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On a Ternary Octonion Algebra

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Abstract

Following a research direction proposed in an earlier work, the ternary octonion algebra $\mathfrak{O}$, which is a ternary composition algebra, is considered. By hand and applying computational linear algebra on matrices, 1-identities and 2-identities of $\mathfrak{O}$ are established. From some of these identities, the non-conservativeness of $\mathfrak{O}$ and of some of its binary reduced algebras, which are binary standard composition algebras of types II and III, is proved. Also from identities of $\mathfrak{O}$, using computational linear algebra based on the representation theory of the symmetric group, ternary enveloping algebras for ternary Maltsev algebras are constructed.

Keywords Composition algebra · Identity · Conservative algebra · Maltsev algebra · Enveloping algebra · Computational linear algebra

Mathematics Subject Classification 17A40 · 17A75 · 17D10 · 15A03

1 Introduction

According to Tomber in [33, p. 1253], the "first appearence in print of a nonassociative algebra occured in an adenda Cayley attached to a paper", of 1845, on elliptic functions. Moreover, he dates the beginnings of Nonassociative algebra to the 1930's, also mentioning Hamilton, who wanted to equip the usual real vector space $\mathbb{R}^3$ with a multiplication having properties similar to those of the multiplication of complex numbers. The attempts culminated in the discovery of quaternions, with the loss of

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commutativity. This aspect paved the way for the study of algebras where other cherished algebraic rules are not valid. For this reason, quaternions should be looked at as the progenerator of nonassociative algebras such as octonions, [33].

Quaternions and octonions are (binary) composition algebras – algebras arising from quadratic forms which permit composition. Among these, the well-known ones are those with identity, that is, Hurwitz algebras. These, over a field of characteristic different from 2, by the generalized Hurwitz Theorem in the work [21] of Jacobson, are isomorphic either to the base field, a separable quadratic extension of the base field, a generalized quaternion algebra or a generalized Cayley algebra. In particular, the dimension of any Hurwitz algebra is 1, 2, 4 or 8. Throughout the years, various generalizations appeared; for instance, Elduque introduced and studied the class of ternary composition algebras in the article [15], and related them to binary ones.

As can be seen by publications such as [1–7], [14–31], the interest in (binary and ternary) composition algebras remains alive. Not only fundamental algebraic properties of composition algebras are still studied, but also their historical evolution, as in [16] by Familton, and their (cutting-edge) applications to: Geodesy, [20] by Inácio, Ditmar, Klees and Farahani; Respiratory Physiology, [26] by Klco, Kollarik and Tatar; Quantum Computing, [9] by Blackman and Stier, [18] by Freedman, Shokrian-Zini and Wang; Artificial Intelligence, [27] by Parcollet, Morchid and Linarès, [32] by Takahashi, Fujita and Hashimoto; Theoretical Physics, [8] by Blackman and Lemurell. All these studies, either pure or applied, offer valuable insights and pathways for future research.

Several research directions were proposed in [3], where a ternary quaternion algebra which appears analogously to the quaternions from the Lie algebra $\mathfrak{sl}(2)$ was considered. In particular, the study of the ternary octonion algebra $\mathfrak{O}$ was expected to lead to an interesting class of ternary alternative algebras. The algebra $\mathfrak{O}$ is an 8-dimensional ternary composition algebra constructed from an 8-dimensional binary composition algebra – an octonion algebra. Aspects of the former algebra were addressed by: Brown and Gray, [14]; Elduque, [15]; Kamiya and Okubo, [22]; Pozhidaev, [28]; Shaw, [31]. Its multiplication can also be found in [1] and [25], works due to, respectively, Beites and Nicolás, and Kaygorodov, Pozhidaev and Saraiva.

Regarding the present work, where computational linear algebra is needed, its structure is as follows. In the next section, 2 – Preliminaries, where some background is presented, known definitions and results related to s-identities, certain classes of (binary and ternary) algebras and the mentioned ternary octonion algebra $\mathfrak{O}$ are collected. Moreover, several aspects concerning some binary reduced algebras of $\mathfrak{O}$, which are binary standard composition algebras of types II and III considered by Beites and Nicolás in [2], are recalled. In addition, the reasons for following the research direction that led to considering the ternary octonion algebra $\mathfrak{O}$, suggested by Beites, Nicolás, Pozhidaev and Saraiva in [3], are explained.

Concerning section 3 – Identities, all 1-identities and 2-identities of $\mathfrak{O}$ are established. More concretely, although some are proved by hand, the random vectors method is applied; briefly speaking, the information about the structure of the space of identities is given by the nullspace of an adequate matrix, and this linear-algebraic data can be translated back into the sought identities, [12]. From the deduced identities of $\mathfrak{O}$, on the one hand, the non-conservativeness of $\mathfrak{O}$ is proved in section 4 – Conservativeness;

on the other hand, using computational linear algebra based on the representation theory of the symmetric group, ternary enveloping algebras for ternary Maltsev algebras are constructed in section 5 – Envelopes.

2 Preliminaries

In what follows, F denotes a field with $\mathrm{char}(F) \neq 2$.

Recall that a *binary algebra* over F is a vector space U over F equipped with a bilinear map $U \times U \to U$ (binary multiplication). A *ternary algebra* over F is a vector space U over F equipped with a trilinear map $U \times U \times U \to U$ (ternary multiplication).

Let A be a binary (respectively, ternary) algebra over F.

An identity of A is an *s-identity* or an *identity of level s*, where $s \in \mathbb{N}$, if the binary (respectively, ternary) multiplication of A appears s times in each term of the identity, [1, 3].

Remark 2.1 An s-identity of A when $\mathrm{char}(F) = 0$ and having only 1 and -1 as coefficients in its terms is also an s-identity of A when $\mathrm{char}(F) \neq 2$.

An s-identity of A can be lifted to an $(s+1)$-identity of A by replacing one argument of the s-identity by a product or embedding the s-identity in a product, [11]. The new identity is a *lifting* of the first one.

Two of the following definitions are related to the conservativeness of an algebra, property introduced by Kantor as a generalization of Jordan algebras.

Let A be a binary algebra over F, with binary multiplication denoted by juxtaposition. A is a *conservative algebra*, [24], if a new binary multiplication $(\cdot,\cdot)_n$ can be defined on the underlying vector space of A in such a way that the following identity holds:

$$\begin{aligned} & b(a(xy)-(ax)y-x(ay))-a((bx)y)+(a(bx))y \\ & +(bx)(ay)-a(x(by))+(ax)(by)+x(a(by)) \\ = \quad & -(a,b)_n(xy)+((a,b)_n x)y+x((a,b)_n y). \end{aligned}$$

Let B be a ternary algebra over F, with ternary multiplication denoted by $(\cdot,\cdot,\cdot)$. Let $\mathcal{S}_3$ denote the symmetric group of degree 3.

B is a *3-Lie algebra*, [17], if the skew-symmetric identity and the ternary Jacobi identity hold, that is, respectively,

$$(x_1,x_2,x_3) = \mathrm{sgn}(\sigma)(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}) \tag{1}$$

for all $\sigma \in \mathcal{S}_3$ and

$$\begin{aligned} ((x_1,x_2,x_3),y_2,y_3) &= ((x_1,y_2,y_3),x_2,x_3) \\ &+(x_1,(x_2,y_2,y_3),x_3)+(x_1,x_2,(x_3,y_2,y_3)). \end{aligned}$$

B is a ternary *Maltsev algebra*, [28], if (1) is valid and the ternary Maltsev identity holds, that is,

$$\begin{aligned}&(((z, x_2, x_3), x_2, x_3), y_2, y_3) - (((z, y_2, y_3), x_2, x_3), x_2, x_3)\\ &= ((z, x_2, x_3), (x_2, y_2, y_3), x_3) + ((z, x_2, x_3), x_2, (x_3, y_2, y_3))\\ &+ ((z, (x_2, y_2, y_3), x_3), x_2, x_3) + ((z, x_2, (x_3, y_2, y_3)), x_2, x_3).\end{aligned}$$

It is a well known fact that every 3-Lie algebra, also called ternary Filippov algebra as in [3], is a ternary Maltsev algebra.

B is a *conservative algebra*, [23], if a new ternary multiplication $(\cdot, \cdot, \cdot)_n$ can be defined on the underlying vector space of B in such a way that the identity holds:

$$((a, b, c)_n, x, y) = (a, b, (c, x, y)) - (c, (a, b, x), y) - (c, x, (a, b, y)).$$

The *commutator algebra* $B^{(-)}$ of B is the ternary algebra with multiplication $(\cdot, \cdot, \cdot)_c$ given by the ternary version of the generalized commutator:

$$(x_1, x_2, x_3)_c = \sum_{\sigma \in \mathcal{S}_3} \operatorname{sgn}(\sigma)(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

Let $\mathbb{O} = (\mathbb{O}, *)$ be the binary octonion algebra over F with: basis $\{e_0, \ldots, e_7\}$, which we call canonical; multiplication table given by $e_i * e_i = -e_0$ for $i \in \{1, \ldots, 7\}$, being e_0 the identity, and the Fano plane in Fig. 1, where the cyclic ordering of each three elements lying on the same line is shown by the arrows.

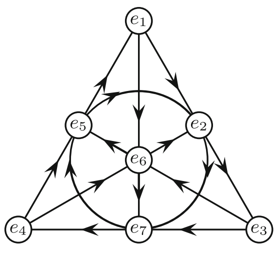
Consider the *ternary octonion algebra* $\mathfrak{O} = (\mathbb{O}, \{\cdot, \cdot, \cdot\})$ over F. The multiplication $\{\cdot, \cdot, \cdot\}$ is defined by

$$\{x, y, z\} = (x * \overline{y}) * z, \tag{2}$$

where $^{-} : x \mapsto \overline{x}$ is the usual involution of $\mathbb{O} = (\mathbb{O}, *)$ and the norm $n(x)$ of $x \in \mathbb{O}$ is given by $n(x)e_0 = x * \overline{x} = \overline{x} * x$.

Multiplication (2) was connected, by Brown and Gray in [14], with three-fold vector cross products. Shaw pointed out in [31] that the mentioned multiplication constitutes one of four distinct multiplications, $(x * \overline{y}) * z$, $x * (\overline{y} * z)$, $(z * \overline{y}) * x$, $z * (\overline{y} * x)$, in the sense that they lie on different $SO(8)$ orbits, for constructing an 8-dimensional real ternary composition algebra out of an 8-dimensional real binary composition

Fig. 1 Fano plane for $\mathbb{O}$



algebra. Elduque, in [15], presented the automorphisms and the derivations of $\mathfrak{O}$, and explained the interest in the Maltsev operations defined by $(x * \overline{y}) * z$ and $x * (\overline{y} * z)$ on a Cayley-Dickson algebra. Pozhidaev proved in [28] that the multiplication

$$[x, y, z] := (x * \overline{y}) * z - (y, z)x + (x, z)y - (x, y)z,$$

where $(\cdot, \cdot)$ denotes the nondegenerate, symmetric, bilinear form given by $(x, y) = \frac{1}{2}(x * \overline{y} + y * \overline{x})$, turns $\mathbb{O}$ into a ternary Maltsev algebra $(\mathbb{O}, [\cdot, \cdot, \cdot])$, which is the main reason for investigating $\mathfrak{O}$ as a ternary alternative algebra.

In addition to the described research, the work [22] of Kamiya and Okubo allows to say that $\mathfrak{O}$ is a *quadratic triple system* and a *weakly commutative triple system*, that is, respectively,

$$\{a, a, b\} = \{b, a, a\} = n(a)b \tag{3}$$

and

$$\{a, b, \{a, a, a\}\} = \{\{a, a, a\}, b, a\} \tag{4}$$

are identities of $\mathfrak{O}$. It is also a *weakly alternative triple system*, that is,

$$\{\{a, b, a\}, b, c\} = \{a, b, \{a, b, c\}\}$$

and

$$\{\{a, b, c\}, b, c\} = \{a, b, \{c, b, c\}\}$$

are identities of $\mathfrak{O}$, [22]; equivalently, from [22, Lemma 16.5],

$$\{a, b, \{b, a, c\}\} = \{\{c, a, b\}, b, a\} = n(a)n(b)c \tag{5}$$

are identities of $\mathfrak{O}$. The identities (3) to (5) are obtained differently in the present work, as consequences of the study of all 1-identities and 2-identities of $\mathfrak{O}$. Recently, the multiplication of $\mathfrak{O}$ was considered by Kaygorodov, Pozhidaev and Saraiva when studying generalized quaternions in [25].

Taking $z = e_0$ (respectively, $x = e_0$) in (2) leads to the *binary reduced algebra* O (respectively, $\mathcal{O}$) of $\mathfrak{O}$ over F, with multiplication in Table 1 (respectively, Table 2 in [2]) denoted by juxtaposition and defined by

$$xy = x * \overline{y} \text{ (respectively, } xy = \overline{x} * y).$$

These anti-isomorphic algebras O of the *standard octonions of type III* and $\mathcal{O}$ of the *standard octonions of type II* are *binary standard composition algebras of types, respectively, III and II over* $\mathbb{O}$, whose identities were studied by Beites and Nicolás in [2].

Table 1 Multiplication table of $\mathfrak{O}$

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	$-e_1$	$-e_2$	$-e_3$	$-e_4$	$-e_5$	$-e_6$	$-e_7$
e_1	e_1	e_0	$-e_3$	e_2	$-e_5$	e_4	$-e_7$	e_6
e_2	e_2	e_3	e_0	$-e_1$	$-e_6$	e_7	e_4	$-e_5$
e_3	e_3	$-e_2$	e_1	e_0	e_7	e_6	$-e_5$	$-e_4$
e_4	e_4	e_5	e_6	$-e_7$	e_0	$-e_1$	$-e_2$	e_3
e_5	e_5	$-e_4$	$-e_7$	$-e_6$	e_1	e_0	e_3	e_2
e_6	e_6	e_7	$-e_4$	e_5	e_2	$-e_3$	e_0	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	e_0

3 Identities

The present section is devoted to the study of the 1-identities and the 2-identities of $\mathfrak{O}$.

3.1 1-identities

Let F be a field with char$(F) \neq 2$. Let $\mathcal{S}_3$ denote the symmetric group of degree 3. The 1-identities of $\mathfrak{O}$ are of the form

$$\sum_{\sigma \in \mathcal{S}_3} \alpha_\sigma \{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\} = 0, \tag{6}$$

with $\alpha_\sigma \in F$.

Theorem 3.1 *Let F be a field with char$(F) = 0$. All 1-identities of $\mathfrak{O}$ are consequences of*

$$\{a, a, b\} - \{b, a, a\} = 0. \tag{7}$$

Proof In order to find the identities of the form in (6), we apply the random vectors method using Maple in characteristic 0. The process begins with the construction of a 14×6 matrix M initialized to zero. The columns of this matrix are labeled by the 3! monomials $\{\cdot, \cdot, \cdot\}$. Furthermore, we think of M as consisting of a 6×6 square matrix on top of an 8×6 matrix. We generate three pseudo-random vectors, each with 8 components, and store the components of the evaluation of the jth monomial $\{\cdot, \cdot, \cdot\}$, $j \in \{1, \ldots, 6\}$, in column j of M, in rows 7 through 14. The computation of the row canonical form of the obtained matrix, with rank less or equal to 6, appears in rows 1 to 6 and completes the first iteration of the algorithm; its bottom 8×6 submatrix is now a null matrix. We repeat this fill and reduce process until the stabilization of the rank of the matrix is reached, in this case 4. The 1-identities satisfied by $\mathfrak{O}$ lie in the nullspace of this matrix, being $\{(-1, -1, 0, 1, 0, 1), (0, 1, -1, -1, 1, 0)\}$ a basis for it. These basis vectors represent the subsequent 1-identities of $\mathfrak{O}$:

$$\begin{aligned} -\{a,b,c\}-\{a,c,b\}+\{b,c,a\}+\{c,b,a\}&=0,\\ \{a,c,b\}-\{b,a,c\}-\{b,c,a\}+\{c,a,b\}&=0. \end{aligned}$$

It is clear that the second identity can be obtained by the action of the transposition $(a\ b)\in\mathcal{S}_3$ over the first one. So, the first identity generates the whole space of 1-identities of $\mathfrak{O}$ under the action of $\mathcal{S}_3$. Finally, observe that the first identity is a linearized form of (7). □

The following result asserts that the 1-identity which implies all 1-identities of $\mathfrak{O}$ if $\mathrm{char}(F)=0$ also holds if $\mathrm{char}(F)\neq 2$.

Theorem 3.2 *Let F be a field with char$(F)\neq 2$. Then (7), that is,*

$$\{a,a,b\}-\{b,a,a\}=0$$

is a 1*-identity of* $\mathfrak{O}$.

Proof As $(a*\overline{a})*b=(b*\overline{a})*a$ by [28, Lemma 2.1 (1)], then the result follows. □

Corollary 3.3 *Let F be a field with char$(F)\neq 2$. Then $\mathfrak{O}$ is a quadratic triple system.*

Proof From Theorem 3.2, it suffices to see that $\{a,a,b\}=(a*\overline{a})*b=n(a)b$. Hence, (3) holds. □

3.2 2-identities

Let F be a field with $\mathrm{char}(F)\neq 2$. Let $\mathcal{S}_5$ denote the symmetric group of degree 5. The 2-identities of $\mathfrak{O}$ are of the form

$$\sum_{\sigma\in\mathcal{S}_5}\big(\alpha_\sigma\{\{x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}\},x_{\sigma(4)},x_{\sigma(5)}\} \\ +\beta_\sigma\{x_{\sigma(1)},\{x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)}\},x_{\sigma(5)}\} \\ +\gamma_\sigma\{x_{\sigma(1)},x_{\sigma(2)},\{x_{\sigma(3)},x_{\sigma(4)},x_{\sigma(5)}\}\}\big)=0, \tag{8}$$

with $\alpha_\sigma,\beta_\sigma,\gamma_\sigma\in F$.

Theorem 3.4 *Let F be a field with char$(F)=0$. All* 2*-identities of* $\mathfrak{O}$ *are consequences of (7), that is,*

$$\{a,a,b\}-\{b,a,a\}=0,$$

and the following four identities

$$\{\{a,b,c\},c,d\}-\{\{a,b,d\},c,c\}=0, \tag{9}$$

$$\{\{a,b,c\},c,d\}-\{a,\{b,c,c\},d\}=0, \tag{10}$$

$$\{\{a, b, c\}, d, e\} + \{\{a, c, b\}, d, e\} - \{\{b, c, e\}, d, a\}$$
$$-\{a, \{b, c, d\}, e\} + \{a, \{b, c, e\}, d\} - \{d, e, \{c, b, a\}\} = 0, \quad (11)$$
$$\sum_{\sigma \in \mathcal{S}_5} sgn(\sigma) \, [\{\sigma(a), \{\sigma(b), \sigma(c), \sigma(d)\}, \sigma(e)\}$$
$$-3\{\sigma(a), \sigma(b), \{\sigma(c), \sigma(d), \sigma(e)\}\}] = 0. \quad (12)$$

Proof We apply the random vectors method using Maple in characteristic 0. Since there are three association types in (8), $\{\{\cdot,\cdot,\cdot\},\cdot,\cdot\}$, $\{\cdot,\{\cdot,\cdot,\cdot\},\cdot\}$ and $\{\cdot,\cdot,\{\cdot,\cdot,\cdot\}\}$, the process begins with the construction of a 368×360 matrix M initialized to zero. The first 120 columns of this matrix are labeled by the 5! monomials $\{\{\cdot,\cdot,\cdot\},\cdot,\cdot\}$, the second 120 columns are labeled by the 5! monomials $\{\cdot,\{\cdot,\cdot,\cdot\},\cdot\}$ and the last 120 columns are labeled by the 5! monomials $\{\cdot,\cdot,\{\cdot,\cdot,\cdot\}\}$. Furthermore, we think of M as consisting of a 360×360 square matrix on top of an 8×360 matrix. We generate five pseudo-random vectors, each with 8 components, and store the 8 components of the evaluation of the jth monomial $\{\{\cdot,\cdot,\cdot\},\cdot,\cdot\}$ in column $j \in \{1,\ldots,120\}$, of the jth monomial $\{\cdot,\{\cdot,\cdot,\cdot\},\cdot\}$ in column $j \in \{121,\ldots,240\}$ and of the jth monomial $\{\cdot,\cdot,\{\cdot,\cdot,\cdot\}\}$ in columnn $j \in \{241,\ldots,360\}$ of M, in rows 361 to 368. The computation of the row canonical form of the obtained matrix, with rank less or equal to 360, appears in rows 1 to 360 and completes the first iteration of the algorithm; its bottom 8×360 submatrix is now a null matrix. We repeat this fill and reduce process until the stabilization of the rank of the matrix is reached, in this case 30. The 2-identities satisfied by $\mathfrak{O}$ lie in the nullspace of this matrix. Moreover, it is a well known fact that they span a left $\mathcal{S}_5$-module of the group ring $\mathbb{Q}\mathcal{S}_5$. So, it suffices to find a set of generators of this module to get a complete description of the 2-identities of $\mathfrak{O}$. The liftings of a linearized form of (7), a linearized form of (9), a linearized form of (10) and the identities (11) and (12) span a left $\mathcal{S}_5$-module of dimension 330 which is contained in the nullspace of the matrix M. Since the rank of M is 30, the result follows. □

The subsequent result shows, in particular, that some of the 2-identities which imply all 2-identities of $\mathfrak{O}$ when $\text{char}(F) = 0$ still hold if $\text{char}(F) \neq 2$.

Theorem 3.5 *Let F be a field with $char(F) \neq 2$. Then (9) and (10), that is,*

$$\{\{a, b, c\}, c, d\} - \{\{a, b, d\}, c, c\} = 0$$

and

$$\{\{a, b, c\}, c, d\} - \{a, \{b, c, c\}, d\} = 0,$$

and

$$\{\{a, b, c\}, b, c\} - \{a, \{b, c, b\}, c\} = 0 \quad (13)$$

are 2-identities of $\mathfrak{O}$.

Proof Invoking [28, Lemma 2.1 (1)], we have:

$$\{\{a,b,c\},c,d\} = (((a*\overline{b})*c)*\overline{c})*d = ((a*\overline{b})*(c*\overline{c}))*d = n(c)(a*\overline{b})*d,$$
$$\{\{a,b,d\},c,c\} = (((a*\overline{b})*d)*\overline{c})*c = ((a*\overline{b})*d)*(\overline{c}*c) = n(c)(a*\overline{b})*d,$$

and

$$\{a,\{b,c,c\},d\} = (a*(\overline{c}*(c*\overline{b})))*d = (a*((\overline{c}*c)*\overline{b}))*d = n(c)(a*\overline{b})*d.$$

Since $\mathbb{O} = (\mathbb{O}, *)$ is alternative, by [28], the right Moufang identity holds and any two elements generate an associative subalgebra. Finally, we have

$$\{\{a,b,c\},b,c\} = (((a*\overline{b})*c)*\overline{b})*c = (a*(\overline{b}*c*\overline{b}))*c = \{a,\{b,c,b\},c\}.$$

□

Corollary 3.6 *Let F be a field with char$(F) \neq 2$. Then $\mathfrak{O}$ is a weakly commutative triple system.*

Proof Recall Remark 2.1. Observe that the 2-identity (4) is a consequence of the 1-identity (7) and the 2-identity (11). In fact, taking $b = c = e = a$ in (11) and then $d = b$, we have

$$\{\{a,a,a\},b,a\} + \{a,\{a,a,a\},b\} - \{a,\{a,a,b\},a\} - \{b,a,\{a,a,a\}\} = 0.$$

On the one hand, due to (7),

$$\{a,\{a,a,a\},b\} - \{b,a,\{a,a,a\}\} = n(a)(\{a,a,b\} - \{b,a,a\}) = 0.$$

On the other hand,

$$\{a,\{a,a,b\},a\} = n(a)\{a,b,a\} = \{a,b,n(a)a\} = \{a,b,\{a,a,a\}\}.$$

Thus, (4) follows. □

Corollary 3.7 *Let F be a field with char$(F) \neq 2$. Then $\mathfrak{O}$ is a weakly alternative triple system.*

Proof Recall Remark 2.1. Notice that the identities in (5) are consequences of the 1-identity (7) and the 2-identities (13), (9), (10) and (11). Replacing a, b, c, d and e by, respectively, c, a, b, a and b in (11), we obtain

$$\{\{c,a,b\},a,b\} + \{\{c,b,a\},a,b\} - \{\{a,b,b\},a,c\} - \{a,b,\{b,a,c\}\}$$
$$+\{c,\{a,b,b\},a\} - \{c,\{a,b,a\},b\} = 0.$$

From (7) and (9), we can write

$$\begin{aligned}\{\{c,b,a\},a,b\} &\overset{(9)}{=} \{\{c,b,b\},a,a\}\\ &\overset{(7)}{=} \{\{b,b,c\},a,a\}\\ &\overset{(9)}{=} \{\{b,b,a\},a,c\}\\ &\overset{(7)}{=} \{\{a,b,b\},a,c\}.\end{aligned}$$

Now, using (13), we obtain

$$\{\{c,a,b\},a,b\} = \{c,\{a,b,a\},b\}.$$

Finally, we arrive at the subsequent first equality and, using (10), to the second one,

$$\{a,b,\{b,a,c\}\} = \{c,\{a,b,b\},a\} = \{\{c,a,b\},b,a\}.$$

At last, by [28, Lemma 2.1 (1)], notice that

$$\{c,\{a,b,b\},a\} = (c * \overline{((a*\overline{b})*b)}) * a = n(a)n(b)c.$$

□

4 Conservativeness

The present section is devoted to the study of the conservativeness of $\mathfrak{O}$, and of some of its binary reduced algebras.

Theorem 4.1 *Let F be a field with char$(F) \neq 2$. Then $\mathfrak{O}$ is not a conservative algebra.*

Proof Assume that $\mathfrak{O}$ is a conservative algebra. Then we can define a new multiplication $\{\cdot,\cdot,\cdot\}_n$ on the underlying vector space of $\mathfrak{O}$ such that

$$\{\{a,b,c\}_n,x,y\} = \{a,b,\{c,x,y\}\} - \{c,\{a,b,x\},y\} - \{c,x,\{a,b,y\}\} \qquad (14)$$

is an identity of $\mathfrak{O}$. Consider the canonical basis of $\mathfrak{O}$, $\{e_0,\ldots,e_7\}$. With $x=y=e_0$ in (14), using Corollary 3.3, we have

$$\{\{e_i,e_j,e_k\}_n,e_0,e_0\} = n(e_0)\{e_i,e_j,e_k\}_n = \{e_i,e_j,e_k\}_n.$$

By the same corollary and since $e_0=\overline{e_0}$ is the identity of $\mathbb{O}=(\mathbb{O},*)$, for $i\neq j$ we get

$$\begin{aligned}&\{e_i,e_j,\{e_k,e_0,e_0\}\} - \{e_k,\{e_i,e_j,e_0\},e_0\} - \{e_k,e_0,\{e_i,e_j,e_0\}\}\\ &= n(e_0)(e_i*\overline{e_j})*e_k - (e_k*\overline{(e_i*\overline{e_j})*e_0})*e_0 - (e_k*\overline{e_0})*((e_i*\overline{e_j})*e_0)\\ &= (e_i*\overline{e_j})*e_k + e_k*(e_i*\overline{e_j}) - e_k*(e_i*\overline{e_j})\\ &= \{e_i,e_j,e_k\}.\end{aligned}$$

So, $\{e_i, e_j, e_k\}_n = \{e_i, e_j, e_k\}$ for $i \neq j$. However, taking $a = c = e_4$, $b = e_0$, $x = e_6$ and $y = e_3$ in (14), this leads to the contradiction $4e_5 = 0$. Thus, $\mathfrak{O}$ is not a conservative algebra. □

Theorem 4.2 *Let F be a field with char$(F) \neq 2$. Then the binary reduced algebra O of $\mathfrak{O}$ is not a conservative algebra.*

Proof Assume that O is a conservative algebra. Then we can define a new multiplication $(\cdot, \cdot)_n$ on the underlying vector space of O such that

$$\begin{aligned} &b(a(xy) - (ax)y - x(ay)) - a((bx)y) + (a(bx))y \\ &\quad +(bx)(ay) - a(x(by)) + (ax)(by) + x(a(by)) \\ &= -(a,b)_n(xy) + ((a,b)_n x)y + x((a,b)_n y) \end{aligned} \tag{15}$$

is an identity of O. By [2, Corollary 4.6.], e_0 is the right identity of O and, from Table 1, it is straightforward that, for all $w \in \mathsf{O}$, $e_0 w = \overline{w}$. With $x = y = e_0$ in (15), we obtain $\overline{(a,b)_n} = ab + ba - a\overline{b} - b\overline{a} + \overline{ab}$. So, for all $a, b \in \mathsf{O}$,

$$(a,b)_n = 2ab + ba - \overline{a}b - \overline{b}a.$$

However, taking $a = b = x = y = e_1$ in (15) leads to the contradiction $5e_0 = -15e_0$. Thus, O is not a conservative algebra. □

Theorem 4.3 *Let F be a field with char$(F) \neq 2$. Then the binary reduced algebra $\mathcal{O}$ of $\mathfrak{O}$ is not a conservative algebra.*

Proof Assume that $\mathcal{O}$ is a conservative algebra. Then we can define a new multiplication $(\cdot, \cdot)_n$ on the underlying vector space of $\mathcal{O}$ such that

$$\begin{aligned} &b(a(xy) - (ax)y - x(ay)) - a((bx)y) + (a(bx))y \\ &\quad +(bx)(ay) - a(x(by)) + (ax)(by) + x(a(by)) \\ &= -(a,b)_n(xy) + ((a,b)_n x)y + x((a,b)_n y) \end{aligned} \tag{16}$$

Observe that e_0 is the left identity of $\mathcal{O}$ since, by [2, Corollary 4.6. 1.], e_0 is the right identity of O, and, [2, Theorem 4.5], $\mathcal{O}$ and O are anti-isomorphic algebras. From [2, Table 2], it is straightforward that, for all $w \in \mathcal{O}$, $we_0 = \overline{w}$. With $x = y = e_0$ in (16), we obtain, for all $a, b \in \mathcal{O}$,

$$(a,b)_n = -ab - ba + \overline{b}a + \overline{b}\overline{a} + \overline{a}\overline{b}.$$

However, taking $a = e_1, b = e_2, x = e_4, y = e_5$ in (16) leads to the contradiction $-5e_2 = 3e_2$. Thus, O is not a conservative algebra. □

5 Envelopes

The present section is devoted to the construction of some ternary enveloping algebras for ternary Maltsev algebras.

Theorem 5.1 *Let F be a field with $char(F) = 0$. Then $\mathfrak{O}^{(-)}$ is a ternary Maltsev algebra which is not a 3-Lie algebra.*

Proof Consider the commutator algebra $\mathfrak{O}^{(-)}$ of $\mathfrak{O}$ by equipping $\mathfrak{O}$ with the ternary commutator

$$\{x_1, x_2, x_3\}_c = \sum_{\sigma \in \mathcal{S}_3} sgn(\sigma)\{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\}.$$

By construction, $\{\cdot, \cdot, \cdot\}_c$ is skew-symmetric and, applying linearized forms of the 1-identity (7) of $\mathfrak{O}$, we have

$$\{a, b, c\}_c = 3(\{a, b, c\} - \{c, b, a\}).$$

We will see that the ternary Maltsev identity in $\mathfrak{O}^{(-)}$, which can be written as

$$\begin{aligned} &\{\{\{z, x_2, x_3\}_c, x_2, x_3\}_c, y_2, y_3\}_c - \{\{\{z, y_2, y_3\}_c, x_2, x_3\}_c, x_2, x_3\}_c \\ &= \{\{z, x_2, x_3\}_c, \{x_2, y_2, y_3\}_c, x_3\}_c + \{\{z, x_2, x_3\}_c, x_2, \{x_3, y_2, y_3\}_c\}_c \\ &+ \{\{z, \{x_2, y_2, y_3\}_c, x_3\}_c, x_2, x_3\}_c + \{\{z, x_2, \{x_3, y_2, y_3\}_c\}_c, x_2, x_3\}_c, \end{aligned} \tag{17}$$

holds in $\mathfrak{O}^{(-)}$. Since $\{\cdot, \cdot, \cdot\}_c$ is skew-symmetric, there are only two association types for 3-identities, namely $\{\{\{\cdot, \cdot, \cdot\}_c, \cdot, \cdot\}_c, \cdot, \cdot\}_c$ (first association type) and $\{\{\cdot, \cdot, \cdot\}_c, \{\cdot, \cdot, \cdot\}_c, \cdot\}_c$ (second association type). Following [12], we use the random vectors method with Maple. For that, we construct a 5048×10080 matrix initialized to zero. The first 5040 columns of M are labeled by the 7! monomials of the first association type whereas the other 5040 are labeled by the 7! monomials of the second association type. Since the structure constants of the ternary algebra $\mathfrak{O}$ belong to the set $\{-1, 0, 1\}$, we perform the needed computations using arithmetic modulo an appropriate prime p, allowing to control memory allocation during the row reduction of large matrices. We generate seven pseudo-random vectors with eight components each, which can be seen as uniformly distributed elements from 0 to $p-1$ in the field with p elements. We allocate the eight components of the jth monomial in column j of M, in rows 5041 through 5048. The computation of the row canonical form of the obtained matrix completes the first iteration of the algorithm. We repeat this fill and reduce process until the stabilization of the rank of the matrix is reached.

The described algorithm was implemented with primes $p = 101, 103$, arriving, in both cases, at a matrix of rank 155. These primes are much larger than the number of arguments in $\{\{\{\cdot, \cdot, \cdot\}_c, \cdot, \cdot\}_c, \cdot, \cdot\}_c$ and $\{\{\cdot, \cdot, \cdot\}_c, \{\cdot, \cdot, \cdot\}_c, \cdot\}_c$, so, the group ring $F\mathcal{S}_7$ will be semisimple and we can expect the rank obtained to be equal to the one in characteristic zero. This assumption is corroborated by the equality of the determined ranks with those distinct primes. Finally, we verified that the linearized form of (17) belongs to the nullspace of M, and so the identity holds in $\mathfrak{O}^{(-)}$.

Finally, we see that $\mathfrak{O}^{(-)}$ is not a 3-Lie algebra. For this, it suffices to see that the ternary Jacobi identity

$$\{\{a, b, c\}_c, d, e\}_c - \{\{a, d, e\}_c, b, c\}_c - \{a, \{b, d, e\}_c, c\}_c - \{a, b, \{c, d, e\}_c\}_c = 0$$

does not hold in $\mathfrak{O}^{(-)}$. Taking $a = e_0$, $b = e_1$, $c = e_2$, $d = e_2$ and $e = e_4$, we obtain $-108e_5 = 0$, a contradiction. □

Remark 5.2 In order to save space and time, computational methods are sometimes implemented using modular arithmetic. In particular, as highlighted in [10], finding the row canonical form of a large matrix may require large amounts of memory and slow down the computations. In Theorem 5.1, the feasibility of the computations was achieved through the implementation in Maple with primes $p = 101, 103$.

Corollary 5.3 *Let F be a field with char$(F) \neq 2, 3$. Then $\mathfrak{O}^{(-)}$ is a ternary Maltsev algebra.*

Proof The result is a consequence of Theorem 5.1 and Remark 2.1. □

In what follows, let F be a field with char$(F) \neq 2, 3$. Let $\mathcal{VO}$ be the variety of ternary algebras over F given by the (1; 2)-identities (of $\mathfrak{O} \in \mathcal{VO}$)

$$(a, a, b) - (b, a, a) = 0, \tag{18}$$
$$((a, b, c), c, d) - ((a, b, d), c, c) = 0, \tag{19}$$
$$((a, b, c), c, d) - (a, (b, c, c), d) = 0, \tag{20}$$
$$\begin{aligned}&((a, b, c), d, e) + ((a, c, b), d, e) - ((b, c, e), d, a)\\ &-(a, (b, c, d), e) + (a, (b, c, e), d) - (d, e, (c, b, a)) = 0,\end{aligned} \tag{21}$$
$$\begin{aligned}&\sum_{\sigma \in \mathcal{S}_5} \mathrm{sgn}(\sigma)\, [(\sigma(a), (\sigma(b), \sigma(c), \sigma(d)), \sigma(e))\\ &-3(\sigma(a), \sigma(b), (\sigma(c), \sigma(d), \sigma(e)))] = 0.\end{aligned} \tag{22}$$

Theorem 5.4 *Let F be a field with char$(F) = 0$. For any $\mathcal{B} \in \mathcal{VO}$, $\mathcal{B}^{(-)}$ is a ternary Maltsev algebra.*

Proof Let $\mathcal{B} \in \mathcal{VO}$. Consider the commutator algebra $\mathcal{B}^{(-)}$ of $\mathcal{B}$ by equipping $\mathcal{B}$ with the ternary commutator

$$(x_1, x_2, x_3)_c = \sum_{\sigma \in \mathcal{S}_3} sgn(\sigma)(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

By construction, $(\cdot, \cdot, \cdot)_c$ is skew-symmetric and, applying linearized forms of $(a, a, b) - (b, a, a) = 0$, we have

$$(a, b, c)_c = 3((a, b, c) - (c, b, a)). \tag{23}$$

The ternary Maltsev identity holds in $\mathcal{B}^{(-)}$ if and only if

$$\begin{aligned}&(((z, x_2, x_3)_c, x_2, x_3)_c, y_2, y_3)_c - (((z, y_2, y_3)_c, x_2, x_3)_c, x_2, x_3)_c\\ &= ((z, x_2, x_3)_c, (x_2, y_2, y_3)_c, x_3)_c + ((z, x_2, x_3)_c, x_2, (x_3, y_2, y_3)_c)_c\\ &+ ((z, (x_2, y_2, y_3)_c, x_3)_c, x_2, x_3)_c + ((z, x_2, (x_3, y_2, y_3)_c)_c, x_2, x_3)_c.\end{aligned} \tag{24}$$

Following [13], we apply computational linear algebra based on the representation theory of the symmetric group $\mathcal{S}_7$ in order to prove that (24) holds in $\mathcal{B}^{(-)}$.

Each identity $I = I(a_1, \cdots, a_7)$ can be written as a sum $I = \sum_{i=1}^{t} I^{(i)}$ over the t distinct inequivalent association types i in level 3. Since each $I^{(i)}$ is an element of the group ring $F\mathcal{S}_7$, the whole identity can be seen as an element of the direct sum of t copies of $F\mathcal{S}_7$, that is, $I \in \oplus_{i=1}^{t} F\mathcal{S}_7^{(i)}$. From the representation theory of the symmetric group $\mathcal{S}_7$, we know that each group ring $F\mathcal{S}_7$ is the direct sum of full matrix rings corresponding to the partitions λ of 7, and we have

$$I \in \bigoplus_{i=1}^{t} \bigoplus_{\lambda} \mathcal{M}_{d_\lambda}^{(i)}(F),$$

where d_λ is the number of standard tableaux associated to λ. For each partition λ, we can extract the t components of I in the matrix rings corresponding to λ, and so the identity I can be seen as a sum of components indexed by λ, that is,

$$I = \sum_{\lambda} I^{(\lambda)}, \ I^{(\lambda)} \in \bigoplus_{i=1}^{t} \mathcal{M}_{d_\lambda}^{(i)}(F).$$

Each component $I^{(\lambda)}$ is represented by a matrix $R(I, \lambda)$ of size $d_\lambda \times td_\lambda$. Left multiplications in the group ring are represented by row operations in the matrix rings. So, the reduced row-echelon form of $R(I, \lambda)$ gives a canonical form for the component $I^{(\lambda)}$. The rank of this matrix is called the rank of the identity I in the representation λ.

This construction can be extended to any finite set of identities $S = \{I_1, \ldots, I_k\}$. For each identity I_j we obtain a matrix of size $d_\lambda \times td_\lambda$. If we stack the k matrices, we obtain a matrix $R(S, \lambda)$ of size $kd_\lambda \times td_\lambda$. The vertical blocks of size $kd_\lambda \times d_\lambda$ of this matrix $R(S, \lambda)$ represent the component in each association type of the representation λ generated by the k identities. We can consider the row canonical form of $R(S, \lambda)$, whose rank is called the rank of the set S of identities in the representation λ.

We will see that, for each partition λ of 7, the rank of the set S of all identities defining $\mathcal{B}^{(-)}$ is equal to the rank of the set $S \cup \{I\}$, where I is the Maltsev identity (24). Notice that S is formed by the liftings of the identities (18) to (22), plus the level 3 identities satisfied by the commutator $(\cdot, \cdot, \cdot)_c$: anticommutative and expansion identities.

Thus, we have to lift the level 1 identity (18) to a level 2 identity. Since there are three association types for the ternary nonassociative operation $(\cdot, \cdot, \cdot)$:

$$1 : ((a, b, c), d, e), \ 2 : (a, (b, c, d), e), \ 3 : (a, b, (c, d, e)),$$

we have that a multilinear identity $I(a, b, c)$ can be lifted to a level 2 identity in six different ways:

$$\begin{gathered} I((a, d, e), b, c), \ I(a, (b, d, e), c), \ I(a, b, (c, d, e)), \\ (I(a, b, c), d, e), \ (d, I(a, b, c), e), \ (d, e, I(a, b, c)). \end{gathered}$$

For level 3, we have to consider a total of 14 association types. The first 12 correspond to the nonassociative ternary operation $(\cdot,\cdot,\cdot)$ and are the following ones:

$$\begin{array}{lll} 1: (((a,b,c),d,e),f,g), & 2: ((a,(b,c,d),e),f,g), & 3: ((a,b,(c,d,e)),f,g), \\ 4: (a,((b,c,d),e,f),g), & 5: (a,(b,(c,d,e),f),g), & 6: (a,(b,c,(d,e,f),g)), \\ 7: (a,b,((c,d,e),f,g)), & 8: (a,b,(c,(d,e,f),g)), & 9: (a,b,(c,d,(e,f,g))), \\ 10: ((a,b,c),(d,e,f),g)), & 11: ((a,b,c),d,(e,f,g)), & 12: (a,(b,c,d),(e,f,g)). \end{array}$$

The other two come from the skew-symmetric ternary commutator $(\cdot,\cdot,\cdot)_c$ and are:

$$13: (((a,b,c)_c,d,e)_c,f,g)_c,\ 14: ((a,b,c)_c,(d,e,f)_c,g)_c.$$

Given a partition λ of 7, let us suppose that the dimension of the corresponding irreducible representation of $\mathcal{S}_7$ is d_λ. For each multilinear identity of level 2, $I(a,b,c,d,e)$, there are eight different ways to lift this identity to a level 3 identity:

$$\begin{array}{l} I((a,f,g),b,c,d,e),\ I(a,(b,f,g),c,d,e),\ I(a,b,(c,f,g),d,e), \\ I(a,b,c,(d,f,g),e),\ I(a,b,c,d,(e,f,g)),\ (f,g,I(a,b,c,d,e)), \\ (f,I(a,b,c,d,e),g),\ (I(a,b,c,d,e),f,g). \end{array}$$

The liftings of the identities (18) to (22), which define the algebra $\mathcal{B} \in \mathcal{VO}$, give $k = 18$ level 3 identities represented by a matrix of size $18d_\lambda \times 14d_\lambda$ with a zero submatrix of size $18d_\lambda \times 2d_\lambda$ corresponding to the association types 13 and 14.

The skew-symmetric ternary commutator satisfies $k = 7$ trivial identities, which come from the anticommutativity of $(\cdot,\cdot,\cdot)_c$. They are the following ones:

$$\begin{array}{l} (((a,b,c)_c,d,e)_c,f,g)_c + (((b,a,c)_c,d,e)_c,f,g)_c = 0, \\ (((a,b,c)_c,d,e)_c,f,g)_c + (((a,c,b)_c,d,e)_c,f,g)_c = 0, \\ (((a,b,c)_c,d,e)_c,f,g)_c + (((a,b,c)_c,e,d)_c,f,g)_c = 0, \\ (((a,b,c)_c,d,e)_c,f,g)_c + (((a,b,c)_c,d,e)_c,g,f)_c = 0, \\ ((a,b,c)_c,(d,e,f)_c,g)_c + ((b,a,c)_c,(d,e,f)_c,g)_c = 0, \\ ((a,b,c)_c,(d,e,f)_c,g)_c + ((a,c,b)_c,(d,e,f)_c,g)_c = 0, \\ ((a,b,c)_c,(d,e,f)_c,g)_c + ((d,e,f)_c,(a,b,c)_c,g)_c = 0. \end{array}$$

Notice that these identities are valid since $\mathrm{char}(F) \neq 2$. They are represented by a matrix of size $7d_\lambda \times 14d_\lambda$ with a zero submatrix of size $7d_\lambda \times 12d_\lambda$ for association types 1-12.

From (23), we have that the expansion of the two noncommutative types, 13 and 14, in terms of the other nonassociative types, 1–12, gives the following two identities (notice that $\mathrm{char}(F) \neq 3$):

$$\begin{aligned} (((a,b,c)_c,d,e)_c,f,g)_c = 27[&(((a,b,c),d,e),f,g) - (g,f,((a,b,c),d,e)) \\ &-((e,d,(a,b,c)),f,g)) + (g,f,(e,d,(a,b,c))) \\ &-(((c,b,a),d,e),f,g) + (g,f,((c,b,a),d,e)) \\ &+((e,d,(c,b,a)),f,g) - (g,f,(e,d,(c,b,a)))], \end{aligned}$$

$$((a,b,c)_c,(d,e,f)_c,g)_c = 27[((a,b,c),(d,e,f),g)-(g,(d,e,f),(a,b,c)))\\ -((a,b,c),(f,e,d),g)+(g,(f,e,d),(a,b,c))\\ -((c,b,a),(d,e,f),g)+(g,(d,e,f),(c,b,a))\\ -((c,b,a),(f,e,d),g)+(g,(f,e,d),(c,b,a))].$$

They are represented by a matrix of size $2d_\lambda \times 14d_\lambda$.

Finally, we only have to consider the linearized form of the Maltsev identity (24). It is represented by a matrix of size $d_\lambda \times 14d_\lambda$ whith a zero submatrix of size $d_\lambda \times 12d_\lambda$ for the first 12 nonassociation types.

Table 2 summarizes the results obtained for the different partitions λ of 7.

Table 2 Level 3 identities

i	λ	d_λ	$\mathcal{B}$	$\mathcal{B}^{(-)}$	Maltsev	New
1	[7]	1	11	13	13	0
2	[61]	6	71	83	83	0
3	[52]	14	167	195	195	0
4	[511]	15	180	210	210	0
5	[43]	14	167	195	195	0
6	[421]	35	420	490	490	0
7	[4111]	20	240	280	280	0
8	[331]	21	252	294	294	0
9	[322]	21	252	294	294	0
10	[3211]	35	420	490	490	0
11	[31111]	15	180	210	210	0
12	[2221]	14	168	196	196	0
13	[22111]	14	168	196	196	0
14	[211111]	6	72	84	84	0
15	[1111111]	1	11	13	13	0

Column 3 gives the dimension of the corresponding representation of $\mathcal{S}_7$. In columns 4 and 5 we have the rank of the identities defining $\mathcal{B}$ and $\mathcal{B}^{(-)}$, respectively. Column 6 gives the rank of the identities defining $\mathcal{B}^{(-)}$ plus the linearized form of the Maltsev identity. Finally, column 7 gives the difference between columns 5 and 6. Since this last column is zero, we can deduce that the Maltsev identity is generated by all identities defining $\mathcal{B}^{(-)}$, and so that it holds in $\mathcal{B}^{(-)}$. All calculations were made using Maple. □

Corollary 5.5 *Let F be a field with char$(F) \neq 2, 3$. For any $\mathcal{B} \in \mathcal{VO}$, $\mathcal{B}^{(-)}$ is a ternary Maltsev algebra.*

Proof The result follows from Theorem 5.4 and Remark 2.1. □

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Declarations

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