Inducing pivot variables and non-centrality parameters in elliptical distributions

Dário Ferreira, Sandra S. Ferreira, Célia Nunes, and Sónia Inácio

Citation: AIP Conference Proceedings **1558**, 833 (2013); doi: 10.1063/1.4825625
View online: http://dx.doi.org/10.1063/1.4825625
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1558?ver=pdfcov
Published by the AIP Publishing

Inducing Pivot Variables and Non-centrality Parameters in Elliptical Distributions

Dário Ferreira*, Sandra S. Ferreira*, Célia Nunes* and Sónia Inácio†

*Department of Mathematics and Center of Mathematics, University of Beira Interior, Covilhã, Portugal
†Department of Mathematics and CMA, Faculty of Science and Technology, New University of Lisbon, Portugal

Abstract. We used inducing pivot variables to derive confidence intervals for the non-centrality parameters of samples with elliptical errors. A numerical application is presented.

Keywords: Pivot variables, inference, variance components
AMS: 62k15, 62E15,62H10, 62H15, 62J10

INTRODUCTION

Inducing pivot variables are functions of statistics and parameters with known distributions which induce probability measures. As it may be seen in [2], these variables may be used to carry out inference. to derive confidence intervals for the non centrality parameters of samples with elliptical errors.

The remainder of this article is arranged as follows. In the next section we present results on spherical and elliptical distributions. Next we introduce the concept of inducing pivot variable and show how to construct confidence intervals and test hypothesis for non-centrality parameters of samples with elliptical errors. Finally we present a numerical application.

SPHERICAL AND ELLIPTICAL DISTRIBUTIONS

Let $\boldsymbol{X}$ be a $k \times 1$ random vector and its support be the set of k-dimensional real vectors

$$R(\boldsymbol{X}) = \mathbb{R}^k. \tag{1}$$

Consider $\boldsymbol{\mu}$ a $k \times 1$ vector and $\boldsymbol{V}$ a $k \times k$ symmetric and positive definite matrix, that is,

$$\boldsymbol{V} = \boldsymbol{V}', \tag{2}$$

where $\boldsymbol{V}'$ denotes the transpose of matrix $\boldsymbol{V}$, and $\boldsymbol{S}'\boldsymbol{V}\boldsymbol{S}$ is positive for any non-zero column vector $\boldsymbol{S}$ of k real numbers. Vector $\boldsymbol{X}$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{V}$ if its joint density function is

$$f(\boldsymbol{X}) = (2\pi)^{-\frac{\boldsymbol{K}}{2}} |det(\boldsymbol{V})|^{-\frac{1}{2}} exp\left(-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{V}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right). \tag{3}$$

We indicate that $\boldsymbol{X}$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{V}$ by

$$\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{V}). \tag{4}$$

The k random variables $X_1, ..., X_k$ constituting the vector $\boldsymbol{X}$ are said to be jointly normal.

The class of spherical distributions is an extension of the class of multivariate normal distributions. Vector $\boldsymbol{X}$ is said to have a spherical distribution if $\boldsymbol{X}$ and $\boldsymbol{QX}$ have the same distribution for all orthogonal $k \times k$ matrices $\boldsymbol{Q}$, see [4]. If $\boldsymbol{X}$ is a continuous random vector with a spherical distribution, then due to the equality

$$\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}, \tag{5}$$

11th International Conference of Numerical Analysis and Applied Mathematics 2013
AIP Conf. Proc. 1558, 833-836 (2013); doi: 10.1063/1.4825625

its density function depends on the argument $\boldsymbol{X}$ through the value of $\boldsymbol{X}'\boldsymbol{X}$ and is of the form $g(\boldsymbol{X}'\boldsymbol{X})$ for some non-negative function $g(\cdot)$. A study about the density of a spherical distribution may be seen, for example in [1]. If two random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same distribution we use the notation $\boldsymbol{X} \overset{d}{=} \boldsymbol{Y}$.

Let the k-vector $\boldsymbol{X}$ be spherically distributed. Then $\boldsymbol{X}$ has the stochastic representation

$$\boldsymbol{X} \overset{d}{=} \boldsymbol{R}\boldsymbol{U}, \tag{6}$$

where $\boldsymbol{U}$ is uniformly distributed on the unit sphere, $\boldsymbol{R} \sim F(\boldsymbol{X})$ is independent of $\boldsymbol{U}$, and $F(\boldsymbol{X})$ is a distribution function over $[0, +\infty[$, see again [4]. The random variable $\boldsymbol{R}$ may be looked upon as a radius. If $\boldsymbol{X} \overset{d}{=} \boldsymbol{R}\boldsymbol{U}$ and $P(\boldsymbol{X} = \boldsymbol{0}) = 0$ then $\|\boldsymbol{X}\| \overset{d}{=} \boldsymbol{R}$, where $\|\cdot\|$ denotes the usual Euclidian norm.

Vector $\boldsymbol{X}$ is said to have an elliptical distribution, with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{V}$, if $\boldsymbol{X}$ has the same distribution as $\boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{Y}$, where $\boldsymbol{Y}$ has a spherical distribution and $\boldsymbol{A}$ is a $k \times p$ matrix, with

$$\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{V} \tag{7}$$

and $rank(\boldsymbol{V}) = p$.

The multivariate normal distribution belongs to the class of elliptical distributions since, if $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{V})$, the vector $\boldsymbol{X}$ can be represented as $\boldsymbol{X} = \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{Y}$, where $\boldsymbol{Y} \sim N(\boldsymbol{0}, \boldsymbol{I})$ and $\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{V}$.

INDUCING PIVOT VARIABLES

As already stated in the introduction, pivot variables are functions of statistics and parameters with known distributions. For example, if S is distributed as the product by γ of a central chi-square with g degrees of freedom, $S \sim \gamma\chi^2_g$, then

$$Z = \frac{S}{\gamma} \tag{8}$$

is distributed as a central chi-square with g degrees of freedom, being therefore a pivot variable. Now, let $\mathscr{B}_r$ be the σ-algebra of the borelian sets in $\mathbb{R}^r$, see [7], and the parameter space $\Theta \in \mathscr{B}_r$. According to [3] the pivot variable

$$\boldsymbol{Z} = g(\boldsymbol{Y}, \boldsymbol{\theta}) \tag{9}$$

is an inducing pivot variable if, for any realization $\boldsymbol{y}$ of $\boldsymbol{Y}$ the function

$$l(\boldsymbol{\theta}|\boldsymbol{y}) = g(\boldsymbol{y}, \boldsymbol{\theta}) \tag{10}$$

has an inverse measurable function $h(\boldsymbol{z}|\boldsymbol{y})$ in $\mathscr{B}_r$.

In the next section we will use inducing pivot variables to derive confidence intervals and test hypothesis for non-centrality parameters when we are dealing with elliptical distributions.

INFERENCE FOR NON-CENTRALITY PARAMETERS

Consider the random vector

$$\boldsymbol{X} \sim N(\boldsymbol{\mu}, \theta\boldsymbol{I}) \tag{11}$$

independent of S, which is distributed as the product by θ of a central chi-square with g degrees of freedom,

$$S \sim \theta\chi^2_g. \tag{12}$$

If we consider

$$W = \frac{\|\boldsymbol{X}\|^2 - \|\boldsymbol{\mu}\|^2}{2\|\boldsymbol{\mu}\|}, \tag{13}$$

with $\|\boldsymbol{\mu}\| \to \infty$, then W will be approximately normally distributed, with mean 0 and variance θ, $W \overset{a}{\sim} N(0, \theta)$, and independent of S, see [5] and [6]. Therefore $\mathscr{F} = g\frac{W^2}{S}$ will have F distribution with 1 and g degrees of freedom and non-centrality parameter δ, $\mathscr{F} \sim F(.|1, g, \delta)$, where

$$\delta = \frac{\|\boldsymbol{\mu}\|^2}{\theta}. \tag{14}$$

Moreover, if

$$T = \sqrt{g}\frac{W}{\sqrt{S}}, \tag{15}$$

T will have t-distribution, with g degrees of freedom, $T \sim t_g$.

Now, if $\boldsymbol{x}$, s, t and w are realizations of $\boldsymbol{X}$, S, T and W respectively, from (13) and (15) we will have

$$\frac{\|\boldsymbol{x}\|^2 - \|\boldsymbol{\mu}\|^2}{2\|\boldsymbol{\mu}\|\sqrt{s}} = \frac{t}{\sqrt{g}}, \tag{16}$$

which is equivalent to having

$$\|\boldsymbol{\mu}\|^2 + \frac{2t\sqrt{s}\|\boldsymbol{\mu}\|}{\sqrt{g}} - \|\boldsymbol{x}\|^2 = 0. \tag{17}$$

Solving (17) in order to $\|\boldsymbol{\mu}\|$, we get

$$\|\boldsymbol{\mu}\| = \frac{-t\sqrt{s}}{\sqrt{g}} + \sqrt{\frac{t^2 s}{g} + \|\boldsymbol{x}\|^2} = -w + \sqrt{w^2 + \|\boldsymbol{x}\|^2}, \tag{18}$$

so

$$\|\boldsymbol{\mu}\|^2 = \left(-w + \sqrt{w^2 + \|\boldsymbol{x}\|^2}\right)^2. \tag{19}$$

If we consider the pivot variable $Z = \frac{S}{\theta} \sim \chi_g^2$ and $W \sim N(0, \theta)$, we may induce probability measures for

$$\delta = \frac{\|\boldsymbol{\mu}\|^2}{\theta} = \frac{\left(-w + \sqrt{w^2 + \|\boldsymbol{x}\|^2}\right)^2}{\frac{s}{Z}}, \tag{20}$$

using large samples $D = \{D_1, ..., D_n\}$, with

$$D_u = \frac{\left(-W_u + \sqrt{W_u^2 + \|\boldsymbol{x}\|^2}\right)^2}{\frac{s}{Z_u}}, \quad u = 1, ..., n, \tag{21}$$

where $Z_u \sim \chi_g^2$, $W_u \sim N(0, \theta_u)$ and $\theta_u = \frac{s}{Z_u}$, $u = 1, ..., n$.

Let $F(x)$ be the distribution of D, x_p the quantile for probability p, F_n be the empirical distribution of D and $x_{n,p}$ the F_n quantile for probability p. Now, using the reverse Glivenko-Cantelli theorem, see [2], we may estimate the quantiles x_p of F_x from the quantiles $x_{n,p}$ of F_n. Therefore, we may construct confidence intervals $[x_{n,\frac{\alpha}{2}}; x_{n,1-\frac{\alpha}{2}}]$; $[0; x_{n,1-\alpha}]$ and $[x_{n,\alpha}; +\infty[$ for δ, with (estimated) confidence level $1-\alpha$. These confidence intervals allow us to test hypothesis

$$H_0 : \delta = \delta_0 \tag{22}$$

against

$$H_{1,j} : \delta \neq \delta_0; \;\; \delta > \delta_0 \;\text{ and }\; \delta < \delta_0. \tag{23}$$

We reject the test hypothesis if δ_0 is not contained in the corresponding confidence interval. Thus, by duality, we obtain tests with (approximate) confidence level α.

NUMERICAL APPLICATION

In order to obtain confidence intervals for δ, defined in (14), when the random vector $\boldsymbol{X}$, in (11), and the sum of squares for the error, S, in (12), are known, we used the R software to generate $Z_u \sim \chi_g^2$ and $\boldsymbol{X}_u$, with components $X_{u,1}, ..., X_{u,10}$, considering

$$X_{u,1}, ..., X_{u,9} \sim N(0, \theta_u) \;\text{ and }\; X_{u,10} \sim N(\sqrt{\theta_u}, \theta_u),$$

where $\theta_u = \frac{s}{Z_u}$ and $u = 1, ..., 10000$. Note that we considered $\delta = 1$ since $\|\boldsymbol{\mu}\|^2 = \theta$.

Next we generated $W_u \sim N(0, \theta_u)$, $u = 1, ..., 10000$, and obtained the sample $D = \{D_1, ..., D_{10000}\}$, with

$$D_u = \frac{\left(-W_u + \sqrt{W_u^2 + \|\boldsymbol{x}\|^2}\right)^2}{\frac{s}{Z_u}}, \;\; u = 1, ..., 10000.$$

The obtained values for $x_{n,0.025}$ and $x_{n,0.975}$ are presented in Table 1 and Table 2, respectively. These tables also show the considered values for S and for g.

TABLE 1. Obtained values for $x_{n,0.025}$.

	S											
g	**0.5**	**1**	**2**	**4**	**8**	**16**	**32**	**64**	**128**	**256**	**512**	**1024**
1	0.8301	0.8289	0.8377	0.8348	0.8257	0.8368	0.8290	0.8311	0.8384	0.8350	0.8378	0.8294
2	0.8373	0.8381	0.8325	0.8385	0.8362	0.8398	0.8387	0.8539	0.8418	0.8399	0.8387	0.8427
4	0.8414	0.8340	0.8385	0.8308	0.8312	0.8310	0.8444	0.8344	0.8303	0.8367	0.8374	0.8251
8	0.8367	0.8389	0.8496	0.8301	0.8354	0.8405	0.8252	0.8274	0.8364	0.8363	0.8368	0.8232
16	0.8323	0.8331	0.8200	0.8271	0.8334	0.8378	0.8453	0.8353	0.8457	0.8285	0.8321	0.8205
32	0.8340	0.8310	0.8399	0.8280	0.8411	0.8232	0.8272	0.8368	0.8435	0.8349	0.8421	0.8334
64	0.8313	0.8339	0.8447	0.8421	0.8301	0.8461	0.8338	0.8373	0.8219	0.8434	0.8302	0.8263
128	0.8350	0.8408	0.8409	0.8412	0.8342	0.8341	0.8339	0.8372	0.8349	0.8313	0.8279	0.8376

TABLE 2. Obtained values for $x_{n,0.975}$.

	S											
g	**0.5**	**1**	**2**	**4**	**8**	**16**	**32**	**64**	**128**	**256**	**512**	**1024**
1	1.1923	1.1796	1.1856	1.1927	1.1933	1.1848	1.1872	1.1903	1.1892	1.1971	1.1927	1.1812
2	1.1854	1.1845	1.1841	1.1826	1.1970	1.1821	1.1854	1.1884	1.1687	1.1765	1.1832	1.1758
4	1.1898	1.1895	1.1793	1.1665	1.1955	1.1748	1.1906	1.1815	1.1955	1.1723	1.1732	1.1932
8	1.1841	1.1824	1.1796	1.1862	1.1863	1.1825	1.1881	1.1694	1.2004	1.1834	1.1811	1.1786
16	1.2057	1.1712	1.1827	1.1990	1.1922	1.1839	1.1814	1.1886	1.1905	1.1846	1.1865	1.1850
32	1.1992	1.1846	1.1892	1.1801	1.1831	1.1900	1.1796	1.1982	1.1854	1.1912	1.1892	1.2043
64	1.1784	1.1795	1.1709	1.1850	1.1784	1.1867	1.1734	1.1852	1.1889	1.1896	1.1873	1.2018
128	1.1935	1.1877	1.1905	1.1927	1.1821	1.1751	1.1697	1.1993	1.1891	1.1745	1.1788	1.1836

These tables point to the stability of the quantiles we chosen. So, the presented method of constructing confidence intervals, for non-centrality parameters, is usefull and apply when we have elliptical density for the error.

ACKNOWLEDGMENTS

This work was partially supported by the Center of Mathematics, University of Beira Interior through the project PEst-OE/MAT/UI0212/2011 and by CMA, Faculty of Science and Technology, New University of Lisbon, through the project PEst- OE/MAT/UI0297/2011.

REFERENCES

1. Fang, K.-T. and Zhang, Y.-T. *Generalized Multivariate Analysis*. Science Press, Beijing, Springer-Verlag, Berlin 1990.
2. Ferreira, D, Ferreira, Sandra, S, Nunes, C, Fonseca, M. and Mexia, J. T. (2013). *Chisquared and Related Inducing Pivot Variables: An Application to Orthogonal Mixed Models*. Communications in Statistics - Theory and Methods. To appear
3. Fonseca, M., Mexia, J. T. and Zmyślony, R. (2007). *Jordan algebras, generating pivot variables and orthogonal normal models*. Journal of Interdisciplinary Mathematics, **1**, 305–326.
4. Kollo, T. and von Rosen, D. *Advanced Multivariate Statistics with Matrices*. Springer-Verlag, Berlin 2005.
5. Mexia J.T. and Oliveira M.M. (2010). *Asymptotic linearity and limit distributions, approximations*. J. Stat. Plan. Infer. **140** (2), 353–357.
6. Nunes C., Oliveira M.M. and Mexia J.T. (2013). *Application domains for the Delta method.* Statistics. **47**(2), 317–328.
7. Williams, D. *Probability with Martingales*. Cambridge University Press 1991.