \lambda_j \delta_j; \end{aligned} \quad (3.1.2)$$

hence

$$K = \frac{4}{\sum_{j=0}^{p-1} \lambda_j \delta_j}. \quad (3.1.3)$$

(In particular, the constant curvature K must be positive.) By Minding theorem ([23, pp.292]), which states that all surfaces with the same constant curvature K are locally isometric, we have

$$\Psi^*g = \frac{\alpha}{(1+z\bar{z})^2} dz d\bar{z}, \quad \text{with } \alpha = \frac{4}{K}. \quad (3.1.4)$$

Set $k^{(j)} = k_0 + \dots + k_j$ and consider the holomorphic map

$$\psi^{(j)} = \psi_0 \oplus \dots \oplus \psi_j : S^2 \rightarrow G_{k^{(j)}}(\mathbb{C}^n).$$

Following remark 2.3.12, we take the composition of $\psi^{(j)}$ with the Plücker embedding of $G_{k^{(j)}}(\mathbb{C}^n)$ to obtain a holomorphic map $\sigma_j : S^2 \rightarrow \mathbb{C}P^{\binom{n}{k^{(j)}}-1}$. This holomorphic map σ_j satisfies

$$\sigma_j^* h_j = \gamma_j dz d\bar{z},$$

where h_j is the Fubini-Study metric on $\mathbb{C}P^{\binom{n}{k^{(j)}}-1}$. Let $\tilde{\sigma}_j : \mathbb{C} \rightarrow \mathbb{C}^{\binom{n}{k^{(j)}}}$ be a holomorphic local section of σ_j . Without loss of generality, we can assume that $\tilde{\sigma}_j$ is nowhere zero, as we can always remove the greatest common divisor of its components. Set

$\beta_j = \|\tilde{\sigma}_j\|^2$, which is polynomial in z and \bar{z} . From equation (2.3.13), we have

$$\partial_z \partial_{\bar{z}} \log \beta_j = \gamma_j, \quad (3.1.5)$$

hence

$$\Psi^* g = \partial_z \partial_{\bar{z}} \log \left(\beta_0^{\lambda_0} \dots \beta_{p-1}^{\lambda_{p-1}} \right) dz d\bar{z}. \quad (3.1.6)$$

Since

$$\partial_z \partial_{\bar{z}} \log(1 + z\bar{z})^\alpha = \frac{\alpha}{(1 + z\bar{z})^2}, \quad (3.1.7)$$

we deduce, by combining (3.1.4), (3.1.6) and (3.1.7), that the following holds on $U_0 = \mathbb{C}$:

$$\partial_z \partial_{\bar{z}} \log \frac{\beta_0^{\lambda_0} \dots \beta_{p-1}^{\lambda_{p-1}}}{(1 + z\bar{z})^\alpha} = 0. \quad (3.1.8)$$

So, $\log \frac{\beta_0^{\lambda_0} \dots \beta_{p-1}^{\lambda_{p-1}}}{(1 + z\bar{z})^\alpha}$ is a harmonic function on \mathbb{C} , which implies that there exists an entire function f such that

$$\log \frac{\beta_0^{\lambda_0} \dots \beta_{p-1}^{\lambda_{p-1}}}{(1 + z\bar{z})^\alpha} = f + \bar{f}. \quad (3.1.9)$$

Exponentiate (3.1.9) to obtain

$$\beta_0^{\lambda_0} \dots \beta_{p-1}^{\lambda_{p-1}} = (1 + z\bar{z})^\alpha e^f e^{\bar{f}}. \quad (3.1.10)$$

Since all polynomials $\beta_j(z, \bar{z})$, with $j \in \{0, \dots, p-1\}$, and the factor $1 + z\bar{z}$ are nonvanishing, the functions

$$z \mapsto \beta_j^{\lambda_j}(z, \bar{z}) \quad z \mapsto (1 + z\bar{z})^\alpha$$

are real analytic on \mathbb{C} , even when λ_j and α are not integers. The corresponding complexifications are given, respectively, by

$$(z, w) \mapsto \beta_j^{\lambda_j}(z, w) \quad (z, w) \mapsto (1 + zw)^\alpha,$$

considering the principal branch of each multivalued function $Z \mapsto Z^q$. Now, take the complexifications of both sides of (3.1.10),

$$F(z, w) = \beta_0^{\lambda_0}(z, w) \dots \beta_{p-1}^{\lambda_{p-1}}(z, w) \quad G(z, w) = (1 + zw)^\alpha e^{f(z)} e^{\bar{f}(w)}.$$

Since $F(z, \bar{z}) = G(z, \bar{z})$, by the uniqueness of complexification (see Lemma 2.2.1) we have

$$\beta_0^{\lambda_0}(z, w) \dots \beta_{p-1}^{\lambda_{p-1}}(z, w) = (1 + zw)^\alpha e^{f(z)} e^{\bar{f}(w)} \quad (3.1.11)$$

on an open set $V \subset \mathbb{C} \times \mathbb{C}$ containing $\{(z, w) : w = \bar{z}\}$.

Set $\mathcal{Z} = \{(z, w) : \beta_0(z, w) \dots \beta_{p-1}(z, w) = 0\}$. It is well known that $\mathbb{C}^2 \setminus \mathcal{Z}$, as the complement of the zero set of a polynomial, is a dense open path-connected subset of \mathbb{C}^2 . So, for each $z \in \mathbb{C}$ and $(z_0, w_0) \in \mathcal{Z}$, there exists a continuous path $\gamma : [0, 1] \rightarrow \mathbb{C}^2$ such that $\gamma(0) = (z, \bar{z})$, $\gamma(1) = (z_0, w_0)$ and $\gamma([0, 1)) \subset \mathbb{C}^2 \setminus \mathcal{Z}$. For each $T \in [0, 1)$, consider the curve $\gamma_T = \gamma|_{[0, T]}$. Being the composition of a polynomial with the principal branch of the complex power multivalued function, each $\beta_j^{\lambda_j}$ admits an analytic continuation along γ_T . Hence, $F(z, w)$ also admits an analytic continuation $F_T(z, w)$ along γ_T . By uniqueness of analytic continuation,

$$|F_T(z, w)| = |\beta_0(z, w)|^{\lambda_0} \dots |\beta_{p-1}(z, w)|^{\lambda_{p-1}}$$

hence $\lim_{T \rightarrow 1} F_T(\gamma(T)) = 0$. This completely characterizes the zeros of the left-hand side of (3.1.11). The same can be applied to the right-hand side in order to conclude that if not empty, the zero set of each β_j (considering each β_j as a polynomial in the independent complex variables z and \bar{z}) coincides with the zero set of the irreducible factor $1 + z\bar{z}$. Hence

$$\beta_j(z, \bar{z}) = (1 + z\bar{z})^{\alpha_j} C_j \quad (3.1.12)$$

for some real constant C_j and nonnegative integer α_j . It follows from the equations (3.1.5), (3.1.7) and (3.1.12) that

$$\gamma_j = \frac{\alpha_j}{(1 + z\bar{z})^2}, \quad j \in \{0, \dots, p-1\}.$$

If $\alpha_j = 0$ for some $j \in \{0, \dots, p-1\}$, then $\gamma_j = 0$, which implies that the holomorphic map σ_j is constant, giving a contradiction with the hypothesis of ψ_0 being full. Hence, $\alpha_j > 0$ for all $j \in \{0, \dots, p-1\}$. Consequently, taking account of (2.3.8) and (2.3.9), we see that all ψ_j are immersions of constant curvature and constant Kähler angle.

If \tilde{g} is any other $U(n)$ -invariant metric on F_{k_0, \dots, k_p} , then, for some positive constants $\tilde{\lambda}_0, \dots, \tilde{\lambda}_{p-1}$, we have

$$\Psi^* \tilde{g} = \sum_{j=0}^{n-1} \tilde{\lambda}_j \gamma_j dz d\bar{z} = \frac{\sum_{j=0}^{n-1} \tilde{\lambda}_j \alpha_j}{(1 + z\bar{z})^2} dz d\bar{z},$$

and then we conclude from (2.4.4) that Ψ has constant curvature

$$K = \frac{4}{\sum_{j=0}^{n-1} \tilde{\lambda}_j \alpha_j}.$$

□

We say that a primitive map $\Psi = (\psi_0, \dots, \psi_p)$ is *full* if all ψ_j are linearly full.

Corollary 3.1.2. *Let $\Psi : S^2 \rightarrow F_{1,\dots,1}$ be a full primitive immersion. If there exists at least one $U(n)$ -invariant metric on $F_{1,\dots,1}$ with respect to which Ψ has constant curvature, then Ψ is unitarily congruent with the $(n-1)$ -Veronese primitive map.*

Proof. Since Ψ is a full primitive map, each ψ_j , with $j \in \{0, \dots, n-1\}$, is a full harmonic map into $\mathbb{C}P^{n-1}$ (see Remark 2.4.2), and the first Gauss bundle of ψ_j is either $\{0\}$ or ψ_{j+1} . Since all harmonic maps from S^2 into $\mathbb{C}P^{n-1}$ are pseudoholomorphic, we can reorder (this corresponds to a unitary congruence on the full flag) if necessary, so that $\psi_0 : S^2 \rightarrow \mathbb{C}P^{n-1}$ is holomorphic and ψ_0, \dots, ψ_n is the corresponding harmonic sequence. By Theorem 3.1.1, ψ_0 is an immersion of constant curvature, hence, by Calabi's result, ψ_0 is unitarily congruent with the $(n-1)$ -Veronese map. □

Example 3.1.3. Let $\psi_0^a : S^2 \rightarrow \mathbb{C}P^2$ be a holomorphic immersion defined by

$$\psi_0^a(z) = [1, az, z^2], \quad a \in \mathbb{C} \setminus \{0\}.$$

Observe that $\psi_0^{V^2} = V^2$. The nowhere zero local section $f_0(z) = (1, az, z^2)$ satisfies

$$|f_0|^2 = 1 + |a|^2|z|^2 + |z|^4, \quad \left\langle \frac{\partial f_0}{\partial z}, f_0 \right\rangle = \bar{z}(|a|^2 + 2|z|^2).$$

Set $f_1 = A'_{\psi_0^a}(f_0)$. We have

$$\begin{aligned} f_1(z) &= \frac{\partial f_0}{\partial z} - \frac{1}{|f_0|^2} \left\langle \frac{\partial f_0}{\partial z}, f_0 \right\rangle f_0 \\ &= \frac{1}{1 + |a|^2|z|^2 + |z|^4} (-\bar{z}(|a|^2 + 2|z|^2), a(1 - |z|^4), z(2 + |a|^2|z|^2)). \end{aligned}$$

This is a nowhere zero local section of $\psi_1^a : S^2 \rightarrow \mathbb{C}P^2$. Since

$$|f_1|^2 = \frac{4|z|^2 + |a|^2(1 + |z|^4)}{1 + |a|^2|z|^2 + |z|^4},$$

we get

$$\gamma_0 = \frac{|f_1|^2}{|f_0|^2} = \frac{4|z|^2 + |a|^2(1 + |z|^4)}{(1 + |a|^2|z|^2 + |z|^4)^2}.$$

Similarly, the local section $f_2 = A'_{\psi_1^a}(f_1)$ of ψ_2^a is given by

$$f_2(z) = \frac{1}{4|z|^2 + |a|^2(1 + |z|^4)} (2|a|^2|z|^2, 4a\bar{z}, 2|a|^2),$$

and

$$|f_2|^2 = \frac{4a^2}{4|z|^2 + a^2(1 + |z|^4)};$$

hence

$$\gamma_1 = \frac{|f_2|^2}{|f_1|^2} = \frac{4|a|^2(1 + |a|^2|z|^2 + |z|^4)}{(4|z|^2 + |a|^2(1 + |z|^4))^2}.$$

Let $\Psi^a = (\psi_0^a, \psi_1^a, \psi_2^a) : S^2 \rightarrow F_{1,1,1}$ be the corresponding primitive immersion. The curvature on S^2 induced by Ψ^a from a $U(3)$ -invariant metric on $F_{1,1,1}$ with parameters λ_0, λ_1 is given by

$$K_{\lambda_0, \lambda_1}(\Psi^a) = -\frac{2}{\lambda_0\gamma_0 + \lambda_1\gamma_1} \partial_z \partial_{\bar{z}} \log(\lambda_0\gamma_0 + \lambda_1\gamma_1).$$

We plot the graph (using *Mathematica* software) of $K_{1,1}(\Psi^a)$ (Figure 3.1), $K_{1,2}(\Psi^a)$ (Figure 3.2) and $K_{1,\frac{1}{2}}(\Psi^a)$ (Figure 3.3) for different values of a , as a function of latitude angle φ on S^2 , that is, taking $|z| = \cot \frac{\varphi}{2}$, with $\varphi \in [0, \pi]$.

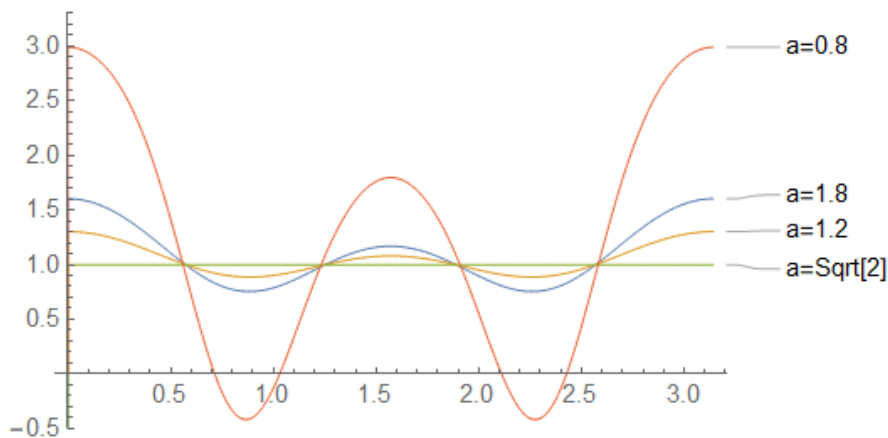


Figure 3.1: Graphs of $K_{1,1}(\Psi^a)$

Example 3.1.4. Consider the holomorphic map $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^6)$ locally spanned by $f_0, g_0 : S^2 \rightarrow \mathbb{C}^6$, where

$$f_0(z) = \left(1, 0, \frac{1}{\sqrt{2}}z, \frac{\sqrt{31}}{2\sqrt{7}}z^2, \frac{9}{2\sqrt{7}}z^2, 0\right), \quad g_0(z) = \left(0, 1, 0, 0, \frac{\sqrt{7}}{\sqrt{2}}z, \frac{1}{2}z^2\right).$$

As observed in [29], this holomorphic map has constant curvature, and its harmonic map sequence ψ_0, ψ_1, ψ_2 is nondegenerate, meaning that $\psi_1, \psi_2 : S^2 \rightarrow G_2(\mathbb{C}^6)$; moreover, ψ_1 and ψ_2 are immersions of nonconstant curvature. Hence, by Theorem 3.1.1, the

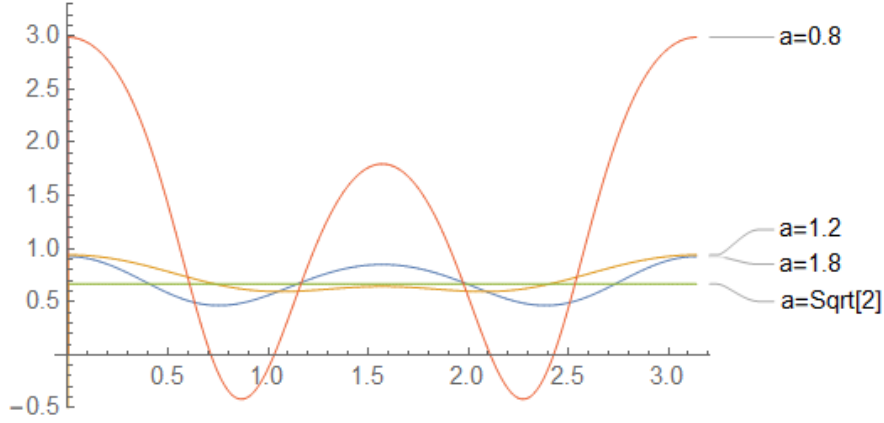


Figure 3.2: Graphs of $K_{1,2}(\Psi^a)$

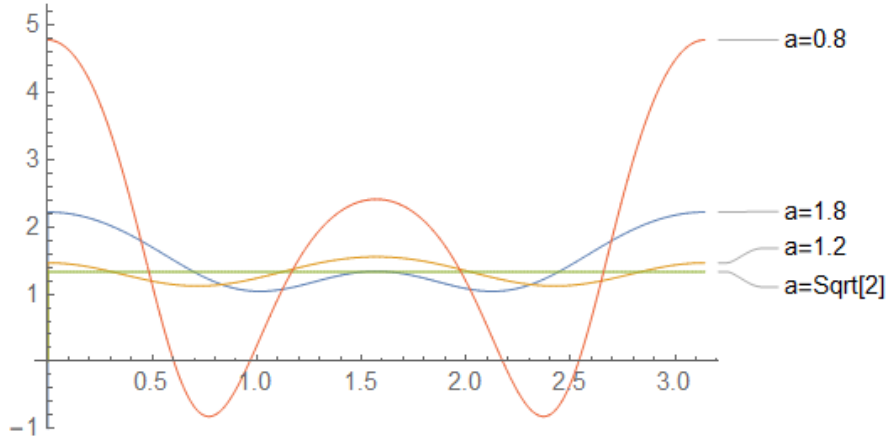


Figure 3.3: Graphs of $K_{1,\frac{1}{2}}(\Psi^a)$

primitive map $\Psi = (\psi_0, \psi_1, \psi_2) : S^2 \rightarrow F_{2,2,2}$ has nonconstant curvature with respect to all $U(6)$ -invariant metrics on $F_{2,2,2}$.

3.2 Which invariant metric maximizes the area?

In this section, we address the question of finding the invariant metric on the flag manifold, under a certain normalization condition, that maximizes the induced area of S^2 by a given primitive immersion, and also provide some examples.

Let $\psi_0 : S^2 \rightarrow G_{k_0}(\mathbb{C}^n)$ be a linearly full holomorphic map, with harmonic sequence ψ_0, \dots, ψ_p . Suppose that the primitive lift $\Psi = (\psi_0, \dots, \psi_p) : S^2 \rightarrow F_{k_0, \dots, k_p}$ is an immersion. For a given choice of $U(n)$ -invariant metric on the flag manifold, we have

seen that, locally, the induced metric on S^2 takes the form

$$\Psi^*g = \sum_{j=0}^{p-1} \lambda_j \gamma_j dz d\bar{z}$$

The induced area of S^2 with respect to this metric is given by

$$A = \frac{1}{2i} \sum_{j=0}^{p-1} \lambda_j \int_{S^2} \gamma_j d\bar{z} \wedge dz = \pi \sum_{j=0}^{p-1} \lambda_j \delta_j, \quad \text{where } \delta_j = \frac{1}{2\pi i} \int_{S^2} \gamma_j d\bar{z} \wedge dz. \quad (3.2.1)$$

Under the normalization

$$\lambda_0^2 + \dots + \lambda_{p-1}^2 = 1,$$

we can now easily address the problem of finding the invariant metric on the flag manifold that maximizes the induced area of S^2 . Consider the vector $\vec{\delta} = (\delta_0, \dots, \delta_{p-1})$ together with the unit vector $\vec{\lambda} = (\lambda_0, \dots, \lambda_{p-1})$, both living in \mathbb{R}^p . Then, under the usual inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^p , the induced area of S^2 is given by $A = \pi \langle \vec{\lambda}, \vec{\delta} \rangle$. Hence, the invariant metric which maximizes the induced area has parameters

$$\lambda_0 = \frac{\delta_0}{\sqrt{\sum_{j=0}^{p-1} \delta_j^2}}, \dots, \lambda_{p-1} = \frac{\delta_{p-1}}{\sqrt{\sum_{j=0}^{p-1} \delta_j^2}}.$$

Example 3.2.1. Consider a full holomorphic map $\psi_0 : S^2 \rightarrow \mathbb{C}P^2$, totally unramified, with primitive lift

$$\Psi = (\psi_0, \psi_1, \psi_2) : S^2 \rightarrow F_{1,1,1}.$$

Since ψ_0 is totally unramified, it follows from [7, Equation 3.25] that the degree δ_i of each ψ_i is given by

$$\delta_0 = \delta_1 = 2, \quad \delta_2 = 0.$$

So, the normalized invariant metric which maximizes the area of Ψ has parameters

$$\lambda_0 = \lambda_1 = \frac{1}{\sqrt{2}}.$$

This is an Einstein metric (see Example 2.4.5).

Example 3.2.2. Let $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^5)$ be the holomorphic map locally spanned by

$$f_0 = (1, 0, 2z, 2z^2, z^2), \quad g_0 = (0, 1, 0, z^2, 0).$$

This map is also investigated in [29]. We have

$$\begin{aligned} g_0 \wedge f_0 &= (e_2 \wedge e_1) + 2z(e_2 \wedge e_3) + 2z^2(e_2 \wedge e_4) + z^2(e_2 \wedge e_5) \\ &\quad + z^2(e_4 \wedge e_1) + 2z^3(e_4 \wedge e_3) + z^4(e_4 \wedge e_5). \end{aligned} \quad (3.2.2)$$

Taking norm square, we obtain

$$\|g_0 \wedge f_0\|^2 = (1 + z\bar{z})^4.$$

From (2.3.13), we obtain,

$$\gamma_0 = \partial_z \partial_{\bar{z}} \log \|f_0 \wedge g_0\|^2 = \partial_z \partial_{\bar{z}} \log(1 + z\bar{z})^4 = \frac{4}{(1 + z\bar{z})^2}.$$

Hence $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^5)$ is a holomorphic immersion of constant curvature $K = 1$. Next we proceed similarly to find γ_1 .

The first order partial derivatives of f_0 and g_0 with respect to z are given by

$$\partial_z f_0(z) = (0, 0, 2, 4z, 2z), \quad \partial_z g_0(z) = (0, 0, 0, 2z, 0).$$

Hence,

$$\partial_z g_0 \wedge \partial_z f_0 = 4z(e_4 \wedge e_3) + 4z^2(e_4 \wedge e_5).$$

Together with (3.2.2), this gives

$$\begin{aligned} g_0 \wedge f_0 \wedge \partial_z g_0 \wedge \partial_z f_0 &= 4z(e_1 \wedge e_2 \wedge e_3 \wedge e_4) - 4z^2(e_1 \wedge e_2 \wedge e_4 \wedge e_5) \\ &\quad + 4z^3(e_2 \wedge e_3 \wedge e_4 \wedge e_5). \end{aligned}$$

Taking norm square, yields

$$\|g_0 \wedge f_0 \wedge \partial_z g_0 \wedge \partial_z f_0\|^2 = 16|z|^2 (1 + |z|^2 + |z|^4).$$

From here we see that the first Gauss bundle ψ_1 has rank 2, so $\psi_1 : S^2 \rightarrow G_2(\mathbb{C}^5)$. Moreover,

$$\begin{aligned} \gamma_1 &= \partial_z \partial_{\bar{z}} \log \|g_0 \wedge f_0 \wedge \partial_z g_0 \wedge \partial_z f_0\|^2 \\ &= \partial_z \partial_{\bar{z}} \log (1 + |z|^2 + |z|^4) \\ &= \frac{1 + |z|^2(4 + |z|^2)}{(1 + |z|^2(1 + |z|^2))^2}. \end{aligned}$$

This shows that ψ_1 is not of constant curvature. Observe also that ψ_1 is not antiholomorphic, otherwise γ_1 would be zero; hence the second Gauss bundle ψ_2 must have rank 1.

We have

$$\delta_0 = \frac{1}{2\pi i} \int_{S^2} \gamma_0 d\bar{z} \wedge dz = 8 \int_0^\infty \frac{r}{(1+r^2)^2} dr = 4 \quad (3.2.3)$$

and

$$\delta_1 = \frac{1}{2\pi i} \int_{S^2} \gamma_1 d\bar{z} \wedge dz = 2 \int_0^\infty \frac{r(1+4r^2+r^4)}{(1+r^2+r^4)^2} dr = 2. \quad (3.2.4)$$

Now, let $\Psi = (\psi_0, \psi_1, \psi_2) : S^2 \rightarrow F_{2,2,1}$ be a primitive lift. From (3.2.3) and (3.2.4), we see that, under the normalization $\lambda_0^2 + \lambda_1^2 = 1$, the metric $\Psi^*g = (\lambda_0\gamma_0 + \lambda_1\gamma_1) dzd\bar{z}$ which maximizes the area of S^2 is given by $\lambda_0 = \frac{2}{\sqrt{5}}$, $\lambda_1 = \frac{1}{\sqrt{5}}$. This is an Einstein metric (see Theorem 2.4.4). The curvature on S^2 induced by Ψ from an invariant metric on $F_{2,2,1}$ is given by

$$K_{\lambda_0, \lambda_1}(\Psi) = -\frac{2}{\lambda_0\gamma_0 + \lambda_1\gamma_1} \partial_z \partial_{\bar{z}} \log(\lambda_0\gamma_0 + \lambda_1\gamma_1).$$

Figure 3.4 shows the graph of $K_{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}(\Psi)$, $K_{\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}}(\Psi)$ and $K_{\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}}(\Psi)$ as a function of the latitude angle φ on S^2 , that is, taking $|z| = \cot \frac{\varphi}{2}$, with $\varphi \in [0, \pi]$.

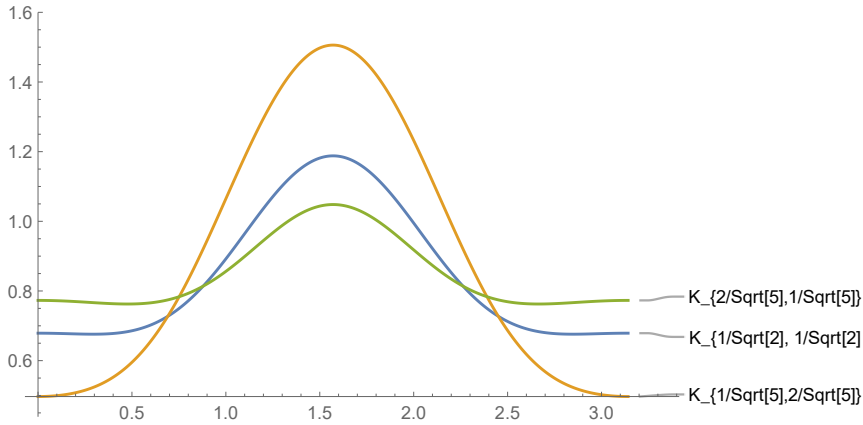


Figure 3.4: Graphs of $K_{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}}(\Psi)$, $K_{\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}}(\Psi)$ and $K_{\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}}(\Psi)$

Example 3.2.3. In [16], the authors found an interesting example of a constant curved holomorphic immersion $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^5)$ of degree $\delta_0 = 6$: this is locally spanned by

$$f_0 = \left(1, 0, -\sqrt{6}z^2, -2z^3, -3z^4\right), \quad g_0 = \left(0, 1, \sqrt{6}z, 3z^2, 4z^3\right).$$

We have,

$$\begin{aligned} f_0 \wedge g_0 &= (e_1 \wedge e_2) + \sqrt{6}z(e_1 \wedge e_3) + 3z^2(e_1 \wedge e_4) + 4z^3(e_1 \wedge e_5) - \sqrt{6}z^2(e_3 \wedge e_2) \\ &\quad - \sqrt{6}z^4(e_3 \wedge e_4) - \sqrt{6}z^5(e_3 \wedge e_5) - 2z^3(e_4 \wedge e_2) + z^6(e_4 \wedge e_5) - 3z^4(e_5 \wedge e_2). \end{aligned} \quad (3.2.5)$$

Taking norm square yield us,

$$\|f_0 \wedge g_0\|^2 = (1 + z\bar{z})^6.$$

From (2.3.13), we obtain,

$$\gamma_0 = \partial_z \partial_{\bar{z}} \log \|f_0 \wedge g_0\|^2 = \partial_z \partial_{\bar{z}} \log(1 + z\bar{z})^6 = \frac{6}{(1 + z\bar{z})^2}.$$

This implies that, $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^5)$ is a holomorphic immersion of constant curvature $K = \frac{2}{3}$. Next we proceed to compute γ_1 .

The first order partial derivatives of f_0 and g_0 with respect to z are given by

$$\partial_z f_0(z) = (0, 0, -2\sqrt{6}z, -6z^2, -12z^3), \quad \partial_z g_0(z) = (0, 0, \sqrt{6}, 6z, 12z^2).$$

Hence,

$$\partial_z f_0 \wedge \partial_z g_0 = -6\sqrt{6}z^2(e_3 \wedge e_4) - 12\sqrt{6}z^3(e_3 \wedge e_5)$$

Together with (3.2.5), this gives

$$\begin{aligned} \partial_z f_0 \wedge \partial_z g_0 \wedge f_0 \wedge g_0 &= -6\sqrt{6}z^2(e_1 \wedge e_2 \wedge e_3 \wedge e_4) + 12\sqrt{6}z^5(e_1 \wedge e_3 \wedge e_4 \wedge e_5) \\ &\quad + 6\sqrt{6}z^6(e_2 \wedge e_3 \wedge e_4 \wedge e_5) - 12\sqrt{6}z^3(e_1 \wedge e_2 \wedge e_3 \wedge e_5). \end{aligned}$$

By taking norm square, we obtain

$$\|\partial_z f_0 \wedge \partial_z g_0 \wedge f_0 \wedge g_0\|^2 = 216|z|^4 (1 + 4|z|^2 + 4|z|^6 + |z|^8).$$

From here we see that the first Gauss bundle ψ_1 has rank 2, so $\psi_1 : S^2 \rightarrow G_2(\mathbb{C}^5)$. Moreover,

$$\begin{aligned} \gamma_1 &= \partial_z \partial_{\bar{z}} \log \|\partial_z f_0 \wedge \partial_z g_0 \wedge f_0 \wedge g_0\|^2 \\ &= \partial_z \partial_{\bar{z}} \log (1 + 4|z|^2 + 4|z|^6 + |z|^8) \\ &= \frac{4(1 + 9|z|^4 + 20|z|^6 + 9|z|^8 + |z|^{12})}{(1 + 4|z|^2 + 4|z|^6 + |z|^8)^2}. \end{aligned}$$

We can see from the above equation that ψ_1 is an immersion of nonconstant curvature and ψ_1 is a non antiholomorphic vector bundle, so $\psi_2 : S^2 \rightarrow \mathbb{C}P^4$ is antiholomorphic

vector subbundle of rank 1 and $\gamma_2 = 0$. Again, the primitive map $\Psi = (\psi_0, \psi_1, \psi_2) : S^2 \rightarrow F_{2,2,1}$ has nonconstant curvature with respect to all $U(5)$ -invariant metrics on $F_{2,2,1}$.

Since $\delta_0 = 6$, and to compute δ_1 , we have

$$\delta_1 = \frac{1}{2\pi i} \int_{S^2} \gamma_1 d\bar{z} \wedge dz = 8 \int_0^\infty \frac{r(1 + 9r^4 + 20r^6 + 9r^8 + r^{12})}{(1 + 4r^2 + 4r^6 + r^8)^2} dr = 4. \quad (3.2.6)$$

Under the normalization $\lambda_0^2 + \lambda_1^2 = 1$, the metric which maximizes the area of S^2 has parameters $\lambda_0 = \frac{3}{\sqrt{13}}$, $\lambda_1 = \frac{2}{\sqrt{13}}$. This is not an Einstein metric (see Theorem 2.4.4). Figure 3.5 shows the graphs of the curvature $K_{\lambda_0, \lambda_1}(\Psi)$ for different values of λ_0, λ_1 , as functions of the latitude angle φ on S^2 .

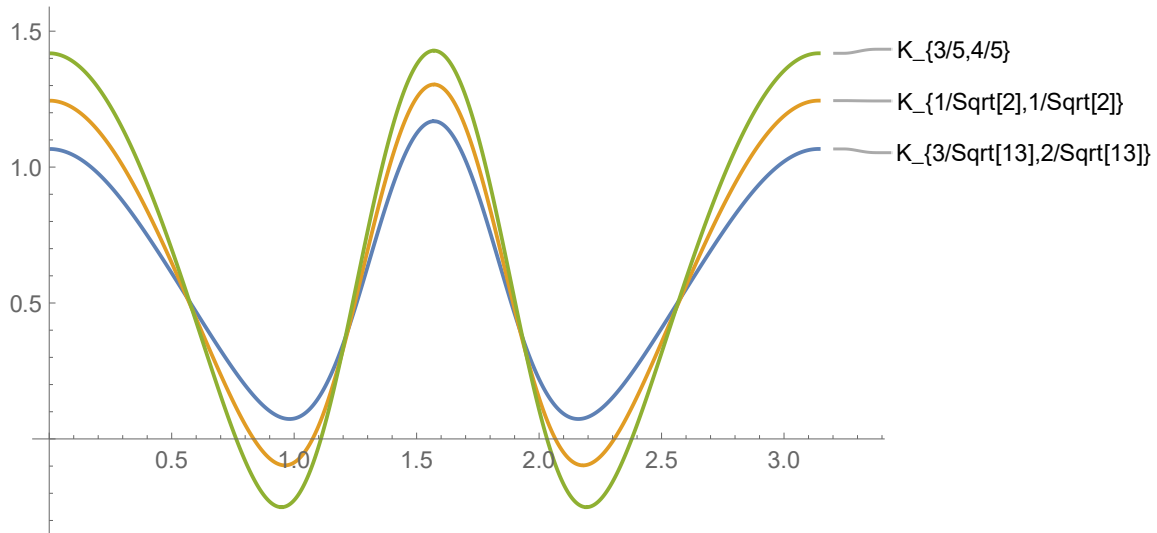


Figure 3.5

3.3 Primitive harmonic immersions of constant curvature of a simply connected surface.

In this section our aim is to generalize Corollary 3.1.2 to primitive harmonic immersion of a simply connected Riemann surface, not necessarily closed. The technique introduced by Q.-S. Chi and Y. Zheng [13] will play an important role in this generalization.

Let M be a simply connected surface and $\psi_0 : M \rightarrow \mathbb{C}P^n$ be a full holomorphic curve. Let $\Psi = (\psi_0, \dots, \psi_n) : M \rightarrow F_{1, \dots, 1}$ be the corresponding primitive map, which is assumed to be an immersion, and $\sigma_j : M \rightarrow \mathbb{C}P^{(n+1)-1}$ be the j -th osculating curve

of ψ_0 . On the full flag manifold $F_{1,\dots,1}$, we fix an invariant metric $g = ds_F^2$ such that

$$\Psi^* ds_F^2 = \sum_{j=0}^{n-1} k_j \gamma_j dz d\bar{z}, \quad \text{where each } k_j \text{ is a positive integer.} \quad (3.3.1)$$

Observe that any Einstein metric on $F_{1,\dots,1}$ gives (3.3.1). From (2.3.16), $\gamma_j = \partial_z \partial_{\bar{z}} \log \|\tilde{\sigma}_j\|^2$, where $\tilde{\sigma}_j^2$ is nonzero section of σ_j .

We define $\eta : M \rightarrow \mathbb{C}P^N$, by

$$\eta = \sigma_0^{k_0} \otimes \sigma_1^{k_1} \otimes \dots \otimes \sigma_{n-1}^{k_{n-1}},$$

where

$$N = L_0^{k_0} \dots L_{n-1}^{k_{n-1}} - 1, \quad \text{with } L_j = \binom{n+1}{j+1}.$$

Here we are denoting

$$\sigma_j^{k_j} = \underbrace{\sigma_j \otimes \dots \otimes \sigma_j}_{k_j \text{ times}}$$

for each $j \in \{0, 1, \dots, n-1\}$. Notice that the map η is *holomorphic*, because this map is defined as a tensor product of *holomorphic osculating curves*.

Lemma 3.3.1. *Under the above assumptions,*

$$\eta^* ds_{\mathbb{C}P^N}^2 = \Psi^* ds_F^2,$$

where $ds_{\mathbb{C}P^N}^2$ stands for the Fubini-Study metric on $\mathbb{C}P^N$.

Proof. Let $\tilde{\eta} : \mathbb{C} \rightarrow \mathbb{C}^{N+1}$ be a local lift of the map η of the form

$$\tilde{\eta} = \tilde{\sigma}_0^{k_0} \otimes \tilde{\sigma}_1^{k_1} \otimes \dots \otimes \tilde{\sigma}_{n-1}^{k_{n-1}},$$

where, for each j , $\tilde{\sigma}_j : \mathbb{C} \rightarrow \mathbb{C}^{L_j}$ is a nonvanishing local section of the osculating curve σ_j . Taking norm square yields

$$\|\tilde{\eta}\|^2 = \|\tilde{\sigma}_0\|^{2k_0} \dots \|\tilde{\sigma}_{n-1}\|^{2k_{n-1}}.$$

Then

$$\partial_z \partial_{\bar{z}} \log \|\tilde{\eta}\|^2 = k_0 \partial_z \partial_{\bar{z}} \log \|\tilde{\sigma}_0\|^2 + \dots + k_{n-1} \partial_z \partial_{\bar{z}} \log \|\tilde{\sigma}_{n-1}\|^2$$

By (2.3.16), we conclude that

$$\eta^* ds_{\mathbb{C}P^N}^2 = \sum_{j=0}^{n-2} k_j \gamma_j = \Psi^* ds_F^2.$$

□

Theorem 3.3.2. *Let M be a simply connected Riemann surface and $\psi_0 : M \rightarrow \mathbb{C}P^n$ be a full holomorphic map, with $n > 1$. If the lift*

$$\Psi = (\psi_0, \dots, \psi_n) : M \rightarrow F_{1, \dots, 1}$$

is a primitive immersion of constant curvature with respect to at least one invariant metric on $F_{1, \dots, 1}$ of the form (3.3.1), then Ψ is locally congruent with the Veronese primitive map.

Proof. Assume Ψ has constant curvature K with respect to at least one invariant metric of the form (3.3.1); then, by Lemma 3.3.1, the holomorphic map η has constant curvature K . Hence, by E. Calabi's result, η is locally unitarily equivalent to a portion of the Veronese map V_N . Now, take local nonvanishing holomorphic sections $\tilde{\sigma}_j$ of σ_j , with $j \in \{0, \dots, n-1\}$, such that η is locally spanned by

$$\tilde{\eta} = \tilde{\sigma}_0^{k_0} \otimes \tilde{\sigma}_1^{k_1} \otimes \dots \otimes \tilde{\sigma}_{n-1}^{k_{n-1}},$$

and

$$\tilde{\eta} = A \tilde{V}_N \tag{3.3.2}$$

for some constant $A \in U(N)$, where \tilde{V}_N is the standard section of the Veronese map. For each positive integer m , set $[m] = \{0, 1, \dots, m-1\}$. Denote by $(\tilde{\sigma}_j)_i$ the i -th component of $\tilde{\sigma}_j$, with $i \in [L_j]$. Comparing components of both sides of (3.3.2) we get, for each multiindex

$$I = (i_0^1, i_0^2, \dots, i_0^{k_0}, i_1^1, i_1^2, \dots, i_1^{k_1}, \dots) \in [L_0]^{k_0} \times [L_1]^{k_1} \times \dots \times [L_{n-1}]^{k_{n-1}}, \tag{3.3.3}$$

the following

$$(\tilde{\eta})_I = (\tilde{\sigma}_0)_{i_0^1} (\tilde{\sigma}_0)_{i_0^2} \dots (\tilde{\sigma}_0)_{i_0^{k_0}} (\tilde{\sigma}_1)_{i_1^1} (\tilde{\sigma}_1)_{i_1^2} \dots (\tilde{\sigma}_{n-1})_{i_{n-1}^{k_{n-1}}} = P^I(z), \tag{3.3.4}$$

where P^I is polynomial in z . We assume, without loss of generality, that $(\tilde{\sigma}_j)_0$ is

nonvanishing for all j . Then

$$\frac{(\tilde{\sigma}_j)_l}{(\tilde{\sigma}_j)_0} = \frac{(\tilde{\sigma}_0)_0^{k_0} \cdots (\tilde{\sigma}_{j-1})_0^{k_{j-1}} (\tilde{\sigma}_j)_l (\tilde{\sigma}_j)_0^{k_j-1} (\tilde{\sigma}_{j+1})_0^{k_{j+1}} \cdots (\tilde{\sigma}_{n-1})_0^{k_{n-1}}}{(\tilde{\sigma}_0)_0^{k_0} \cdots (\tilde{\sigma}_{j-1})_0^{k_{j-1}} (\tilde{\sigma}_j)_0^{k_j} (\tilde{\sigma}_{j+1})_0^{k_{j+1}} \cdots (\tilde{\sigma}_{n-1})_0^{k_{n-1}}} = \frac{P^{I(l,j)}}{P^{I(0,j)}}, \quad (3.3.5)$$

where $I(l, j)$ is the multindex of the form (3.3.3) with $i_j^1 = l \in [L_j]$ and all the other slots equal to zero. Observe that $I(0, j) = \vec{0}$. The equality (3.3.5) implies that

$$\tilde{\sigma}_j = \frac{(\tilde{\sigma}_j)_0}{P^{\vec{0}}} (P^{I(0,j)}, P^{I(1,j)}, P^{I(2,j)}, \dots, P^{I(L_j-1,j)}),$$

hence

$$\|\tilde{\sigma}_j\|^2 = \frac{|(\tilde{\sigma}_j)_0|^2}{|P^{\vec{0}}|^2} \sum_{l \in [L_j]} |P^{I(l,j)}|^2 \quad (3.3.6)$$

Observe also from (3.3.4)

$$|P^{\vec{0}}|^2 = \prod_{j \in [n]} |(\tilde{\sigma}_j)_0|^{2k_j}. \quad (3.3.7)$$

From (3.3.2) we have

$$\|\tilde{\eta}\|^2 = \|A\tilde{V}_N\|^2 = (1 + z\bar{z})^N. \quad (3.3.8)$$

On the other hand, from (3.3.6) and (3.3.7),

$$\begin{aligned} \|\tilde{\eta}\|^2 &= \|\tilde{\sigma}_0\|^{2k_0} \cdots \|\tilde{\sigma}_{n-1}\|^{2k_{n-1}} \\ &= \frac{1}{|P^{\vec{0}}|^{2(k_0+\dots+k_{n-1}-1)}} \prod_{j \in [n]} \left(\sum_{l \in [L_j]} |P^{I(l,j)}|^2 \right)^{k_j}. \end{aligned} \quad (3.3.9)$$

Equating (3.3.8) and (3.3.9), we obtain

$$(1 + z\bar{z})^N |P^{\vec{0}}|^{2(k_0+\dots+k_{n-1}-1)} = \prod_{j \in [n]} \left(\sum_{l \in [L_j]} |P^{I(l,j)}|^2 \right)^{k_j} \quad (3.3.10)$$

Both sides of (3.3.10) are polynomial functions in z and \bar{z} . Since $\mathbb{C}[z, \bar{z}]$ is a unique factorization domain and $(1 + z\bar{z})$ is irreducible in $\mathbb{C}[z, \bar{z}]$, it follows that, for each $j \in [n]$,

$$\sum_{l \in [L_j]} |P^{I(l,j)}|^2 = G_j(z, \bar{z})(1 + z\bar{z})^{N_j}, \quad (3.3.11)$$

for some nonnegative integer N_j , where G_j is a polynomial function in z and \bar{z} not

divisible by $1 + z\bar{z}$.

Each G_j is real since the remaining terms in equation (3.3.11) are real. On the other hand, observe that (3.3.10) and (3.3.11) give

$$(1 + z\bar{z})^N |P^{\bar{0}}|^{2(k_0 + \dots + k_{n-1}-1)} = (1 + z\bar{z})^{k_0 N_0 + \dots + k_{n-1} N_{n-1}} G_0^{k_0}(z, \bar{z}) \dots G_{n-1}^{k_{n-1}}(z, \bar{z}).$$

Since $P^{\bar{0}}$ is holomorphic (polynomial in z), the factor $1 + z\bar{z}$ does not divide $|P^{\bar{0}}|^2$, consequently

$$|P^{\bar{0}}|^{2(k_0 + \dots + k_{n-1}-1)} = G_0^{k_0}(z, \bar{z}) \dots G_{n-1}^{k_{n-1}}(z, \bar{z}). \quad (3.3.12)$$

It follows from (3.3.12), together with the reality of G_j , that $G_j(z, \bar{z}) = |h_j(z)|^2$ for some polynomial $h_j(z)$ in z . Hence, from (3.3.11),

$$\sum_{l \in [L_j]} |P^{l(j)}|^2 = |h_j(z)|^2 (1 + z\bar{z})^{N_j}. \quad (3.3.13)$$

From (3.3.6) and (3.3.13) we have

$$\|\tilde{\sigma}_j\|^2 = |H_j(z)|^2 (1 + z\bar{z})^{N_j} \quad (3.3.14)$$

where H_j is the holomorphic function defined by

$$H_j = \frac{h_j(\sigma_j)_0}{P^{\bar{0}}}.$$

Since, for any holomorphic function H ,

$$\partial_z \partial_{\bar{z}} \log |H(z)|^2 = 0,$$

from (3.3.14) we obtain

$$\gamma_j = \partial_z \partial_{\bar{z}} \log \|\tilde{\sigma}_j\|^2 = \frac{N_j}{(1 + |z|^2)^2}. \quad (3.3.15)$$

Taking $j = 0$, this shows that ψ_0 has constant curvature, then it is locally congruent with a Veronese map. This shows that Ψ is locally congruent with the Veronese primitive map. \square

Chapter 4

Classification of primitive immersions of constant curvature into flag manifolds

In this chapter, we classify primitive immersions of constant curvature from the two-sphere S^2 into the low dimensional flag manifolds $F_{2,1,1}$ and $F_{2,2,1}$. Our methods involve harmonic sequences and harmonic diagrams, as developed in [10, 44], and the generalization of the notion of absolute value type function [21, 35, 41], which we called *generalized absolute value type function*. We start by describing some constructions of new primitive immersions of constant curvature from existing ones. Such constructions will enable us to reduce our problem to the classification of primitive immersions of constant curvature which are lifts of full harmonic maps from S^2 into the Grassmannians $G_2(\mathbb{C}^4)$ and $G_2(\mathbb{C}^5)$, in the sense of Remark 2.4.2.

4.1 Homogeneous projections and adding constants

In this section, we discuss the construction of new primitive immersions of constant curvatures from existing ones.

Let $\psi_0 : M \rightarrow G_{k_0}(\mathbb{C}^n)$ be a linearly full holomorphic map. Consider its harmonic sequence ψ_0, \dots, ψ_p and the corresponding primitive lift

$$\Psi = (\psi_0, \dots, \psi_p) : M \rightarrow F_{k_0, \dots, k_p}.$$

For a given choice of numbers $0 \leq q_0 < \dots < q_{l-1} < q_l = p$, define the *homogeneous projection*

$$\Pi : F_{k_0, \dots, k_p} \rightarrow F_{k^0, \dots, k^l},$$

by

$$\Pi(\psi_0, \dots, \psi_p) = (\psi^0, \dots, \psi^l),$$

where, for each $j \in \{0, \dots, l\}$, setting $q_{-1} = -1$,

$$k^j = k_{q_{j-1}+1} + \dots + k_{q_j}, \quad \psi^j = \psi_{q_{j-1}+1} \oplus \dots \oplus \psi_{q_j}. \quad (4.1.1)$$

The following provides a generalization of Theorem 3.1.1:

Proposition 4.1.1. *Let $\psi_0 : S^2 \rightarrow G_{k_0}(\mathbb{C}^n)$ be a linearly full holomorphic map, with*

harmonic sequence ψ_0, \dots, ψ_p . If the primitive lift $\Psi = (\psi_0, \dots, \psi_p) : S^2 \rightarrow F_{k_0, \dots, k_p}$ is an immersion and there exists at least one $U(n)$ -invariant metric on F_{k_0, \dots, k_p} with respect to which Ψ has constant curvature, then $\tilde{\Psi} = \Pi \circ \Psi : S^2 \rightarrow F_{k^0, \dots, k^l}$ is a primitive immersion of constant curvature with respect to all $U(n)$ -invariant metrics on F_{k^0, \dots, k^l} .

Proof. For ψ^j defined by (4.1.1), observe that the second fundamental form

$$A'_{\psi^j} : \psi^j \rightarrow (\psi^j)^\perp$$

is given by:

$$A'_{\psi^j}(v) = A'_{\psi_{q_j}} \circ \pi_{q_j}(v),$$

where π_{q_j} is the Hermitian projection onto ψ_{q_j} . Hence the image of A'_{ψ^j} is contained in $\psi_{q_j+1} \subset \psi^{j+1}$, which means that $\tilde{\Psi} = \Pi \circ \Psi$ defines a primitive map into F_{k^0, \dots, k^l} . Moreover, in view of (2.3.6), we have

$$\operatorname{tr} A'_{\psi^j} (A'_{\psi^j})^* = \operatorname{tr} A'_{\psi_{q_j}} (A'_{\psi_{q_j}})^* = \gamma_{q_j}. \quad (4.1.2)$$

By Theorem 3.1.1, each ψ_j , with $j \in \{0, \dots, p-1\}$ is an immersion of constant curvature and constant Kähler angle. This implies that each function γ_j , on a local chart (U, z) , takes the form $\gamma_j = \frac{\alpha_j}{(1+z\bar{z})^2}$, for some positive constant α_j . Now, take any $U(n)$ -invariant metric g on F_{k^0, \dots, k^l} . By (2.4.3) and (4.1.2), on a local complex chart (U, z) , $\tilde{\Psi}^* g = \sum_{j=0}^{l-1} \lambda_j \gamma_{q_j} dz d\bar{z}$, for some positive numbers λ_j , with $j \in \{0, \dots, l-1\}$. Then

$$\tilde{\Psi}^* g = \sum_{j=0}^{l-1} \frac{\lambda_j \alpha_{q_j}}{(1+z\bar{z})^2} dz d\bar{z},$$

which implies that $\tilde{\Psi}$ is an immersion of constant curvature. \square

Given a primitive map $\Psi = (\psi_0, \dots, \psi_p)$ into F_{k_0, \dots, k_p} , we describe three other processes of obtaining new primitive maps from Ψ :

Remark 4.1.2. Given a primitive map $\Psi = (\psi_0, \dots, \psi_p)$ into F_{k_0, \dots, k_p} , we describe three other processes of obtaining new primitive maps from Ψ :

1. *Adding a constant.* The smooth map $\Psi \oplus_r \mathbb{C}^m = (\psi_0, \dots, \psi_r \oplus \mathbb{C}^m, \dots, \psi_p)$ defines a primitive map into $F_{k_0, \dots, k_r+m, \dots, k_p}$.
2. *Adding a primitive map.* More generally, let $\Phi = (\varphi_0, \dots, \varphi_p)$ be another primitive map into F_{s_0, \dots, s_p} . Then $\Psi \oplus \Phi = (\psi_0 \oplus \varphi_0, \dots, \psi_p \oplus \varphi_p)$ is primitive map into $F_{k_0+s_0, \dots, k_p+s_p}$.
3. *Shifting.* $\tau\Psi = (\psi_p, \psi_0, \dots, \psi_{p-1})$ defines a primitive map into $F_{k_p, k_0, \dots, k_{p-1}}$.

4.2 Primitive immersions into $F_{2,1,1}$

In this section, we classify primitive immersions of constant curvature from S^2 into the flag manifold $F_{2,1,1}$. These primitive immersions are associated to full harmonic maps from S^2 into the complex Grassmannian $G_2(\mathbb{C}^4)$ of isotropy order $r \geq 2$. We also compare our results with [41] and find consistent.

Theorem 4.2.1. *Let $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^4)$ be a linearly full harmonic map of isotropy order $r \geq 2$, with first and second ∂' -Gauss bundles ψ_1 and ψ_2 , respectively. Assume that the primitive map $\Psi = (\psi_0, \psi_1, \psi_2) : S^2 \rightarrow F_{2,1,1}$ is an immersion. If there exists at least one $U(4)$ -invariant metric on $F_{2,1,1}$ with respect to which Ψ has constant curvature, then Ψ is unitarily equivalent to one of the following curves:*

1. $\Psi_1 = (V_0^3 \oplus V_1^3, V_2^3, V_3^3)$;
2. $\Psi_2 = (V_0^2 \oplus \mathbb{C}, V_1^2, V_2^2)$ (taking $\mathbb{C}^4 = \mathbb{C}^3 \oplus \mathbb{C}$);
3. $\Psi_3 = (V_0^3 \oplus V_3^3, V_1^3, V_2^3)$.

For a $U(4)$ -invariant metric on $F_{2,1,1}$ with parameters $\lambda_0, \lambda_1, \lambda_2$, using the notations of (2.4.3), the metrics induced by Ψ_1, Ψ_2 and Ψ_3 , respectively, have curvatures

$$K(\Psi_1) = \frac{4}{4\lambda_1 + 3\lambda_2}, \quad K(\Psi_2) = \frac{2}{\lambda_0 + \lambda_1}, \quad K(\Psi_3) = \frac{4}{3\lambda_0 + 4\lambda_1 + 3\lambda_2}.$$

In the case $r = \infty$, the map Ψ is locally congruent to either Ψ_1 or Ψ_2 , otherwise it is locally congruent to Ψ_3 .

Proof. By dimension constraints, either $r = \infty$ or 2.

Suppose that ψ_0 is holomorphic, i.e., $r = \infty$. Since Ψ has constant curvature with respect to at least one $U(4)$ -invariant metric on $F_{2,1,1}$, then Ψ has constant curvature with respect to all $U(4)$ -invariant metrics on $F_{2,1,1}$ and each ψ_i is an immersion of constant curvature (see Theorem 3.1.1). The fundamental harmonic diagram for this case is the following:

$$\psi_0 \longrightarrow \psi_1 \longrightarrow \psi_2,$$

with $\text{rank } \psi_0 = 2$ and $\text{rank } \psi_1 = \text{rank } \psi_2 = 1$. Set $f_0 = \underline{\ker} A'_{\psi_0}$ and $f_1 = f_0^\perp \cap \psi_0$. It is well-known that $f_0 = \underline{\ker} A'_{\psi_0}$ is a holomorphic subbundle of ψ_0 [10, Proposition 2.2].

Thus, the fundamental diagram admits the following refinement:

$$\begin{array}{ccccc} f_1 & \longrightarrow & f_2 & \longrightarrow & f_3 \\ \uparrow & & & & \\ f_0 & & & & \end{array} . \quad (4.2.1)$$

Here we are denoting $f_2 = \psi_1$ and $f_3 = \psi_2$. In this diagram, the horizontal arrows are nonzero, otherwise ψ_0 would not be linearly full. Now, if the vertical arrow is nonzero, then $f_0 : S^2 \rightarrow \mathbb{C}P^3$ is a linearly full holomorphic map. Since $f_2 = \psi_1$ has constant curvature, then, by [7, Theorem 5.4], f_0 is unitarily equivalent to the Veronese map V_0^3 , and, consequently, Ψ is unitarily equivalent to $\Psi_1 = (V_0^3 \oplus V_1^3, V_2^3, V_3^3)$. Since, by Theorem 2.3.13,

$$\gamma_1^3 = \frac{4}{(1 + z\bar{z})^2}, \quad \gamma_2^3 = \frac{3}{(1 + z\bar{z})^2}$$

the metric induced on S^2 by Ψ_1 is

$$\Psi_1^* ds_{F_{2,1,1}}^2 = \frac{4\lambda_1 + 3\lambda_2}{(1 + z\bar{z})^2} dzd\bar{z},$$

and the corresponding curvature is given by

$$K(\Psi_1) = \frac{4}{4\lambda_1 + 3\lambda_2}.$$

Now, if the vertical arrow vanishes, then f_0 is a constant one-dimensional complex subspace, which we denote by \mathbb{C} , and the diagram (4.2.1) becomes

$$\begin{array}{ccccc} f_1 & \longrightarrow & f_2 & \longrightarrow & f_3 \\ \mathbb{C} & & & & \end{array} .$$

In this case, f_1 is holomorphic and linearly full map from S^2 into $\mathbb{C}P^2$. Again, since $f_2 = \psi_1$ has constant curvature, then, by [7, Theorem 5.4], f_1 is unitarily equivalent to the Veronese map V_0^2 , and, consequently, Ψ is unitarily equivalent to $\Psi_2 = (V_0^2 \oplus \mathbb{C}, V_1^2, V_2^2)$. Since

$$\gamma_0^2 = \gamma_1^2 = \frac{2}{(1 + z\bar{z})^2},$$

the induced metric is given by

$$\Psi_2^* ds_{F_{2,1,1}}^2 = \frac{2(\lambda_0 + \lambda_1)}{(1 + z\bar{z})^2},$$

and the corresponding curvature by

$$K(\Psi_2) = \frac{2}{\lambda_0 + \lambda_1}.$$

Suppose now that ψ_0 has isotropy order $r = 2$. We have the following fundamental harmonic diagram

$$\begin{array}{ccccc} & & \curvearrowright & & \\ \psi_0 & \longrightarrow & \psi_1 & \longrightarrow & \psi_2 \end{array} \quad (4.2.2)$$

with $\text{rank } \psi_0 = 2$ and $\text{rank } \psi_1 = \text{rank } \psi_2 = 1$. Observe that we cannot apply here Theorem 3.1.1 to conclude that each ψ_i has constant curvature, because ψ_0 is not anymore holomorphic.

In (4.2.2), the backward arrow is nonzero, because ψ_0 has isotropy order $r = 2$. Set $f_0 = \underline{\ker} A'_{\psi_0}$ and $f_1 = f_0^\perp \cap \psi_0$, so that $\psi_0 = f_0 \oplus f_1$. By [10, Proposition 2.2], we know that $f_0 = \underline{\ker} A'_{\psi_0}$ is a holomorphic subbundle of ψ_0 . Hence, we have the following refinement of (4.2.2):

$$\begin{array}{ccccc} f_1 & \longrightarrow & f_2 & \longrightarrow & f_3 \\ \uparrow & & & & \curvearrowright \\ f_0 & & & & \end{array} . \quad (4.2.3)$$

By Proposition 2.3.10, the smooth map $f_1 : S^2 \rightarrow \mathbb{C}P^3$ is harmonic. If all the arrows in this diagram are nonzero, then $f_1 : S^2 \rightarrow \mathbb{C}P^3$ would be a harmonic map of finite isotropy order, which we know to be impossible due to the topology of S^2 (see Remark 2.3.11). Hence, in (4.2.3), the vertical arrow from f_0 to f_1 must be zero, and we get

$$\begin{array}{ccccc} f_1 & \longrightarrow & f_2 & \longrightarrow & f_3 \\ & & & & \curvearrowright \\ f_0 & & & & \end{array} .$$

In view of (2.4.3), the metric induced on S^2 by $\Psi_3 = (f_0 \oplus f_1, f_2, f_3)$ from the invariant metric on $F_{2,1,1}$ with parameters $\lambda_0, \lambda_1, \lambda_2$ is equal to the metric induced on S^2 by $\tilde{\Psi} = (f_1, f_2, f_3, f_0)$ from the metric on $F_{1,1,1,1}$ with parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, regardless of the value of $\lambda_3 > 0$. Hence $\tilde{\Psi}$ is an immersion of constant curvature. By Theorem 3.1.1, we conclude that $f_2 = \psi_1$ has constant curvature, then, by [7, Theorem 5.4], f_1 is unitarily equivalent to the Veronese map V_0^3 , hence f_0 is unitarily congruent to V_3^3 , and the result follows. \square

Remark 4.2.2. A careful inspection of the proof of Theorem 4.3.4 shows that any other primitive immersion $\Psi : S^2 \rightarrow F_{2,1,1}$ of constant curvature with respect to at least one $U(4)$ -invariant metric on $F_{2,1,1}$ can be obtained from Veronese primitive immersions by operating with homogeneous projections, adding constants, adding primitive maps, and shifts. For example, in diagram (4.2.3), the case where the arrows $f_0 \rightarrow f_1$ and $f_2 \rightarrow f_3$ vanish leads to the primitive immersion into $F_{2,1,1}$ of constant curvature given by $\Psi_1 \oplus \Psi_2$, where $\Psi_1 = (V_0^1, V_1^1, 0)$ and $\Psi_2 = \tau^2 \Psi_1 = (V_1^1, 0, V_0^1)$.

Proposition 4.2.3. More generally, suppose that

$$\Psi = (\psi_0, \psi_1, \dots, \psi_{n-1}) : S^2 \rightarrow F_{2,1,\dots,1}$$

is a primitive immersion of constant curvature with respect to at least one $U(n)$ -invariant metric on $F_{2,1,\dots,1}$, where $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^n)$ is a linearly full harmonic map of isotropy order $r \geq n - 1$, with harmonic sequence $\{\psi_i\}_{i \in \mathbb{Z}}$. By straightforward extension of the previous proof, Ψ is unitarily congruent to one of the following curves:

1. $\Psi_1 = (V_0^{n-1} \oplus V_1^{n-1}, V_2^{n-1}, \dots, V_{n-1}^{n-1})$;
2. $\Psi_2 = (V_0^{n-2} \oplus \mathbb{C}, V_1^{n-2}, \dots, V_{n-2}^{n-2})$;
3. $\Psi_3 = (V_0^{n-1} \oplus V_{n-1}^{n-1}, V_2^{n-1}, \dots, V_{n-2}^{n-1})$.

In the next two remarks, we will compare our result with the results of [41].

Remark 4.2.4. In [41, Theorem A], the authors provided a complete classification of holomorphic immersions of constant curvature K from S^2 into $G_2(\mathbb{C}^4)$. Any such immersion $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^4)$ is unitarily congruent to one of the following minimal immersions (locally spanned by f_0 and g_0 , in terms of the canonical complex coordinate z on $\mathbb{C} = S^2 \setminus \{0\}$):

1. $f_0 = (1, 0, z, 0)$ and $g_0 = (0, 1, 0, 0)$, with $K = 4$. This holomorphic immersion is not linearly full.
2. $f_0 = (1, 0, z^2 \cos 2t, \sqrt{2}z \sin t)$ and $g_0 = (0, 1, \sqrt{2}z \cos t, 0)$ for some $t \in [0, \pi)$, with $K = 2$. Clearly, ψ_0 is not linearly full if $t = 0$, so we take $t \in (0, \pi)$. Straightforward computations show that

$$\gamma_0 = \partial_z \partial_{\bar{z}} \log \|f_0 \wedge g_0\|^2 = \frac{2}{(1 + z\bar{z})^2},$$

which confirms that ψ_0 has constant curvature $K = 2$, and that

$$\|f_0 \wedge g_0 \wedge \partial_z f_0 \wedge \partial_z g_0\|^2 = 2 \sin^2 t \cos^2 t.$$

Hence the first Gauss bundle ψ_1 of ψ_0 has rank 1 if $t = \frac{\pi}{2}$ and it has rank 2 otherwise. For $t = \frac{\pi}{2}$, we have $f_0 = V_0^2$ and g_0 is constant, hence ψ_0 (with $t = \frac{\pi}{2}$) generates the primitive lift Ψ_2 of Theorem 4.2.1.

3. $f_0 = (1, 0, \sqrt{3}z^2, 0)$ and $g_0 = (0, 1, \sqrt{8/3}z, \sqrt{1/3}z)$, with $K = \frac{4}{3}$. A straightforward computation shows that

$$\|f_0 \wedge g_0 \wedge \partial_z f_0 \wedge \partial_z g_0\|^2 = 4|z|^2.$$

Hence the first Gauss bundle of ψ has rank 2.

4. $f_0 = (1, 0, 2z^3, \sqrt{3}z^2)$ and $g_0 = (0, 1, \sqrt{3}z^2, 2z)$, with $K = 1$. Consider the Plücker embedding $\iota : G(2, 4) \hookrightarrow \mathbb{C}P^5$. A straightforward computation shows that, up to unitarily congruence,

$$\iota \circ \psi_0 = \iota \circ (V_0^3 \oplus V_1^3).$$

Hence ψ_0 generates the primitive immersion Ψ_1 in Theorem (4.2.1).

Remark 4.2.5. In [41, Theorem B], the authors provided a complete classification for constant curved minimal immersions $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^4)$ which are neither holomorphic nor antiholomorphic. Any such immersion is unitarily congruent to one of the following minimal immersions (locally spanned by f_0 and g_0 , in terms of the canonical chart (U_0, z)):

1. $f_0(z) = (1, z, 0, 0)$ and $g_0(z) = (0, 0, 1, \bar{z})$, with $K = 2$. This minimal immersion has second ∂' -Gauss bundle $\psi_2 = \{0\}$.
2. $f_0(z) = (1, \sqrt{2}z, z^2, 0)$ and $g_0(z) = (\bar{z}^2, -\sqrt{2}\bar{z}, 1, 0)$, with $K = 1$. This minimal immersion is not full.
3. $f_0(z) = (1, \sqrt{3}z, \sqrt{3}z^2, z^3)$ and $g_0(z) = (\bar{z}^3, -\sqrt{3}\bar{z}^2, \sqrt{3}\bar{z}, -1)$, with $K = \frac{2}{3}$. This is precisely $\psi_0 = V_0^3 \oplus V_3^3$, and its harmonic sequence yields Ψ_3 in Theorem 4.2.1.
4. $f_0(z) = (1, \sqrt{3}z, \sqrt{3}z^2, z^3)$ and $g_0(z) = (\sqrt{3}\bar{z}^2, \bar{z}(z\bar{z} - 2), 1 - 2z\bar{z}, \sqrt{3}z)$, with $K = \frac{2}{5}$. We have

$$\|f_0 \wedge g_0 \wedge \partial_z f_0 \wedge \partial_z g_0\|^2 = 9(1 + z\bar{z})^8.$$

Hence the first ∂' -Gauss bundle has rank 2.

From Remarks 4.2.4 and 4.2.5 we conclude that our results are consistent with the classification results in [41].

4.3 Primitive immersions into $F_{2,2,1}$

In this section, we will discuss the classification of all primitive immersions of constant curvature from S^2 into the flag manifold $F_{2,2,1}$.

Theorem 4.3.1. *Let $\Psi = (\psi_0, \psi_1, \psi_2) : S^2 \rightarrow F_{2,2,1}$ be a primitive immersion of constant curvature with respect to at least one $U(5)$ -invariant metric on $F_{2,2,1}$. Here $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^5)$ is a linearly full holomorphic map with first and second ∂' -Gauss bundles ψ_1 and ψ_2 . Then, Ψ is unitarily equivalent to the curve*

$$(V_0^2 \oplus V_0^1, V_1^2 \oplus V_1^1, V_2^2) \quad (\text{taking } \mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2). \quad (4.3.1)$$

For a $U(5)$ -invariant metric on $F_{2,2,1}$ with parameters $\lambda_0, \lambda_1, \lambda_2$, using the notations of (2.4.3), the metric induced by Ψ has constant curvature $K(\Psi) = \frac{4}{3\lambda_0 + 2\lambda_1}$.

Proof. Suppose that $\Psi = (\psi_0, \psi_1, \psi_2)$ has constant curvature with respect to at least one $U(5)$ -invariant metric on $F_{2,2,1}$. Since $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^5)$ is holomorphic, then by Theorem 3.1.1, Ψ has constant curvature w.r.t all $U(5)$ -invariant metrics on $F_{2,2,1}$ and each ψ_i has constant curvature. We have the following fundamental harmonic diagram

$$\psi_0 \longrightarrow \psi_1 \longrightarrow \psi_2,$$

with $\text{rank } \psi_0 = \text{rank } \psi_1 = 2$ and $\text{rank } \psi_2 = 1$. Now, define the line subbundles

$$f_2 = \underline{\ker} A'_{\psi_1}, \quad f_3 = f_2^\perp \cap \psi_1, \quad f_1 = \underline{\text{Im}} A''_{\psi_1}|_{f_3}, \quad f_0 = f_1^\perp \cap \psi_0, \quad f_4 = \underline{\text{Im}} A'_{\psi_1}.$$

Then we have the following refinement of the fundamental diagram

$$\begin{array}{ccccc} f_1 & \longrightarrow & f_3 & \longrightarrow & f_4 \\ \uparrow & \searrow & \uparrow & & \\ f_0 & \longrightarrow & f_2 & & \end{array} \quad (4.3.2)$$

where $\psi_0 = f_0 \oplus f_1$, $\psi_1 = f_2 \oplus f_3$ and $\psi_2 = f_4$. Due to our assumptions on the harmonic sequence ψ_0, ψ_1, ψ_2 , all the horizontal arrows are nonzero. Observe that f_3 and f_4 are linearly full harmonic maps in $\mathbb{C}P^p$, for some $p \in \{2, 3, 4\}$. On the other hand, since $\psi_2 = f_4$ has constant curvature, $f_3, f_4 : S^2 \rightarrow \mathbb{C}P^p$ belong to a Veronese sequence: $f_3 = V_{p-1}^p$ and $f_4 = V_p^p$, up to unitary congruence. We analyze these three cases separately.

Case $p = 4$. We have

$$\begin{array}{ccccc}
 f_1 & \longrightarrow & V_3^4 & \longrightarrow & V_4^4 \\
 \uparrow & & \searrow & & \uparrow \\
 f_0 & \longrightarrow & f_2 & &
 \end{array} . \tag{4.3.3}$$

Now, take the composition of ψ_0 with the Plücker embedding of $G_2(\mathbb{C}^5) \hookrightarrow \mathbb{C}P^9$ to obtain a holomorphic map $\sigma_0 : S^2 \rightarrow \mathbb{C}P^9$. Since ψ_0 has constant curvature, then σ_0 also has constant curvature and it coincides, up to unitary congruence, with the Veronese map in $\mathbb{C}P^k$, for some $k \in \{1, \dots, 9\}$. Let $\hat{\sigma}_0 : \mathbb{C} \rightarrow \mathbb{C}^{10}$ be a holomorphic local section of σ_0 . Without loss of generality, we assume that $\hat{\sigma}_0$ is nonvanishing, $\|\hat{\sigma}_0(0)\| = 1$ and that its components are holomorphic polynomials. Then

$$\|\hat{\sigma}_0\|^2 = (1 + z\bar{z})^k \quad \text{for some } k \in \{1, \dots, 9\}. \tag{4.3.4}$$

Consider the local section $v_0 = (1, 2z, \sqrt{6}z^2, 2z^3, z^4)$ of V_0^4 . In view of (4.3.3), we have

$$f_0 \oplus f_1 \oplus f_2 = V_0^4 \oplus V_1^4 \oplus V_2^4 = \underline{\text{span}}\{v_0, \partial_z v_0, \partial_z^2 v_0\};$$

hence

$$\hat{\sigma}_0 = P_0(v_0 \wedge \partial_z v_0) + P_1(v_0 \wedge \partial_z^2 v_0) + P_2(\partial_z v_0 \wedge \partial_z^2 v_0) \tag{4.3.5}$$

for some polynomials P_0, P_1, P_2 . Consider the local sections v_1 and v_2 of V_1^4 and V_2^4 , respectively, defined as in (2.3.11). From (2.3.20), we have the following

$$\partial_z v_0 = v_1 + \frac{4\bar{z}}{1 + z\bar{z}}v_0, \quad \partial_z v_1 = v_2 + \frac{2\bar{z}}{1 + z\bar{z}}v_1. \tag{4.3.6}$$

By using (4.3.6), after straightforward computation we deduce that

$$\partial_z^2 v_0 = v_2 + \frac{6\bar{z}}{(1 + z\bar{z})}v_1 + \frac{12\bar{z}^2}{(1 + z\bar{z})^2}v_0.$$

Hence,

$$\begin{aligned}
 v_0 \wedge \partial_z v_0 &= v_0 \wedge v_1, & v_0 \wedge \partial_z^2 v_0 &= \frac{6\bar{z}}{1 + z\bar{z}}v_0 \wedge v_1 + v_0 \wedge v_2 \\
 \partial_z v_0 \wedge \partial_z^2 v_0 &= \frac{12\bar{z}^2}{(1 + z\bar{z})^2}v_0 \wedge v_1 + \frac{4\bar{z}}{(1 + z\bar{z})}v_0 \wedge v_2 + v_1 \wedge v_2.
 \end{aligned}$$

Replacing these formulas in (4.3.5), we obtain

$$\hat{\sigma}_0 = \left(P_0 + \frac{6\bar{z}P_1}{1+z\bar{z}} + \frac{12\bar{z}^2P_2}{(1+z\bar{z})^2} \right) v_0 \wedge v_1 + P_2 (v_1 \wedge v_2) + \left(P_1 + \frac{4\bar{z}P_2}{1+z\bar{z}} \right) v_0 \wedge v_2.$$

Taking the norm square (recall that v_0, v_1, v_2 are mutually orthogonal)

$$\begin{aligned} \|\hat{\sigma}_0\|^2 &= \left| P_0 + \frac{6\bar{z}P_1}{1+z\bar{z}} + \frac{12\bar{z}^2P_2}{(1+z\bar{z})^2} \right|^2 \|v_0\|^2 \|v_1\|^2 + |P_2|^2 \|v_1\|^2 \|v_2\|^2 \\ &\quad + \left| P_1 + \frac{4\bar{z}P_2}{1+z\bar{z}} \right|^2 \|v_2\|^2 \|v_0\|^2. \end{aligned} \quad (4.3.7)$$

From Theorem 2.3.13 we know that

$$\frac{\|v_1\|^2}{\|v_0\|^2} = \gamma_0^4 = \frac{4}{(1+z\bar{z})^2}, \quad \frac{\|v_2\|^2}{\|v_1\|^2} = \gamma_1^4 = \frac{6}{(1+z\bar{z})^2}.$$

Together with (4.3.7), and since $\|v_0\|^2 = (1+z\bar{z})^4$, this gives

$$\begin{aligned} \|\hat{\sigma}_0\|^2 &= 4 \left| P_0 (1+z\bar{z})^3 + 6\bar{z}(1+z\bar{z})^2 P_1 + 12\bar{z}^2(1+z\bar{z}) P_2 \right|^2 \\ &\quad + 96 \left| (1+z\bar{z}) P_2 \right|^2 + 24 \left| (1+z\bar{z})^2 P_1 + 4\bar{z}(1+z\bar{z}) P_2 \right|^2. \end{aligned} \quad (4.3.8)$$

Combining (4.3.4) and (4.3.8),

$$\begin{aligned} (1+z\bar{z})^{k-2} &= 4 \left| P_0 (1+z\bar{z})^2 + 6\bar{z}(1+z\bar{z}) P_1 + 12\bar{z}^2 P_2 \right|^2 \\ &\quad + 96 |P_2|^2 + 24 \left| (1+z\bar{z}) P_1 + 4\bar{z} P_2 \right|^2. \end{aligned} \quad (4.3.9)$$

The case $k = 1$ is impossible, because the right-hand side of (4.3.9) is polynomial in z, \bar{z} . For $k = 2$, equation (4.3.9) yields

$$\begin{aligned} 1 &= 4 \left| P_0 (1+z\bar{z})^2 + 6\bar{z}(1+z\bar{z}) P_1 + 12\bar{z}^2 P_2 \right|^2 \\ &\quad + 96 |P_2|^2 + 24 \left| (1+z\bar{z}) P_1 + 4\bar{z} P_2 \right|^2. \end{aligned}$$

From this we see that the three adding terms on the right-hand side must be bounded. But this occurs if, and only if, all the polynomials $P_0(z)$, $P_1(z)$ and $P_2(z)$ are zero, giving a contradiction.

Suppose that $k > 2$. Recall that $\mathbb{C}[z, \bar{z}]$ is a unique factorization domain and $(1+z\bar{z})$ is irreducible in $\mathbb{C}[z, \bar{z}]$. Observe that if we expand the right-hand side of (4.3.9), all the terms except

$$|P_2|^2 (96 + 384z\bar{z} + 576z^2\bar{z}^2)$$

are divisible by $(1+z\bar{z})$. Since, for $k > 2$, the factor $(1+z\bar{z})$ divides the left-hand side of (4.3.9), it must divide $|P_2|^2 (96 + 384z\bar{z} + 576z^2\bar{z}^2)$ as well. But P_2 is a holo-

morphic polynomial, hence, if $P_2 \neq 0$, $|P_2|^2$ is not divisible by $(1 + z\bar{z})$; moreover, $(96 + 384z\bar{z} + 576z^2\bar{z}^2)$ is not divisible by $(1 + z\bar{z})$. Hence, we must have $P_2 = 0$. In this case, equation (4.3.9) simplifies to

$$(1 + z\bar{z})^{k-4} = 4|P_0(1 + z\bar{z}) + 6\bar{z}P_1|^2 + 24|P_1|^2. \quad (4.3.10)$$

From the equation (4.3.10), observe that for $k = 3$ the left-hand side is no longer a polynomial, whereas the right-hand is polynomial. Therefore, $k = 3$ is not possible.

For $k = 4$, equation (4.3.10) yields

$$1 = 4|P_0(1 + z\bar{z}) + 6\bar{z}P_1|^2 + 24|P_1|^2.$$

From this we see that both terms $|P_1|^2$ and $|P_0(1 + z\bar{z}) + 6\bar{z}P_1|^2$ on the right-hand side must be bounded. But this occurs if, and only if, the polynomials $P_0(z)$ and $P_1(z)$ are both zero, giving a contradiction.

For $k > 4$ we must have $P_1 = 0$ by applying the similar arguments as those used to show that $P_2 = 0$. Consequently, $\hat{\sigma}_0 = P_0(v_0 \wedge v_1)$. This implies that

$$\psi_0 = f_0 \oplus f_1 = V_0^4 \oplus V_1^4,$$

which leads to the following refinement of the diagram (4.3.3)

$$\begin{array}{ccc} V_1^4 & & V_3^4 \longrightarrow V_4^4 \\ \uparrow & \searrow & \uparrow \\ V_0^4 & & V_2^4 \end{array} . \quad (4.3.11)$$

Here the first \mathcal{D}' -Gauss bundle of ψ_0 is V_2^4 , leading to a contradiction, since we are assuming that the first \mathcal{D}' -Gauss bundle of ψ_0 has rank 2.

Case $p = 3$. Let us consider the case $f_3 = V_2^3$. The diagram (4.3.2) will then take the following form

$$\begin{array}{ccccc} f_1 & \longrightarrow & V_2^3 & \longrightarrow & V_3^3 \\ \uparrow & & \searrow & & \uparrow \\ f_0 & \longrightarrow & f_2 & & \end{array} . \quad (4.3.12)$$

Again, by taking the composition of $\psi_0 = f_0 \oplus f_1$ with the Plücker embedding of $G_2(\mathbb{C}^5) \hookrightarrow \mathbb{C}P^9$ we obtain a holomorphic map $\sigma_0 : S^2 \rightarrow \mathbb{C}P^9$. Since ψ_0 has constant curvature, then σ_0 also has constant curvature and it coincides with the Veronese map

in $\mathbb{C}P^k$, for some $k \in \{1, \dots, 9\}$. Let $\hat{\sigma}_0 : \mathbb{C} \rightarrow \mathbb{C}^{10}$ be a holomorphic local section of σ_0 . Without loss of generality, we assume that $\hat{\sigma}_0$ is nonvanishing, $\|\hat{\sigma}_0(0)\| = 1$ and that its components are holomorphic polynomials. Then (4.3.4) holds, that is, $\|\hat{\sigma}_0\|^2 = (1 + z\bar{z})^k$ for some $k \in \{1, \dots, 9\}$.

Consider the orthogonal decomposition

$$\mathbb{C}^5 = V_0^3 \oplus V_1^3 \oplus V_2^3 \oplus V_3^3 \oplus \text{span}\{e\},$$

where $e = (0, 0, 0, 0, 1)$. Consider the section $v_0 = (1, \sqrt{3}z, \sqrt{3}z^2, z^3, 0)$ of V_0^3 , and let v_1, v_2 and v_3 be local sections of V_1^3, V_2^3 , and V_3^3 , respectively, defined by (2.3.11).

Observe that $\hat{\sigma}_0$ is a section of

$$\underline{\text{span}}\{v_0 \wedge e, v_0 \wedge \partial_z v_0, \partial_z v_0 \wedge e\}.$$

Hence,

$$\hat{\sigma}_0 = P_0(v_0 \wedge \partial_z v_0) + P_1(v_0 \wedge e) + P_2(\partial_z v_0 \wedge e), \quad (4.3.13)$$

where P_0, P_1 and P_2 are polynomials. Since, in view of (2.3.20),

$$\partial_z v_0 = v_1 + \frac{3\bar{z}}{1 + z\bar{z}}v_0,$$

from (4.3.13) we get

$$\hat{\sigma}_0 = P_0(v_0 \wedge v_1) + \left(P_1 + \frac{3\bar{z}}{(1 + z\bar{z})}P_2\right)v_0 \wedge e + P_2(v_1 \wedge e). \quad (4.3.14)$$

Taking the norm square,

$$\|\hat{\sigma}_0\|^2 = |P_0|^2 \|v_0\|^2 \|v_1\|^2 + \left|P_1 + \frac{3\bar{z}}{(1 + z\bar{z})}P_2\right|^2 \|v_0\|^2 + |P_2|^2 \|v_1\|^2. \quad (4.3.15)$$

For $v_0 = (1, \sqrt{3}z, \sqrt{3}z^2, z^3, 0)$, we have

$$\|v_0\|^2 = (1 + z\bar{z})^3;$$

and Theorem 2.3.13 gives

$$\frac{\|v_1\|^2}{\|v_0\|^2} = \gamma_0^3 = \frac{3}{(1 + z\bar{z})^2}.$$

Hence, from (4.3.15), and the fact that $\|\hat{\sigma}_0\|^2 = (1 + z\bar{z})^k$, we have

$$(1 + z\bar{z})^k = 3|P_0|^2(1 + z\bar{z})^4 + |P_1(1 + z\bar{z}) + 3\bar{z}P_2|^2(1 + z\bar{z}) + 3|P_2|^2(1 + z\bar{z}),$$

for some $k \in \{1, \dots, 9\}$, hence

$$(1 + z\bar{z})^{k-1} = 3|P_0|^2(1 + z\bar{z})^3 + |P_1(1 + z\bar{z}) + 3z\bar{z}P_2|^2 + 3|P_2|^2. \quad (4.3.16)$$

For $k = 1$, this equation simplifies to

$$1 = 3|P_0|^2(1 + z\bar{z})^3 + |P_1(1 + z\bar{z}) + 3z\bar{z}P_2|^2 + 3|P_2|^2.$$

Again, the three terms on the right-hand side must be bounded, and this forces a contradiction. So we have $k \geq 2$.

For $k \geq 2$, we see from (4.3.16) that $(1 + z\bar{z})$ must divide $|P_2|^2$. But P_2 is a polynomial in z , hence $|P_2|^2$ is not divisible by $(1 + z\bar{z})$, unless $P_2 = 0$. Substituting this in (4.3.16), we obtain

$$(1 + z\bar{z})^{k-3} = 3|P_0|^2(1 + z\bar{z}) + |P_1|^2. \quad (4.3.17)$$

From (4.3.17), we see that $k \geq 3$, because the right-hand side is polynomial. If $k > 3$, then $P_1 = 0$ with the similar arguments given above to show that $P_2 = 0$. Thus, $\hat{\sigma}_0 = P_0(v_0 \wedge v_1)$, hence $\psi_0 = f_0 \oplus f_1 = V_0^3 \oplus V_1^3$. This implies that the first \mathcal{D}' -Gauss bundle of ψ_0 has rank 1, which is a contradiction with the hypothesis of our theorem. For $k = 3$, equation (4.3.17) yields

$$1 = 3|P_0|^2(1 + z\bar{z}) + |P_1|^2.$$

Since both adding terms on the right-hand side must be bounded, we see that $P_0 = 0$ and P_1 is a constant polynomial. Replacing $P_0 = P_2 = 0$ in (4.3.13), we get $\hat{\sigma}_0 = P_1(v_0 \wedge e)$, hence $\psi_0 = V_0^3 \oplus \text{span } e$. In this case, we would have $\psi_1 = V_1^3$, which has rank 1, leading to a contradiction with our assumptions.

Case $p = 2$. Let us consider the case $f_3 = V_1^2$. The diagram (4.3.2) will then take the following form

$$\begin{array}{ccccc} f_1 & \longrightarrow & V_1^2 & \longrightarrow & V_2^2 \\ \uparrow & & \searrow & & \uparrow \\ f_0 & \longrightarrow & f_2 & & \end{array} . \quad (4.3.18)$$

We take the orthogonal decomposition $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2$, with

$$\mathbb{C}^3 = V_0^2 \oplus V_1^2 \oplus V_2^2.$$

Since there is no arrow from f_0 to V_1^2 , we see that f_0 has zero component along V_0^2 , hence f_0 is a holomorphic line bundle in \mathbb{C}^2 . Let $g_0 : \mathbb{C} \rightarrow \mathbb{C}^2$ be a local holomorphic section of f_0 , and consider the section $v_0 = (1, \sqrt{2}z, z^2, 0, 0)$ of V_0^2 . Observe that g_0 is not constant, otherwise the first ∂' -Gauss bundle of ψ_0 would have rank 1; then

$$\mathbb{C}^2 = \underline{\text{span}}\{g_0, \partial_z g_0\}.$$

Use the Plücker embedding of $G_2(\mathbb{C}^5) \hookrightarrow \mathbb{C}P^9$ to obtain a holomorphic map $\sigma_0 : S^2 \rightarrow \mathbb{C}P^9$. Let $\hat{\sigma}_0 : \mathbb{C} \rightarrow \mathbb{C}^{10}$ be a holomorphic local section of σ_0 . Denote by e a constant unit vector spanning the image of \mathbb{C}^2 under the Plücker embedding. Without loss of generality, we assume that $\hat{\sigma}_0$ is nonvanishing and that its components are holomorphic polynomials.

We have

$$\hat{\sigma}_0 = P_0(v_0 \wedge g_0) + P_1 e, \quad (4.3.19)$$

for some polynomials P_0 and P_1 . Since $v_0 = (1, \sqrt{2}z, z^2, 0, 0)$, then $\|v_0\|^2 = (1 + z\bar{z})^2$. Taking the norm square in the above equation, we obtain

$$\|\hat{\sigma}_0\|^2 = |P_0|^2 \|v_0\|^2 \|g_0\|^2 + |P_1|^2 = (1 + z\bar{z})^2 |P_0|^2 \|g_0\|^2 + |P_1|^2. \quad (4.3.20)$$

On the other hand, since ψ_0 has constant curvature, then $\|\hat{\sigma}_0\|^2 = (1 + z\bar{z})^k$ for some $k \in \{1, \dots, 9\}$, hence

$$(1 + z\bar{z})^k = (1 + z\bar{z})^2 |P_0|^2 \|g_0\|^2 + |P_1|^2. \quad (4.3.21)$$

Hence $|P_1|^2$ must be divisible by $(1 + z\bar{z})$, which implies $P_1 = 0$. Replacing this in (4.3.19) gives $\hat{\sigma}_0 = P_0(v_0 \wedge g_0)$, therefore $\psi_0 = f_0 \oplus f_1 = f_0 \oplus V_0^2$, implying $f_1 = V_0^2$, and this gives the following refinement of the diagram (4.3.18)

$$\begin{array}{ccc} V_0^2 & \longrightarrow & V_1^2 & \longrightarrow & V_2^2 \\ & & & & \\ f_0 & \longrightarrow & f_2 & & \end{array} \quad (4.3.22)$$

Replacing $P_1 = 0$ in (4.3.21) we see that

$$\|g_0\|^2 = \frac{(1 + z\bar{z})^{k-2}}{|P_0|^2}.$$

Hence

$$\partial_z \partial_{\bar{z}} \log \|g_0\|^2 = \frac{k-2}{(1+z\bar{z})^2},$$

which implies that f_0 has constant curvature, hence $f_0 = V_0^1$, up to unitary congruence. We conclude that

$$\Psi = (V_0^2 \oplus V_0^1, V_1^2 \oplus V_1^1, V_2^2).$$

The induced metric from the flag $F_{2,2,1}$ and the corresponding curvature are then given as follows:

$$\Psi^* ds_{F_{2,2,1}}^2 = \frac{3\lambda_0 + 2\lambda_1}{(1+z\bar{z})^2}, \quad K(\Psi) = \frac{4}{3\lambda_0 + 2\lambda_1}.$$

□

Remark 4.3.2. Let $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^6)$ be a linearly full holomorphic map whose harmonic sequence $\psi_0, \psi_1, \psi_2, \psi_3$ gives rise to a primitive immersion

$$\Psi = (\psi_0, \psi_1, \psi_2, \psi_3) : S^2 \rightarrow F_{2,2,1,1}$$

of constant curvature with respect to at least one $U(6)$ -invariant metric on $F_{2,2,1,1}$. Then the proof of Theorem 4.3.1 can be straightforwardly adapted to conclude that Ψ is unitarily congruent to the following curve

$$(V_0^3 \oplus V_0^1, V_1^3 \oplus V_1^1, V_2^3, V_3^3).$$

However an open question remains corresponding the flag $F_{2,2,2}$.

In the following remark, we will compare our result with [33].

Remark 4.3.3. In [33, Theorem 4.2] and [34], the authors classified all nonsingular holomorphic curves from S^2 into $G_2(\mathbb{C}^5)$ with constant curvature. They showed that the possible values for the constant curvature are $4, 2, \frac{4}{3}, 1, \frac{4}{5}$. Next we compare their results with our Theorem 4.3.1. It is easy to check that the curves in [33, Theorem 4.2] with constant curvature $K = 4$ and $K = 2$ are not linearly full. Moreover, the curves with constant curvature $K = \frac{4}{5}$ and $K = 1$ have the first ∂' -Gauss bundle with nonconstant curvature; consequently, by Theorem 3.1.1, their primitive lifts are not of constant curvature. We study with more detail the remaining cases. According to [33, Theorem 4.2], any nonsingular holomorphic immersion $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^5)$ of constant curvature $K = \frac{4}{3}$ is unitarily congruent to one of the following minimal immersions (locally spanned by f_0 and g_0 , in terms of the canonical chart (U_0, z)):

1. $f_0(z) = (1, 0, a\sqrt{3-a^2}z, a^{-1}(a^2-1)^{3/2}z, az^2)$ and $g_0(z) = (0, 1, 0, a^{-1}z, 0)$, where

$1 \leq a \leq \sqrt{3}$. Straightforward computations show the following:

$$\|g_0 \wedge f_0\|^2 = (1 + z\bar{z})^3,$$

which confirms that ψ_0 has constant curvature $K = \frac{4}{3}$; moreover

$$\|\partial_z g_0 \wedge \partial_z f_0 \wedge g_0 \wedge f_0\|^2 = 3 + 4z\bar{z} - a^2(1 + (-3 + a^2)(z\bar{z})^2),$$

which shows that the first ∂' -Gauss bundle ψ_1 has rank 2, and

$$\begin{aligned} \gamma_1 &= \partial_z \partial_{\bar{z}} \log \|\partial_z g_0 \wedge \partial_z f_0 \wedge g_0 \wedge f_0\|^2 \\ &= \frac{4(-3 + a^2)(-1 + a^2 z\bar{z}(-3 + a^2 - z\bar{z}))}{(-3 + a^2 - 4z\bar{z} + a^2(-3 + a^2)(z\bar{z})^2)}. \end{aligned}$$

Hence $\psi_1 : S^2 \rightarrow G_2(\mathbb{C}^5)$ is not of constant curvature, but for $a = 1$, $f_0(z) = (1, 0, \sqrt{2}z, 0, z^2)$ and $g_0(z) = (0, 1, 0, z, 0)$, hence $\psi_0 = V_0^2 \oplus V_0^1$, which agrees with (4.3.1).

2. $f_0(z) = (1, 0, az, 0, \sqrt{a^4 - 3a^2 + 3}z^2)$, and

$$g_0(z) = \left(0, 1, 0, \frac{1}{\sqrt{a^4 - 3a^2 + 3}}z, \sqrt{\frac{(2 - a^2)^3}{a^4 - 3a^2 + 3}}z\right),$$

where $0 \leq a \leq \sqrt{2}$. Similarly to the previous case, the ∂' -Gauss bundle $\psi_1 : S^2 \rightarrow G_2(\mathbb{C}^5)$ has constant curvature if and only if $a = \sqrt{2}$. This case corresponds again to $\psi_0 = V_0^2 \oplus V_0^1$.

In the following result, we classify primitive lifts of constant curvature into the flag $F_{2,2,1}$ which are associated to harmonic maps of finite isotropy order from S^2 into the complex Grassmannian $G_2(\mathbb{C}^5)$. The technique of generalized absolute value type function developed in Section 2.1.1 will play an important role in the proof.

Theorem 4.3.4. *Let $\Psi = (\psi_0, \psi_1, \psi_2) : S^2 \rightarrow F_{2,2,1}$ be a primitive immersion, where $\psi_0 : S^2 \rightarrow G_2(\mathbb{C}^5)$ is a harmonic map of isotropy order 2; ψ_1 and ψ_2 are the first and second ∂' -Gauss bundles of ψ_0 . If there exists at least one $U(5)$ -invariant metric on $F_{2,2,1}$ corresponding to which Ψ has constant curvature, then Ψ is unitarily equivalent to the curve*

$$(V_0^4 \oplus V_3^4, V_1^4 \oplus V_4^4, V_2^4); \quad (4.3.23)$$

consequently, Ψ has constant curvature with respect to all $U(5)$ -invariant metrics on $F_{2,2,1}$. For a $U(5)$ -invariant metric on $F_{2,2,1}$ with parameters $\lambda_0, \lambda_1, \lambda_2$, using the notations of (2.4.3), the metric induced by Ψ has constant curvature $K(\Psi) = \frac{2}{2\lambda_0 + 3\lambda_1 + 3\lambda_2}$.

Proof. We have the following fundamental harmonic diagram

$$\psi_0 \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longrightarrow \end{array} \psi_1 \longrightarrow \psi_2. \quad (4.3.24)$$

Both ψ_0 and ψ_1 are of rank 2, while ψ_2 is of rank 1.

The induced metric is given by

$$\Psi^* ds_{F_{2,2,1}}^2 = \eta^2 dz d\bar{z}, \quad \text{where } \eta^2 = \lambda_0 \gamma_0 + \lambda_1 \gamma_1 + \lambda_2 \gamma_2. \quad (4.3.25)$$

Define $\phi = \eta dz$, so that

$$\Psi^* ds_{F_{2,2,1}}^2 = \phi \bar{\phi}. \quad (4.3.26)$$

Set

$$\alpha_1 = \underline{\ker} A'_{\psi_1}, \quad \beta_1 = \alpha_1^\perp \cap \psi_1. \quad (4.3.27)$$

Note that α_1 is a holomorphic vector subbundle of ψ_1 and β_1 is an antiholomorphic vector subbundle of ψ_1 . Set $\beta_2 = \psi_2$. Since A''_{ψ_1} is an antiholomorphic isomorphism between ψ_1 and ψ_0 , we can define

$$\beta_0 = \underline{\text{Im}} A''_{\psi_1}|_{\beta_1}, \quad \alpha_0 = \beta_0^\perp \cap \psi_0.$$

Again, α_0 is a holomorphic vector subbundle of ψ_0 and β_0 is an antiholomorphic vector subbundle of ψ_0 . Hence, we have the following refinement of the diagram (4.3):

$$\begin{array}{ccccc} \beta_0 & \longrightarrow & \beta_1 & \longrightarrow & \beta_2 \\ \uparrow & \searrow & \uparrow & & \uparrow \\ \alpha_0 & \xrightarrow{\quad} & \alpha_1 & & \end{array}. \quad (4.3.28)$$

We choose a unitary frame $\{e_1, \dots, e_5\}$, where e_1, e_2, e_3, e_4, e_5 are local sections of $\alpha_1, \beta_1, \beta_2, \alpha_0, \beta_0$, respectively. In view of (4.3.28), we deduce that the Maurer-Cartan $\mathfrak{u}(5)$ -valued one-form $W = (\omega_i^j)$ corresponding to the above unitary frame is given as

follows

$$\begin{bmatrix} de_1 \\ de_2 \\ de_3 \\ de_4 \\ de_5 \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{-1}\nu_1 & a_1^2\phi & 0 & b_1^4\bar{\phi} & b_1^5\bar{\phi} \\ -a_1^2\bar{\phi} & \sqrt{-1}\nu_2 & a_2^3\phi & 0 & b_2^5\bar{\phi} \\ 0 & -a_2^3\bar{\phi} & \sqrt{-1}\nu_3 & a_3^4\phi & 0 \\ -b_1^4\phi & 0 & -a_3^4\bar{\phi} & \sqrt{-1}\nu_4 & a_4^5\phi \\ -b_1^5\bar{\phi} & -b_2^5\bar{\phi} & 0 & -a_4^5\bar{\phi} & \sqrt{-1}\nu_5 \end{bmatrix}}_W \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}. \quad (4.3.29)$$

Here ν_i are real valued 1-forms; a_i^j and b_i^j are complex valued functions. Under our assumptions (ψ_1 and ψ_2 are the first and second ∂' -Gauss bundles of ψ_0 , with ranks 2 and 1, respectively), all horizontal arrows in the diagram (4.3.28) are nonzero. The backward arrow from β_2 to α_0 is also nonzero, because ψ_0 has finite isotropy order. Hence the complex functions b_1^4 , a_2^3 , a_3^4 and b_2^5 are not zero except at isolated points. Since $\Psi^* ds_{F_{2,2,1}}^2 = \phi\bar{\phi}$, we have the following normalization

$$\lambda_1 |a_2^3|^2 + \lambda_0 |b_1^4|^2 + \lambda_0 |b_1^5|^2 + \lambda_0 |b_2^5|^2 + \lambda_2 |a_3^4|^2 = 1. \quad (4.3.30)$$

By Proposition 2.3.10, β_1 is harmonic. We claim that β_1 is a linearly full harmonic map in $\mathbb{C}P^4$. In fact, suppose that β_1 takes values in $\mathbb{C}P^3$. In this case, we have $\mathbb{C}^4 = \beta_1 \oplus \beta_2 \oplus \alpha_0 \oplus C$ for some line bundle C , hence one of the following harmonic diagrams hold:

$$0 \longrightarrow \beta_1 \longrightarrow \beta_2 \longrightarrow \alpha_0 \longrightarrow C \longrightarrow 0, \quad 0 \longrightarrow C \longrightarrow \beta_1 \longrightarrow \beta_2 \longrightarrow \alpha_0 \longrightarrow 0.$$

Due to dimensions constraints, either β_1 is holomorphic or α_0 is anti-holomorphic. Both of cases lead to a contradiction, because all the horizontal arrows in diagram (4.3.28) are nonzero, thus establishing our claim.

Now, since β_1 is a linearly full harmonic map in $\mathbb{C}P^4$, it belongs to the harmonic sequence g_0, g_1, g_2, g_3, g_4 associated to some full holomorphic map $g_0 : S^2 \rightarrow \mathbb{C}P^4$. Clearly, we must have $g_1 = \beta_1$, $g_2 = \beta_2$, and $g_3 = \alpha_0$. Choose a local unitary frame E_1, E_2, E_3, E_4, E_5 such that E_j spans g_{j-1} , with $j \in \{1, 2, 3, 4, 5\}$, and

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix} = \underbrace{\begin{bmatrix} u & 0 & 0 & 0 & v \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\bar{v} & 0 & 0 & 0 & \bar{u} \end{bmatrix}}_A \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix} \quad (4.3.31)$$

where u and v are complex functions satisfying $|u|^2 + |v|^2 = 1$.

We claim that $u = 0$ and will prove it later. Assume now $u = 0$. Then we have $e_1 = E_5$, $e_5 = E_1$, and $e_i = E_j$ for $i, j = 2, 3, 4$. Hence, diagram (4.3.28) simplifies to

$$\begin{array}{ccccc}
 g_0 & \longrightarrow & g_1 & \longrightarrow & g_2 \\
 & & & & \searrow \\
 g_3 & \longleftarrow & g_4 & &
 \end{array}
 \tag{4.3.32}$$

with $\psi_0 = g_0 \oplus g_3$, $\psi_1 = g_1 \oplus g_4$, and $\psi_2 = g_2$. Consider the primitive map

$$\tilde{\Psi} = (g_0, g_1, g_2, g_3, g_4) : S^2 \rightarrow F_{1,1,1,1,1}.$$

The metric on S^2 induced by Ψ from the U(5)-invariant metric on $F_{2,2,1}$ with parameters $\lambda_0, \lambda_1, \lambda_2$, using the notations of (2.4.3),

$$\Psi^* ds_{F_{2,2,1}}^2 = (\lambda_0 \gamma_0 + \lambda_1 \gamma_1 + \lambda_2 \gamma_2) dz d\bar{z},$$

coincides with the metric on S^2 induced by $\tilde{\Psi}$ from the U(5)-invariant metric on $F_{1,1,1,1,1}$ with parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, regardless the value of $\lambda_3 > 0$. Hence, $\tilde{\Psi}$ has constant curvature, which means, by Corollary 3.1.2, that $\tilde{\Psi}$ is the 4-Veronese primitive map, up to holomorphic isometry, inducing the metric

$$\tilde{\Psi}^* ds_{F_{1,1,1,1,1}}^2 = (\lambda_0 \gamma_0^4 + \lambda_1 \gamma_1^4 + \lambda_2 \gamma_2^4 + \lambda_3 \gamma_3^4) dz d\bar{z},$$

where the γ_j^n , with $j \in \{0, 1, 2, 3\}$, are defined by (2.3.19), which implies that

$$K(\Psi) = K(\tilde{\Psi}) = \frac{2}{4\lambda_0 + 3\lambda_1 + 3\lambda_2}.$$

Now the proof of claim that $u = 0$. Since g_0, g_1, g_2, g_3, g_4 is a harmonic sequence in $\mathbb{C}P^4$, the Maurer-Cartan $\mathfrak{u}(5)$ -valued one-form $\tilde{W} = (\tilde{\omega}_i^j)$ associated to the unitary moving frame E_1, E_2, E_3, E_4, E_5 assumes the standard form

$$\begin{array}{l}
 \begin{bmatrix} dE_1 \\ dE_2 \\ dE_3 \\ dE_4 \\ dE_5 \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{-1}\mu_1 & c_1^2\phi & 0 & 0 & 0 \\ -\overline{c_1^2\phi} & \sqrt{-1}\mu_2 & c_2^3\phi & 0 & 0 \\ 0 & -\overline{c_2^3\phi} & \sqrt{-1}\mu_3 & c_3^4\phi & 0 \\ 0 & 0 & -\overline{c_3^4\phi} & \sqrt{-1}\mu_4 & c_4^5\phi \\ 0 & 0 & 0 & -\overline{c_4^5\phi} & \sqrt{-1}\mu_5 \end{bmatrix}}_{\tilde{W}} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix}
 \end{array}
 \tag{4.3.33}$$

where μ_i are real-valued 1 forms, and c_i^j are complex functions satisfying $c_i^j = -\overline{c_j^i}$.

Denote the column matrix $\{e_1, e_2, e_3, e_4, e_5\}$ by e and $\{E_1, E_2, E_3, E_4, E_5\}$ by E . We have

$$e = AE,$$

where A is the $U(5)$ -valued function defined by (4.3.31). Differentiating and applying the definitions of W and \tilde{W} , we obtain

$$dA = WA - A\tilde{W}. \quad (4.3.34)$$

By comparing the first entry of the first row of equation (4.3.34), we obtain

$$du = u\sqrt{-1}(\nu_1 - \mu_1) - b_1^5\bar{\phi}.$$

After taking conjugate we have

$$d\bar{u} = \bar{u}\sqrt{-1}(\mu_1 - \nu_1) \quad \text{mod } dz$$

Similarly by comparing the last entry of the first row and after taking the conjugate, we get

$$d\bar{v} = \bar{v}\sqrt{-1}(\mu_5 - \nu_1) \quad \text{mod } dz.$$

Therefore, from Lemma 2.1.4, $|u|$ and $|v|$ are of absolute type functions. Together with (4.3.29), (4.3.31) and (4.3.33), equation (4.3.34) also gives:

$$\begin{aligned} a_1^2 &= uc_1^2, & b_1^4 &= -vc_4^5, & a_2^3 &= c_2^3 \\ b_2^5 &= vc_1^2, & a_3^4 &= c_3^4, & a_4^5 &= uc_4^5. \end{aligned} \quad (4.3.35)$$

Since $|u|$ and $|v|$ are absolute value type functions, then by Lemma 2.1.4, together with the Maurer-Cartan structure equations for W and \tilde{W} ,

$$\begin{aligned} \frac{1}{4}\Delta_{S^2} \log |u|^2 \phi \wedge \bar{\phi} &= \sqrt{-1}d(\mu_1 - \nu_1) = d\tilde{w}_1^1 - d\omega_1^1 \\ &= \tilde{w}_1^2 \wedge \tilde{w}_2^1 - (\omega_1^2 \wedge \omega_2^1 + \omega_1^4 \wedge \omega_4^1 + \omega_1^5 \wedge \omega_5^1) \\ &= -|c_1^2|^2 \phi \wedge \bar{\phi} + |a_1^2|^2 \phi \wedge \bar{\phi} + |b_1^4|^2 \bar{\phi} \wedge \phi + |b_1^5|^2 \bar{\phi} \wedge \phi \\ &= \left(-|c_1^2|^2 + |a_1^2|^2 - |b_1^4|^2 - |b_1^5|^2\right) \phi \wedge \bar{\phi}. \end{aligned} \quad (4.3.36)$$

By using (4.3.30), we have

$$-|b_1^4|^2 - |b_1^5|^2 = -\frac{1}{\lambda_0} (1 - \lambda_1|a_2^3|^2 - \lambda_0|b_2^5|^2 - \lambda_2|a_3^4|^2). \quad (4.3.37)$$

Plugging this into (4.3.36), using (4.3.35) and the normalization $|u|^2 + |v|^2 = 1$, we

obtain

$$\frac{1}{4}\Delta_{S^2}\log|u|^2 = \frac{1}{\lambda_0}\left(-1 + \lambda_1|c_2^3|^2 + \lambda_2|c_3^4|^2\right). \quad (4.3.38)$$

Similarly, we compute

$$\begin{aligned} \frac{1}{4}\Delta_{S^2}\log|v|^2\phi\wedge\bar{\phi} &= \sqrt{-1}d(\mu_5 - \nu_1) = d\tilde{w}_5^5 - d\omega_1^1 \\ &= \tilde{w}_5^4\wedge\tilde{w}_4^5 - (\omega_1^2\wedge\omega_2^1 + \omega_1^4\wedge\omega_4^1 + \omega_1^5\wedge\omega_5^1) \\ &= -|c_4^5|^2\bar{\phi}\wedge\phi + |a_1^2|^2\phi\wedge\bar{\phi} + |b_1^4|^2\bar{\phi}\wedge\phi + |b_1^5|^2\bar{\phi}\wedge\phi \\ &= \left(|c_4^5|^2 + |a_1^2|^2 - |b_1^4|^2 - |b_1^5|^2\right)\phi\wedge\bar{\phi}. \end{aligned} \quad (4.3.39)$$

Plugging (4.3.37) into (4.3.39), using (4.3.35) and the normalization $|u|^2 + |v|^2 = 1$, we obtain

$$\frac{1}{4}\Delta_{S^2}\log|v|^2 = |c_1^2|^2 + |c_4^5|^2 + \frac{1}{\lambda_0}\left(-1 + \lambda_1|c_2^3|^2 + \lambda_2|c_3^4|^2\right). \quad (4.3.40)$$

Now, let $\tilde{\sigma}_i$ be a nonzero holomorphic local section of the i -th osculating curve of $g_0 : S^2 \rightarrow \mathbb{C}P^4$, and set $\xi_i = \|\tilde{\sigma}_i\|^2$. In view of (2.3.13), we have, for $1 \leq i \leq 4$,

$$|c_i^{i+1}|^2\phi\bar{\phi} = \partial_z\partial_{\bar{z}}\log\xi_{i-1}dzd\bar{z},$$

Since $\Delta_{S^2} = \frac{4}{\eta^2}\partial_z\partial_{\bar{z}}$, where η^2 is the conformal factor defined by (4.3.25), we get

$$|c_i^{i+1}|^2 = \frac{1}{4}\Delta_{S^2}\log\xi_{i-1}. \quad (4.3.41)$$

It follows from (4.3.38), (4.3.40) and (4.3.41) that

$$\begin{aligned} \frac{1}{4}\Delta_{S^2}\log\frac{|u|^2}{\xi_1^{\lambda_1/\lambda_0}\xi_2^{\lambda_2/\lambda_0}} &= -\frac{1}{\lambda_0} \\ \frac{1}{4}\Delta_{S^2}\log\frac{|v|^2}{\xi_0\xi_3\xi_1^{\lambda_1/\lambda_0}\xi_2^{\lambda_2/\lambda_0}} &= -\frac{1}{\lambda_0}. \end{aligned} \quad (4.3.42)$$

As quotients of generalized absolute value type functions in S^2 , we know that

$$\frac{|u|^2}{\xi_1^{\lambda_1/\lambda_0}\xi_2^{\lambda_2/\lambda_0}}$$

and

$$\frac{|v|^2}{\xi_0\xi_3\xi_1^{\lambda_1/\lambda_0}\xi_2^{\lambda_2/\lambda_0}}$$

are generalized absolute value type functions on S^2 . Then it follows from Lemma 2.1.6 together with (4.3.42) that

$$\begin{aligned} \text{order} \left(\frac{|u|^2}{\xi_1^{\lambda_1/\lambda_0} \xi_2^{\lambda_2/\lambda_0}} \right) &= \frac{4}{K\lambda_0} \\ \text{order} \left(\frac{|v|^2}{\xi_0 \xi_3 \xi_1^{\lambda_1/\lambda_0} \xi_2^{\lambda_2/\lambda_0}} \right) &= \frac{4}{K\lambda_0}. \end{aligned}$$

where K is the constant curvature of Ψ ; moreover,

$$\begin{aligned} |u|^2 &= \frac{|g(z)|^2}{(1+z\bar{z})^{\frac{4}{K\lambda_0}}} \xi_1^{\lambda_1/\lambda_0} \xi_2^{\lambda_2/\lambda_0} \\ |v|^2 &= \frac{|h(z)|^2}{(1+z\bar{z})^{\frac{4}{K\lambda_0}}} \xi_0 \xi_3 \xi_1^{\lambda_1/\lambda_0} \xi_2^{\lambda_2/\lambda_0} \end{aligned} \quad (4.3.43)$$

where

$$\begin{aligned} |g(z)|^2 &= c^2 \frac{|z-z_1|^{2p_1} \dots |z-z_m|^{2p_m}}{|z-z_{m+1}|^{2q_1} \dots |z-z_{m+k}|^{2q_k}} \\ |h(z)|^2 &= d^2 \frac{|z-w_1|^{2r_1} \dots |z-w_n|^{2r_n}}{|z-w_{n+1}|^{2s_1} \dots |z-w_{n+j}|^{2s_j}} \end{aligned}$$

with $c, d \in \mathbb{C}$. Since $|u|^2 + |v|^2 = 1$, equation (4.3.43) gives

$$(1+z\bar{z})^{\frac{4}{K\lambda_0}} = \xi_1^{\lambda_1/\lambda_0} \xi_2^{\lambda_2/\lambda_0} (|g(z)|^2 + |h(z)|^2 \xi_0 \xi_3). \quad (4.3.44)$$

Recall that each function $z \mapsto \xi_j(z, \bar{z})$ is nonvanishing for $z \in \mathbb{C}$. By applying arguments similar to those used in the proof of Theorem 3.1.1, we can interpret equality (4.3.44) as an equality between analytic functions in two independent complex variables z, w to conclude that, if not empty, the zero set of the polynomial $\xi_i(z, w)$, with $i \in \{1, 2\}$, coincides in \mathbb{C}^2 with the zero set of the irreducible polynomial $1+zw$. Thus, for some nonnegative real numbers α_1, α_2 and positive real numbers c_1, c_2 , we have

$$\xi_1(z, \bar{z}) = c_1 (1+z\bar{z})^{\alpha_1}, \quad \xi_2(z, \bar{z}) = c_2 (1+z\bar{z})^{\alpha_2}.$$

This indicates that the harmonic map $g_2 : S^2 \rightarrow \mathbb{C}P^4$ is an immersion of constant curvature, hence, by [7, Theorem 5.4], the harmonic sequence g_0, g_1, g_2, g_3, g_4 is the 4-Veronese sequence, up to isometry. Consequently, all the $\xi_i(z, \bar{z})$, with $i = 0, 1, 2, 3$, are of the form $c_i(1+z\bar{z})^{\alpha_i}$. Thus, equation (4.3.44) becomes

$$(1+z\bar{z})^\alpha = C_1 \left(|g(z)|^2 + C_2 |h(z)|^2 (1+z\bar{z})^{\alpha'} \right), \quad (4.3.45)$$

where C_1, C_2 are positive real numbers, $\alpha = \frac{4}{K\lambda_0} - \frac{\lambda_1\alpha_1 + \lambda_2\alpha_2}{\lambda_0}$, and $\alpha' = \alpha_0 + \alpha_3$.

Now, suppose that $\alpha > 0$. Complexifying (see Lemma 2.2.1) both sides of (4.3.45), we obtain an equality between two analytic functions in the independent complex variables z, w . The left-hand side will vanish whenever $w = -\frac{1}{z}$; then, in view of the general form of $|g(z)|^2$, equality (4.3.45) can occur only if $g(z) = 0$. This implies that $u = 0$, and we are done.

Suppose that $\alpha \leq 0$. From (4.3.45) we obtain

$$1 = C_1 \left(|g(z)|^2 (1 + z\bar{z})^{-\alpha} + C_2 |h(z)|^2 (1 + z\bar{z})^{\alpha' - \alpha} \right).$$

This implies that $|h(z)|^2 (1 + z\bar{z})^{\alpha' - \alpha}$ is bounded, which occurs if, and only if, $h(z) = 0$ (hence $v = 0$), because $\alpha' - \alpha > 0$. Then diagram (4.3.28) becomes

$$\begin{array}{ccccc} & g_4 & & g_1 & \longrightarrow & g_2 \\ & \uparrow & & \uparrow & & \uparrow \\ g_3 & & & g_0 & & \searrow \end{array}$$

This leads to a contradiction with the assumption that ψ_0, ψ_1, ψ_2 is the harmonic sequence generated by ψ_0 ; in fact, if this last diagram were valid, then \mathcal{D}' -Gauss bundle of $\psi_0 = g_3 \oplus g_4$ would be zero. \square

In the following remarks, we discuss how other primitive lifts of constant curvature can be obtained from existing ones.

Remark 4.3.5. A careful inspection of the proofs of Theorem 4.3.1 and Theorem 4.3.4 shows that, besides (4.3.1) and (4.3.23), any other primitive immersion $\Psi : S^2 \rightarrow F_{2,2,1}$ of constant curvature with respect to at least one $U(5)$ -invariant metric on $F_{2,2,1}$ can be obtained by operating with homogeneous projections, adding constants, adding primitive maps, and shifts on primitive immersions of constant curvature into other flag manifolds ($F_{1,1} = \mathbb{C}P^1$, $F_{2,1}$, $F_{1,1,1}$, $F_{2,2} = G_2(\mathbb{C}^4)$, $F_{2,1,1}$, $F_{1,1,1,1}$, $F_{1,1,1,1,1}$) which are primitive lifts of linearly full harmonic maps. For example, (4.3.11) corresponds to the primitive immersion $\Psi_1 : S^2 \rightarrow F_{2,2,1}$ of constant curvature given by $\Psi_1 = (V_0^4 \oplus V_1^4, V_2^4 \oplus V_3^4, V_4^4)$. This is not a primitive lift of $\psi_0 = V_0^4 \oplus V_1^4$, in the sense that $V_2^4 \oplus V_3^4$ is not the first \mathcal{D}' -Gauss bundle of ψ_0 . However, we have $\Psi_1 = \Pi \circ \tilde{\Psi}_1$ with $\tilde{\Psi}_1 : S^2 \rightarrow F_{1,1,1,1,1}$ given by $\tilde{\Psi}_1 = (V_0^4, V_1^4, V_2^4, V_3^4, V_4^4)$. For another example, consider f_0 constant in (4.3.22). This case corresponds to the primitive immersion $\Psi_2 : S^2 \rightarrow F_{2,2,1}$ of constant curvature given by $\Psi_2 = (V_0^2 \oplus \mathbb{C}, V_1^2 \oplus \mathbb{C}, V_2^2)$, which is obtained from $\tilde{\Psi}_2 : S^2 \rightarrow F_{1,1,1}$ given by $\tilde{\Psi}_2 = (V_0^2, V_1^2, V_2^2)$ by adding constants.

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Chapter 5

Conclusions and topics for future research

In this final chapter, we summarize the main results of this thesis and suggest possible directions for future research.

5.1 Summary

An important class of twistor lifts for harmonic maps from Riemann surfaces into symmetric spaces G/H is that of *primitive maps* into generalized flag manifolds G/K , with $K \subset H$. Primitive maps are harmonic with respect to all G -invariant metrics on the generalized flag manifold. Following the work of many researchers on minimal immersions with constant curvature of surfaces into symmetric spaces, led by the pioneering work of E. Calabi, it is natural to investigate primitive immersions of constant curvature from Riemann surfaces into generalized flag manifolds, when these are equipped with G -invariant metrics. In this thesis, we discussed this question in detail. A key observation is that generalized flag manifolds admit infinite invariant metrics, whereas symmetric spaces have a unique invariant metric (up to scaling). This leads to a question: if a primitive immersion from a Riemann surface into a generalized flag manifold has constant curvature with respect to at least one invariant metric, does it necessarily have constant curvature with respect to all invariant metrics?

In Chapter 3, we addressed this problem in the special of primitive immersions into the flag manifold

$$F_{k_0, \dots, k_p} := \frac{U(n)}{U(k_0) \times \dots \times U(k_p)}$$

which are associated with pseudoholomorphic maps from the two-sphere S^2 into the complex Grassmannian $G_{k_0}(\mathbb{C}^n)$. In Theorem 3.1.1, we proved that *if any such primitive immersion lift $\Psi = (\psi_0, \dots, \psi_p)$ from S^2 has constant curvature with respect to at least one invariant metric, then it has constant curvature with respect to all invariant metrics; moreover, each $\psi_j : S^2 \rightarrow G_{k_j}(\mathbb{C}^n)$ is a minimal immersion of constant curvature with constant Kähler angle.* Since all harmonic maps from S^2 into $\mathbb{C}P^{n-1}$ are pseudoholomorphic, we concluded (see Corollary 3.1.2) that *any full primitive immersion from S^2 into the full flag manifold which has constant curvature with respect to at least one invariant metric is unitarily equivalent to the primitive lift of a Veronese map.* Additionally, we extended this corollary mentioned above to the case where the

domain is a general simply connected Riemann surface. However, in this case, we were only able to prove the result for some specific metrics, as described in (3.3.1). On the way, we considered the problem of finding the invariant metric on the flag manifold, under a certain normalization condition, that maximizes the induced area of S^2 by a given primitive immersion. The results of this section were already submitted to a peer-reviewed international journal (to appear in "Results in Mathematics").

In Chapter 4, we classified primitive immersions of constant curvature from S^2 into low dimensional flag manifolds, specifically $F_{2,1,1}$ and $F_{2,2,1}$. We started by describing some constructions of new primitive immersions of constant curvature from existing ones. These constructions enabled us to reduce our problem to the classification of primitive immersions of constant curvature which are lifts of full harmonic maps from S^2 into the Grassmannians $G_2(\mathbb{C}^4)$ and $G_2(\mathbb{C}^5)$, in the sense of Remark 2.4.2. We proved that any such primitive immersion of constant curvature with respect to at least one invariant metric from S^2 into $F_{2,1,1}$ is unitarily congruent to one of three mutually noncongruent curves (see Theorem 4.2.1), while any such primitive immersion of constant curvature from S^2 into $F_{2,2,1}$ is unitarily congruent to one of two mutually noncongruent curves (see Theorem 4.3.1 and Theorem 4.3.4). The results of this section were recently published [43].

5.2 Possible future research directions

There are several potential research directions that can be explored in relation to this thesis. Next we discuss in detail two of these.

5.2.1 Primitive immersions of constant curvature of S^2 into $F_{2,2,2}^{\mathbb{H}}$

Denote by \mathbb{H} the division ring of quaternions. Consider the identification of \mathbb{C}^2 with \mathbb{H} given by

$$(a, b) \in \mathbb{C}^2 \cong a + bj \in \mathbb{H},$$

where j is a unit quaternion ($j^2 = -1$, with $ij = -ji$). This extends to an identification of \mathbb{C}^{2n} with \mathbb{H}^n . Let $J : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ be the conjugate linear map representing the left multiplication by j :

$$J(z_1, z_2, \dots, z_{2n-1}, z_{2n}) = (-\bar{z}_2, \bar{z}_1, \dots, -\bar{z}_{2n}, \bar{z}_{2n-1}).$$

Let $\mathbb{H}P^{n-1}$ be the quaternionic projective space. This is a symmetric $\mathrm{Sp}(n)$ -space

which can be totally geodesically embedded in $G_2(\mathbb{C}^{2n})$:

$$\mathbb{H}P^{n-1} \cong \{V \in G_2(\mathbb{C}^{2n}) : JV = V\}.$$

Let M be a Riemann surface and $\psi_0 : M \rightarrow \mathbb{H}P^{n-1}$ be a harmonic map, with harmonic sequence $\{\psi_k\}_{k \in \mathbb{Z}}$. Due to the J -structure, we have (see [5] for details):

$$\psi_k = J\psi_{-k}, \quad \text{for all } k \in \mathbb{Z}. \quad (5.2.1)$$

Consider now the case $n = 3$. Minimal immersions of constant curvature from S^2 into $\mathbb{H}P^2$ have been classified in [26]. One possible direction for future research is the investigation of primitive immersions of constant curvature from S^2 into the twistor spaces of $\mathbb{H}P^2$. The challenging case occurs when a given harmonic map $\psi_0 : S^2 \rightarrow \mathbb{H}P^2 \subset G_2(\mathbb{C}^6)$ has finite isotropy order r and it is nondegenerate. This case leads to the following fundamental harmonic diagram (by [5, Proposition 3.2], we have $r = 2$):

$$\begin{array}{c} \curvearrowleft \\ J\psi_1 = \psi_{-1} \longrightarrow \psi_0 \longrightarrow \psi_1 \\ \curvearrowright \end{array}, \quad (5.2.2)$$

with $\text{rank } \psi_1 = \text{rank } \psi_{-1} = 2$. Throughout the remaining of this section we will assume that ψ_0 has finite isotropy order and it is nondegenerate. Set

$$F_{2,2,2}^{\mathbb{H}} = \{(V_{-1}, V_0, V_1) \in F_{2,2,2} : V_{-1} = JV_1, V_0 = JV_0\};$$

as a homogeneous space, we have

$$F_{2,2,2}^{\mathbb{H}} \cong \frac{\text{Sp}(3)}{\text{Sp}(1) \times \text{U}(2)}.$$

Associated to ψ_0 we have the primitive lift $\Psi = (\psi_{-1}, \psi_0, \psi_1) : S^2 \rightarrow F_{2,2,2}^{\mathbb{H}}$.

We have

$$T_{\Psi}^{\mathbb{C}} F_{2,2,2}^{\mathbb{H}} \subset \sum_{k \neq s} \text{Hom}(\psi_k, \psi_s).$$

Any $\text{Sp}(3)$ -invariant metric g on $F_{2,2,2}^{\mathbb{H}}$ takes the following form (see [2]): given

$$\xi = \sum_{k \neq s} \xi_{ks}, \eta = \sum_{k \neq s} \eta_{ks} \in T_{\Psi} F_{2,2,2}^{\mathbb{H}},$$

Then

$$\underbrace{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}}_e = \underbrace{\begin{bmatrix} q_{11} & q_{12} & 0 & 0 & q_{15} & q_{16} \\ q_{21} & q_{22} & 0 & 0 & q_{25} & q_{26} \\ q_{31} & q_{32} & 0 & 0 & q_{35} & q_{36} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ q_{61} & q_{62} & 0 & 0 & q_{65} & q_{66} \end{bmatrix}}_A \underbrace{\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix}}_E, \quad (5.2.5)$$

where q_{ij} are complex functions. Since $A \in \text{Sp}(3)$, the entries satisfy the following relations:

$$\begin{aligned} q_{11} &= \overline{q_{26}}, & q_{12} &= \overline{q_{25}}, & q_{15} &= -\overline{q_{22}}, & q_{16} &= -\overline{q_{21}} \\ q_{31} &= \overline{q_{66}}, & q_{32} &= \overline{q_{65}}, & q_{35} &= -\overline{q_{62}}, & q_{36} &= -\overline{q_{61}}. \end{aligned} \quad (5.2.6)$$

Using these relations, and the Maurer-cartan structure equations for both frames, the matrix A can be further simplified to:

$$A = \begin{bmatrix} q_{11} & 0 & 0 & 0 & q_{15} & q_{16} \\ q_{21} & q_{22} & 0 & 0 & 0 & q_{26} \\ q_{31} & q_{32} & 0 & 0 & 0 & q_{36} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ q_{61} & 0 & 0 & 0 & q_{65} & q_{66} \end{bmatrix}. \quad (5.2.7)$$

Differentiating both sides of $e = AE$, we obtain:

$$dA = WA - A\overline{W},$$

where W and \overline{W} are the Maurer-Cartan matrix for e and E respectively. It would be both interesting and challenging (currently, this is a work in progress) to show from this that all the elements of one row or column of the matrix A satisfy equation (2.1.2). Then one could eventually simplify the matrix A by using the technique of generalized absolute value type functions, as we did in the proof of Theorem 4.3.4. We conjecture that, after all the simplifications, one might conclude that Ψ is unitarily congruent to

$$(V_0^5 \oplus V_3^5, V_1^5 \oplus V_4^5, V_2^5 \oplus V_5^5).$$

5.2.2 Minimal immersions of constant curvature of S^2 into $U(n)$

Another particularly interesting direction for future research is the investigation of minimal immersions of constant curvature from the Riemann sphere into the unitary group $U(n)$. The first nontrivial case occurs for $n = 3$. To the best of our knowledge, this is an open problem. To approach this problem, the strategy we propose requires an understanding of some fundamental concepts from the loop group theory for harmonic maps (see [1, 24, 46, 48, 50] for details).

Suppose $\psi : S^2 \rightarrow U(n)$ is a harmonic map. Then, associated to this map there is an *extended solution* $\Psi : S^2 \rightarrow \Omega U(n)$, where $\Omega U(n)$ denotes the *based loop group* of $U(n)$:

$$\Omega U(n) = \{\gamma : S^1 \rightarrow U(n) : \gamma \text{ is smooth and } \gamma(1) = \text{Id}\}.$$

The harmonic map is recovered from its extended solution by the formula

$$\psi(z) = \Psi(z)(-1).$$

Consider the *Grassmannian model* $\text{Gr}^{(n)}$ of $\Omega U(n)$. The extended solution Ψ corresponds to a smooth map

$$W = \Psi \mathcal{H}_+^n : S^2 \rightarrow \text{Gr}^{(n)},$$

where \mathcal{H}_+^n is the closed subspace of the Hilbert space

$$\mathcal{H} = L^2(S^1, \mathbb{C}^n) = \text{Span}\{\lambda^i e_j : i \in \mathbb{Z}, j \in 1, \dots, n\}$$

defined by

$$\mathcal{H}_+^n = \text{Span}\{\lambda^i e_j : i \in \mathbb{N}, j \in 1, \dots, n\}.$$

For each $z \in S^2$, the space $W(z)$ is *shift-invariant*, i.e. it is closed under multiplication by $\lambda \in S^1$:

$$\lambda W \subset W. \tag{5.2.8}$$

The harmonicity equations lead to the following properties of W :

$$\partial_{\bar{z}} \Gamma(W) \subset \Gamma(W), \quad \lambda \partial_z \Gamma(W) \subset \Gamma(W), \tag{5.2.9}$$

where $\Gamma(W)$ is the space of all smooth sections of W . The first condition implies that W is a holomorphic vector subbundle of the trivial bundle $S^2 \times \mathcal{H}$.

According to K. Uhlenbeck [50, Theorem 11.5], any harmonic map $\psi : S^2 \rightarrow U(n)$ has *finite uniton number*, i.e., it admits an *algebraic* extended solution, i.e., an extended

solution Ψ with finite Taylor series expansion:

$$\Psi = \sum_{i=0}^r \lambda^i A_i,$$

where $A_i : S^2 \rightarrow \mathfrak{u}(\mathfrak{n}) \otimes \mathbb{C}$ are smooth maps. This implies that W satisfies the following inclusion

$$\lambda^r \mathcal{H}_+^n \subset W \subset \mathcal{H}_+^n.$$

From [48, Corollary 2.10], we can assume that

$$\bigcup_{z \in S^2} W(z) \text{ spans } \mathcal{H}_+^n. \quad (5.2.10)$$

Moreover, according to [48, Proposition 2.12], it follows that $r < n$. Hence, the holomorphic map $W : S^2 \rightarrow \text{Gr}^{(n)}$ can be expressed as

$$W = V + \lambda^r \mathcal{H}_+^n \quad (5.2.11)$$

where $V : S^2 \rightarrow G_k(\sum_{j=0}^{r-1} \lambda^j \mathbb{C}^n)$, for some $k \in \{1, \dots, rn\}$, with $r < n$. More precisely, due to the holomorphicity of W , the vector subbundle V has constant rank off a discrete set of points; then, we can fill out the zeros in order to get a globally defined holomorphic vector bundle over S^2 . Due to the shift-invariance property (5.2.8) and the harmonicity conditions (5.2.9), we have:

$$a) \lambda V \subset V \pmod{\lambda^r \mathcal{H}_+^n}; \quad b) \lambda \partial_z \Gamma(V) \subset \Gamma(V) \pmod{\lambda^r \mathcal{H}_+^n}. \quad (5.2.12)$$

Consider on $U(n)$ and $G_k(\sum_{j=0}^{r-1} \lambda^j \mathbb{C}^n)$ the standard invariant metrics. We claim (and we will establish this claim later) the following:

Lemma 5.2.1. *The smooth map $\psi : S^2 \rightarrow U(n)$ is a minimal immersion if and only if $V : S^2 \rightarrow G_k(\sum_{j=0}^{r-1} \lambda^j \mathbb{C}^n)$ is a linearly full holomorphic minimal immersion; moreover, if this is the case, the corresponding induced metrics on S^2 coincide up to scaling.*

Therefore, *classifying minimal immersions $\psi : S^2 \rightarrow U(n)$ of constant curvature reduces to classifying linearly full holomorphic minimal immersions of constant curvature from S^2 into finite dimensional Grassmannians satisfying the conditions (5.2.12).*

In particular, consider a minimal immersion $\psi : S^2 \rightarrow U(3)$ of constant curvature. Then the *uniton number* r can be either 1 or 2. Due to equation (5.2.11), we have the following descriptions:

Case $r = 1$. In this case, we have $W = V + \lambda \mathcal{H}_+^3$, where $V : S^2 \rightarrow G_k(\mathbb{C}^3)$ is a linearly full holomorphic immersion of constant curvature, for some $k \in \{1, 2\}$.

For $k = 1$, $G_1(\mathbb{C}^3) = \mathbb{C}P^2$. Then, by Calabi's result, V is unitarily equivalent to V_0^2 .

Consider now $k = 2$, i.e. $V : S^2 \rightarrow G_2(\mathbb{C}^3)$. In this case, $V^\perp : S^2 \rightarrow G_1(\mathbb{C}^3) = \mathbb{C}P^2$ is an antiholomorphic minimal immersion of constant curvature. Hence, by Calabi's result, V^\perp is unitarily congruent either with V_2^2 , hence $V = V_0^2 \oplus V_1^2$, or with V_1^1 , hence $V = V_0^1 \oplus C$, where C is constant subspace of \mathbb{C}^3 .

Case $r = 2$. We have $W = V + \lambda^2 \mathcal{H}_+^3$, with $V : S^2 \rightarrow G_k(\mathbb{C}^3 + \lambda \mathbb{C}^3)$ for some $k \in \{1, \dots, 5\}$. This case presents both significant challenges and interesting opportunities for exploration (work in progress).

Proof of Lemma 5.2.1: Let us compute the second fundamental form \mathcal{A}'_W of W . Observe first that the action of $U(n)$ on \mathbb{C}^n induces an action of $\Omega U(n)$ on \mathcal{H} which is isometric with respect to the L^2 inner product. Since $W = \Psi \mathcal{H}_+^n$, where $\Psi : S^2 \rightarrow \Omega U(n)$ is an extended solution of $\psi : S^2 \rightarrow U(n)$, the sections $\Psi(\lambda^i e_j)$, with $i \in \mathbb{N}$ and $j \in \{1, \dots, n\}$, form a unitary frame for W ; while the sections $\Psi(\lambda^{-l} e_k)$, with $l \in \mathbb{N} \setminus \{0\}$ and $k \in \{1, \dots, n\}$, form a unitary frame for W^\perp . Set $s = \Psi(\lambda^i e_j)$; we have

$$\mathcal{A}'_W(s) = \pi_{W^\perp} \partial_z s = \pi_{W^\perp} \partial_z \Psi(\lambda^i e_j) = \pi_{W^\perp} \Psi \Psi^{-1} \partial_z \Psi(\lambda^i e_j)$$

We have (see [1, 24, 46, 48, 50]) $\Psi^{-1} \Psi_z = (1 - \lambda^{-1}) A'_\psi$, where $A'_\psi = \frac{1}{2} \psi^{-1} \partial_z \psi$. Then

$$\begin{aligned} \mathcal{A}'_W(s) &= \pi_{W^\perp} \Psi (1 - \lambda^{-1}) A'_\psi(\lambda^i e_j) \\ &= \Psi \pi_{H_+^n} \Psi^{-1} \Psi (1 - \lambda^{-1}) A'_\psi(\lambda^i e_j) \\ &= \Psi \pi_{H_+^n} (1 - \lambda^{-1}) A'_\psi(\lambda^i e_j). \end{aligned}$$

Hence,

$$\mathcal{A}'_W(s) = \begin{cases} -\lambda^{-1} \Psi A'_\psi(e_j) & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}.$$

After straightforward computation we obtain

$$\langle \mathcal{A}'_W(\Psi(\lambda^i e_j)), \Psi(\lambda^{-l} e_k) \rangle_{L^2} = \begin{cases} -2\pi \langle A'_\psi(e_j), e_k \rangle & \text{if } i = 0, l = -1 \\ 0 & \text{if } i \geq 1, l \leq -1 \end{cases}. \quad (5.2.13)$$

Also note that, for $V : S^2 \rightarrow G_k(\sum_{j=0}^{r-1} \lambda^j \mathbb{C}^n)$ defined by $W = V + \lambda^r \mathcal{H}_+^n$, then

$$\mathcal{A}'_V(s) = \pi_{V^\perp} \circ \partial_z s = \pi_{W^\perp} \circ \partial_z s = \mathcal{A}'_W(s), \quad (5.2.14)$$

for all $s \in \Gamma(V) \subset \Gamma(W)$. Placing (5.2.14) and (5.2.13) together gives

$$\begin{aligned} V^* ds_{G_k(\sum_{j=0}^{r-1} \lambda^j \mathbb{C}^n)}^2 &= \operatorname{tr} \mathcal{A}'_V (\mathcal{A}'_V)^* dz d\bar{z} = \operatorname{tr} \mathcal{A}'_W (\mathcal{A}'_W)^* dz d\bar{z} \\ &= 4\pi^2 \operatorname{tr} A_z^\psi (A_z^\psi)^* dz d\bar{z} = \pi^2 \psi^* ds_{\mathbb{U}(n)}^2, \end{aligned}$$

and the result follows.

5.2.3 Other open questions

1. Another possible future direction is to investigate further generalizations of Theorem 3.1.1. Observe that we have proved Theorem 3.3.2 for invariant metrics of a very specific form. It would be really interesting to see whether the result will be true for an arbitrary choice of invariant metric.
2. K. Uhlenbeck [50] showed that all harmonic map from S^2 into the unitary group $\mathbb{U}(n)$ have finite uniton number. From other domains, it may exist harmonic maps which do not have finite uniton number: for instance, the Clifford solution does not have finite uniton number (see Remark 2.3.11). Is it possible to extended the classification results of this thesis to primitive minimal immersions of finite uniton number from an arbitrary Riemann surface? To address this question, it may be useful the criteria for finiteness of the uniton number developed in [1, 44].
3. The theory of harmonic maps from the two-torus T^2 into Riemannian symmetric spaces is very interesting, as it brings into play the methods of integrable systems theory [24]. Is it possible to obtain the complete classification of primitive immersions of constant curvature of T^2 into low dimensional flag manifolds?

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