Convergence of asymptotic systems with unbounded delays with applications to Cohen-Grossberg neural networks and Lotka-Volterra systems

Ahmed Osama Mohamed Sayed Sayed Elmwafy

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Orientador: Prof. Doutor César Augusto Teixeira Marques da Silva
Co-orientador: Prof. Doutor José Joaquim Martins Oliveira

Júri:
Prof. Doutora Paulo Jorge da Silva Almeida
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PELO CONHECIMENTO E PELO TRABALHO, ASPIRAMOS ÀS COISAS MAIS ELEVADAS
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To Mona, and Mona.

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Abstract

This thesis studies the dynamics of two prominent classes of models: Cohen-Grossberg neural network (CGNN) and Lotka-Volterra ecological systems. Firstly, we investigate the global exponential stability and the existence of a periodic solution of a general differential equation with unbounded distributed delays. The main stability criterion depends on the dominance of the non-delay terms over the delay terms. The criterion for the existence of a periodic solution is obtained by applying the coincidence degree theorem. We use the main results to obtain criteria for the existence and global exponential stability of periodic solutions of a generalized higher-order periodic CGNN model with discrete-time varying delays and infinite distributed delays. Moreover, we study the convergence of asymptotic systems in nonautonomous CGNN models. We derive stability results under conditions where the non-delay terms asymptotically dominate the delay terms. In the second part, we explore Lotka–Volterra-type ecological models with delays. We investigate the concept of permanence of a general delay differential system and apply it to a general Lotka-Volterra type model. Moreover, we obtain a partial result on the convergence of the system to its asymptotic systems. Additionally, we provide a comparison with results in the literature and numerical examples to illustrate the effectiveness of some of our results.

Keywords

Functional differential equation; Cohen-Grossberg neural network; Lotka-Volterra model; unbounded delay; periodic solution; stability; convergence of models; asymptotic system.

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Resumo Alargado

Esta tese estuda a dinâmica de duas classes proeminentes de modelos: modelos de redes neurais de Cohen-Grossberg e sistemas ecológicos de Lotka-Volterra. Na primeira parte da tese, fornecemos condições suficientes para a estabilidade exponencial global e a existência de uma solução periódica do seguinte sistema diferencial geral com atrasos distribuídos sem limites Cohen-Grossberg de ordem superior com atrasos discretos e distribuídos não necessariamente limitados

$$x_i'(t) = a_i(t, x_i(t))\big[-b_i(t, x_i(t)) + f_i(t, x_t)\big], \quad t \geq 0,\ i = 1, \dots, n.$$

O critério para a existência de soluções periódicas é obtido através da aplicação do teorema do grau de coincidência. O principal critério de estabilidade depende da dominância dos termos sem atraso sobre os termos com atraso. Usamos os principais resultados para obter critérios para a existência de soluções periódicas e estabilidade exponencial global de um modelo de rede neural periódica generalizada num espaço de fase conveniente. Este modelo é suficientemente geral para incluir diferentes modelos de ordem baixa e alta. Além disso, estudamos a convergência de sistemas assintóticos nos seguintes modelos de redes neuronais não autónomas de Cohen-Grossberg

$$\begin{aligned} x_i'(t) = a_i(t, x_i(t))\Bigg[&-b_i(t, x_i(t)) + I_i(t) \\ &+ F_i\Bigg(\sum_{p=1}^{P}\sum_{j,l=1}^{n} c_{ijlp}(t) h_{ijlp}\big(x_j(t-\tau_{ijp}(t)), x_l(t-\tilde{\tau}_{ilp}(t))\big), \\ &\sum_{p=1}^{P}\sum_{j,l=1}^{n} d_{ijlp}(t) f_{ijlp}\Bigg(\int_{-\infty}^{0} g_{ijp}(x_j(t+s))\, d\eta_{ijp}(s), \\ &\int_{-\infty}^{0} \tilde{g}_{ilp}(x_l(t+s))\, d\tilde{\eta}_{ilp}(s)\Bigg)\Bigg)\Bigg], \quad t \geq 0,\ i = 1, \dots, n. \end{aligned}$$

em que $n, P, Q \in \mathbb{N}$ e $a_i : [0, +\infty) \times \mathbb{R} \to (0, +\infty)$, $b_i : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $c_{ijlp}, d_{ijlq}, I_i : [0, +\infty) \to \mathbb{R}$, $\tau_{ijp}, \tilde{\tau}_{ilp} : [0, +\infty) \to [0, +\infty)$, $h_{ijlp}, f_{ijlq} : \mathbb{R}^2 \to \mathbb{R}$, $F_i, G_i, g_{ijq}, \tilde{g}_{ilq} : \mathbb{R} \to \mathbb{R}$ são funções contínuas, e η_{ijq}, $\tilde{\eta}_{ilq} : (-\infty, 0] \to \mathbb{R}$ são funções limitadas não decrescentes tais que $\eta_{ijq}(0) - \eta_{ijq}(-\infty) = 1$ e $\tilde{\eta}_{ilq}(0) - \tilde{\eta}_{ilq}(-\infty) = 1$, para cada $i, j, l = 1, \dots, n$, $p = 1, \dots, P$, e $q = 1, \dots, Q$.

Notamos que este modelo é suficientemente geral para englobar modelos de redes neurais de Cohen-Grossberg de ordem baixa e alta, modelos de Hopfield e modelos estáticos. Obtemos resultados de estabilidade sob condições em que os termos sem atraso dominam assimptoticamente os termos com atraso.

Na segunda parte da tese, introduzimos o conceito de permanência e introduzimos condições suficientes para a permanência do seguinte sistema diferencial com atrasos ilimitados

$$x_i'(t) = x_i(t)\left[g_i(t, x_i(t)) - f_i(t, x_t)\right], \quad t \geq 0,\ i = 1, \ldots, n,$$

num espaço de fase conveniente onde $x_i(t)$ representa a densidade populacional da i-ésima espécie no tempo t. Além disso, aplicamos o nosso resultado de permanência ao seguinte modelo geral do tipo Lotka-Volterra

$$\begin{aligned} x_i'(t) &= x_i(t)\Bigg[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)h_{ij}\big(x_j(t)\big) - \sum_{j=1}^{n} b_{ij}(t)f_j\big(x_j(t-\tau_{ij}(t))\big) \\ &\quad - \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{t} D_{ij}(t,\rho)\theta_j(x_j(\rho))d\rho\Bigg], \quad t \geq 0,\ i = 1, \ldots, n, \end{aligned}$$

onde $h_{ij}, f_{ij}, \theta_{ij}, \tau_{ij} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $D_{ij} : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+$ são funções contínuas, e $r_i, a_{ij}, b_{ij}, c_{ij} \in C(\mathbb{R}, \mathbb{R}_0^+)$, para $i, j = 1, \cdots, n$. Devido ao significado biológico, as condicões inciais serão tomadas num conjunto espaço de funções não negativas. Adicionalmente, obtemos um resultado parcial sobre convergência do modelo geral de Lotka-Volterra para os seus sistemas assintóticos. Por último, mas não menos importante, apresentamos uma comparação com resultados da literatura e exemplos numéricos para ilustrar a eficácia de alguns dos nossos resultados.

Palavras-chave

Equação diferencial funcional; rede neural de Cohen-Grossberg; modelo de Lotka-Volterra; atraso não limitado; solução periódica; estabilidade; convergência de modelos; sistema assintótico.

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Acronyms and Initialisms

UBI	Universidade da Beira Interior
CMA	Centro de Matemática e Aplicações
RFDE	Retarded Functional Differential Equation
DDE	Delay Differential Equation
CGNN	Cohen-Grossberg Neural Network

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List of Symbols

$BC(X, Y)$	Space of all bounded continuous mappings from X to Y.
Dom L	domain of a mapping L
Ker_L	kernel of a mapping L
Im_L	image of a mapping L
dim_L	dimension of a mapping L
$codim_L$	codimension of a mapping L
∂U	boundary of a set U
$\overline{U}$	closure of a set U
$A \oplus B$	direct sum of two sets A and B
$\deg_B$	Brouwer degree

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Chapter 1

Introduction

Delay differential equations (DDEs) are indispensable tools for analysis and prediction via mathematical modeling in various areas of life sciences such as neural networks [69], epidemiology [39], immunology [45] and population dynamics [55]. These systems often involve delays due to many factors depending on the process we are modeling such as the maturation time in ecological systems, the incubation time in infectious diseases, or the transmission time of signals between neurons in neural network models.

Most of the results obtained before the era of Volterra regarding DDEs were concerned with special properties of very specific equations. Since Volterra [58] has started his research on predator-prey models and viscoelasticity, he has formulated some general functional differential equations that account for the past states of the system. Due to the considerations of meaningful models in engineering, studying the DDEs has attracted many researchers. Studying the behavior of solutions of DDEs, particularly understanding their stability and convergence properties (see [28, 37, 54] and references therein), is critical for predicting and controlling dynamics of real-world systems. This was enough motivation to focus on the convergence of asymptotic systems with unbounded delays, with applications to two prominent classes of models: Cohen-Grossberg neural network (CGNN) models and Lotka-Volterra ecological systems.
The first part of this thesis, namely Chapter 2, is divided into two parts: the first concerns studying the stability of general non-autonomous systems with distributed delays and the second is about studying the convergence of asymptotic systems in non-autonomous CGNN models, which include both infinite discrete time-varying and distributed delays.

In the first part of Chapter 2, in Section 2.1, we study a general differential system with delays. Many researchers have been developing work on general systems with delay [7, 23, 53, 65].

In [23], Faria established sufficient conditions for the asymptotic and exponential stability of non-autonomous linear differential systems with infinite distributed delays. Yskak, in [65], studied the stability of solutions of a certain periodic linear differential system with infinite distributed delays. In [7], Berezansky and Braverman established sufficient conditions for the global attractivity of the following non-autonomous neural networks with distributed delays

$$\begin{aligned} x_i'(t) = g_i(t)\Bigg[& -x_i(t) \\ & + \int_{h_{i1}(t)}^{t} \cdots \int_{h_{in}(t)}^{t} f_i(x_1(\tau_1), \dots, x_n(\tau_n))\, d_{\tau_1} r_{i1}(t,\tau_1) \cdots d_{\tau_n} r_{in}(t,\tau_n) \Bigg], \end{aligned} \tag{1.1}$$

for $i = 1, \dots, n$, and applied it to Bidirectional Associative Memory (BAM) neural network models. In order to apply to BAM models as well, Oliveira [53] obtained sufficient conditions for the asymptotic stability of the following general system with infinite distributed delays

$$x_i'(t) = -a_i(t)x_i(t - \tau_i(t)) + h_i\big(t, x(t - \tau_{i1}(t)), \dots, x(t - \tau_{im}(t))\big) + f_i(t, x_t), \tag{1.2}$$

for $i = 1, \dots, n$. None of the general systems, (1.1) or (1.2), can be applied to CGNN type models.

The neural network introduced and studied by Cohen and Grossberg [15] can be described by the following system of ordinary differential equations,

$$\frac{dx_i(t)}{dt} = -a_i(x_i(t))\big[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(x_j(t)) + I_i\big], \ t \geq 0, \ i = 1, \dots, n$$

where n is a natural number that indicates the number of neurons and, for each $i, j = 1, \dots, n$, $x_i(t)$ is the ith neuron state at time t, $a_i(.)$ denotes the amplification functions, $b_i(.)$ is the self-signal functions, $f_j(.)$ is the activation functions, c_{ij} represents the strengths of connectivity between neurons i and j, and, I_i denotes the input from outside of the system.
Differential equations modeling neural networks should incorporate time delays due to synaptic transmission time between neurons or, in the case of artificial neural networks, communication time among amplifiers, in order to be more realistic. However, it is known that delays lead to oscillation or instability [44], thus it is worth studying stability of neural network models with delays.

In recent decades, many authors have studied low-order neural network models. However, it is important to investigate high-order neural network models because, in comparison with low-order systems, they exhibit stronger approximation properties, faster convergence speeds, and higher fault tolerance [26, 60].
Recently, some research has been conducted on the stability of high-order neural network models, with many of them concerning discrete-time models, and the majority of these studies focusing on the Hopfield-type models (see [6, 9, 33, 52, 60] and references therein). In [16], criteria for the exponential stability of a discrete-time high-order

Cohen-Grossberg model were established. Subsequently, in [17], the situation involving impulses was studied. As pointed out by Mohamad and Golpassamy [48], the dynamics of continuous-time and discrete-time models are not necessarily similar. Therefore, it is worthwhile to investigate continuous-time high-order CGNN models.
In [64], the stability of a fractional high-order CGNN model with delays was examined. However, this study was limited to the autonomous situation with finite delays. Recently, Aouiti and Driti [3] obtained sufficient conditions for the existence and global exponential stability of an almost automorphic solution of a general Cohen-Grossberg BAM neural network model that can include high-order models. Nevertheless, the model studied in [3] is not general enough to include the non-autonomous continuous-time high-order CGNN with unbounded delays studied in [40]. In [40], the authors claimed to have obtained a criterion for the existence and exponential stability of a periodic solution for the studied model. However, as we explain in Remark 2.2.10, their proof contains certain issues.

Motivated by the discussion mentioned above, Chapter 2 consists of two main parts based on the results we obtained in [19, 20]. In [20], we focus on deriving criteria for the existence of periodic solutions and global exponential stability for a general class of DDEs with unbounded delays, these results are then applied to high-order CGNN models. In [19], we investigate the global convergence of asymptotic systems in non-autonomous CGNN models, providing new insights into the behavior of solutions with infinite delays.

As referred, in the first part of chapter 2, namely, in Section 2.1, we study the existence of periodic solutions and global exponential stability for a general class of DDEs with unbounded delays, in general settings, which includes as a particular situation, many of the low-order and high-order Cohen-Grossberg type models recently studied. Namely, we study the global stability of the following general non-autonomous differential system with infinite delays

$$x_i'(t) = a_i(t, x_i(t))\big[-b_i(t, x_i(t)) + f_i(t, x_t)\big], \quad t \geq 0,\, i = 1, \dots, n, \tag{1.3}$$

in a convenient phase space. To address the issues encountered in the proof of [40, Theorem 3.1], we establish a general and easily verifiable criterion for the existence of a periodic solution of (1.3). In particular, this criterion is applicable to the model investigated in [40].

Previously, in [24], a criterion for the existence and exponential stability of an equilibrium of

$$x_i'(t) = a_i(t, x_t)\big[-b_i(x_i(t)) + f_i(x_t)\big], \quad t \geq 0,\, i = 1, \dots, n,$$

was established. In [38], a criterion for the existence and exponential stability of an anti-periodic solution for the delay system (1.3) with impulses on time scales was obtained. However, despite the fact that the current study is conducted without impulses on the time scale $\mathbb{R}$, a less restrictive setting is assumed. For instance, in [38], functions a_i must be even, and b_i and f_i must be odd in the second variable - symmetry conditions that are not required here.

Despite the main goal being to apply the obtained results to Cohen-Grossberg type models, system (1.3) is general enough to be applied to other models such as Lotka-Volterra type models.

In the second part of Chapter 2, namely Subsection 2.2.2, we study the global convergence of systems describing CGNN type models. Since their introduction, CGNN models have become a prominent subject of investigation. The dynamical properties of CGNN models, such as stability, instability, and periodic oscillation, have been extensively studied for both theoretical and practical applications. For instance, in [18], the author examined the global convergence of the model to ensure that the trajectories of the network do not exhibit chaotic behavior, thereby maintaining its functionality as an associative memory or optimization solver. Furthermore, the globally convergent dynamics indicate that the neural network algorithm will ensure convergence to an optimal solution from any initial guess when used as an optimization solver.

Several significant studies have been conducted in this area. For an autonomous CGNN model, sufficient conditions were given for the coexistence and local stability of multiple equilibrium points [41]. In [3], the authors established sufficient conditions for the existence and global exponential stability of an almost automorphic solution of a general Cohen–Grossberg Bidirectional Associative Memory neural network with finite discrete delays and infinite distributed delays. In [13], the authors studied the stability and instability of an equilibrium point of a CGNN model with infinite discrete time-varying delays. The existence of multi-periodic solutions of a generalized CGNN was studied in [59]. The existence and exponential stability of an almost periodic solution of a CGNN with infinite distributed delays were explored in [62]. In [12], the global exponential stability of a periodic solution of a delayed CGNN was investigated but in the case of discontinuous activation functions. Additionally, a stochastic CGNN with delays was studied in [61], and criteria for the exponential stability of a discrete time high-order CGNN model with impulses were established in [17].

To our knowledge, there are few studies on the global convergence of systems in the context of neural network models [50, 63, 66, 67, 72]. As illustrated by a simple example in [67], the dynamic behavior of a system is generally not determined by the dynamics of its asymptotic systems (see Definition 2.2.2). Therefore, it is valuable to identify situations where the dynamics of a system can be inferred by studying the dynamics of one of its asymptotic systems. In [50, 63, 67, 72], sufficient conditions are provided to ensure the global convergence of systems in various delayed Hopfield neural network models. In [66], the convergence of systems was examined for the following low-order CGNN with finite discrete time-varying delays:

$$x_i'(t) \;=\; a_i(x_i(t))\Big[-b_i(t,x_i(t))+\sum_{j=1}^{n}c_{ij}(t)f_j\big(x_j(t)\big)+\sum_{j=1}^{n}b_{ij}(t)f_j(t-\tau_{ij}(t))+I_i(t)\Big],$$

where the coefficients asymptotically converge to periodic functions. Motivated by these studies, in Subsection 2.2.2, the convergence of systems within the following generalized high-order CGNN model with both infinite discrete time-varying and distributed delays:

$$\begin{aligned} x_i'(t) = a_i(t,x_i(t))\Big[&-b_i(t,x_i(t))+I_i(t)\\ &+F_i\Big(\sum_{p=1}^{P}\sum_{j,l=1}^{n}c_{ijlp}(t)h_{ijlp}\big(x_j(t-\tau_{ijp}(t)),x_l(t-\tilde{\tau}_{ilp}(t))\big),\\ &\quad\sum_{p=1}^{P}\sum_{j,l=1}^{n}d_{ijlp}(t)f_{ijlp}\Big(\int_{-\infty}^{0}g_{ijp}(x_j(t+s))\,d\eta_{ijp}(s),\\ &\qquad\int_{-\infty}^{0}\tilde{g}_{ilp}(x_l(t+s))\,d\tilde{\eta}_{ilp}(s)\Big)\Big)\Big],\quad t\geq 0,\ i=1,\dots,n, \end{aligned} \tag{1.4}$$

is obtained, where $n, P, Q \in \mathbb{N}$ and $a_i : [0,+\infty)\times\mathbb{R}\to(0,+\infty)$, $b_i : [0,+\infty)\times\mathbb{R}\to\mathbb{R}$, $c_{ijlp}, d_{ijlq}, I_i : [0,+\infty)\to\mathbb{R}$, $\tau_{ijp}, \tilde{\tau}_{ilp} : [0,+\infty)\to[0,+\infty)$, $h_{ijlp}, f_{ijlq} : \mathbb{R}^2\to\mathbb{R}$, $F_i, G_i, g_{ijq}, \tilde{g}_{ilq} : \mathbb{R}\to\mathbb{R}$ are continuous functions, and $\eta_{ijq}, \tilde{\eta}_{ilq} : (-\infty,0]\to\mathbb{R}$ are non-decreasing bounded functions such that $\eta_{ijq}(0)-\eta_{ijq}(-\infty)=1$ and $\tilde{\eta}_{ilq}(0)-\tilde{\eta}_{ilq}(-\infty)=1$, for each $i,j,l=1,\dots,n$, $p=1,\dots,P$, and $q=1,\dots,Q$.
We note that model (1.4) is sufficiently general to encompass both low-order [66] and high-order [20] CGNN models, Hopfield models [50], and static models [49].

The main objective is to achieve the global stability of model (1.4) through the stability of one of its asymptotic systems, which may be autonomous, periodic, or almost periodic, and thus easier to study than the original system (1.4). For instance, model (1.4) itself may not be periodic, but it can have a periodic asymptotic system. In this case, an important goal is to provide sufficient conditions to ensure that all solutions of (1.4) converge to a periodic function assuming that the asymptotic system has a globally attractive periodic solution.

The main novelties in this chapter are as follows:

i. The global exponential stability criterion, Theorem 2.1.3, obtained for the general nonautonomous delay differential equation (1.3). We emphasize the proof method employed, which, without resorting to Lyapunov functionals, enabled the derivation of a new stability criterion. This criterion is easily verifiable, (see condition (2.3) or (2.11)), and applicable to a broad range of neural network models as seen in Section 2.2. Notice that this criterion can be applied to nonautonomous situations where we do not have periodicity.

ii. The provided criterion for the existence of periodic solutions of (1.3), Theorem 2.1.9. We stress that the established criterion is easily verifiable, as outlined in hypothesis (H5*). More, unlike the conditions in [38], where only anti-periodic solutions are considered, our criterion does not need the knowledge of the upper limit magnitude, $\beta_i^*(t)$, in hypothesis (H3*). We stress that this study allowed us to solve the problems in [40].

iii. The results presented for the new general neural network model (2.31), in Section 2.2, which includes a broad range of low-order and high-order Cohen-Grossberg, Hopfield, and static neural network type models found in the existing literature [3, 40, 42, 49, 68]. We highlight the generalization of the main result in [40], as derived by Corollary 2.2.9.

In the second part of this thesis, namely in Chapter 3, in Section 3.1, we discuss the concept of permanence for the following general delay differential system

$$x_i'(t) = x_i(t)\left[g_i(t, x_i(t)) - f_i(t, x_t)\right], \quad t \geq 0,\ i = 1, \dots, n, \tag{1.5}$$

in a convenient phase space where $x_i(t)$ represents the population density of the i^{th} species at time t. A biological system is called sustainable if it posses two important properties, namely, ultimate boundedness and permanence. The first one refers to the existence of a compact domain within the state space of the system, where each motion enters it within a finite time and remains in this domain thereafter, the principle of persistence indicates that during the species interaction, species do not become extinct; regardless of their initial population sizes, from a specific point in time, their populations will surpass certain predetermined positive values. A system that possesses both of these features is called permanent [32].
The concept of permanence for delay differential equations has been attracted many researchers (see [2, 22, 34, 57] and references therein). Permanence conditions for biological models described by ordinary differential equations are well investigated (see for example [10, 11, 36, 43] and the references therein). However, it is important

to recognize that many realistic models must account for additional factors such as delays. For example, delays may arise due to the dependence of the growth rates of some species on the past states of others, variations in reproduction and mortality rates across different age groups, or similar temporal dependencies (see for example [2, 22, 34, 57] and references therein). In [34], the authors investigated the following nonautuonomous $n-$species Lotka-Volterra model with impulses

$$x_i'(t) = x_i(t)\bigg(a_i(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t)\bigg). \tag{1.6}$$

Moreover, in [35], sufficient conditions on the permanence and global attractivity are established for the following nonautuonomous Lotka-Volterra model with unbounded delays and impulses

$$x_i'(t) = x_i(t)\bigg(a_i(t) - b_{ii}(t)x_i(t) - \sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij}(t)\int_{-\infty}^{0} k_i(s)x(t+s)ds\bigg). \tag{1.7}$$

In [5], a nonlinear Lotka-Volterra systems with discrete delays, of the form

$$\begin{aligned} u_i'(t) \quad &= \quad u_i(t)\bigg[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)u_{ij}(t) - \sum_{j=1}^{n} b_{ij}(t)f_j\big(u_j(t)\big) \\ &\qquad - \sum_{j=1}^{n} c_{ij}(t)g_j\big(u_j(t-\delta_j(t))\big)\bigg], \qquad t \geq 0,\, j = 1,\dots,n, \end{aligned} \tag{1.8}$$

was studied. Last but not least, as example of research focusing on studying the dynamics of general Lotka-Volterra models, in [21], a criterion for the global attractivity of a positive equilibrium of the following $n-$dimensional non-autonomous Lotka–Volterra systems with distributed delays was given

$$x'(t) = r_i(t)x_i(t)\big[\alpha_i - b_i x_i(t) - \sum_{j=1}^{n} l_{ij}\int_{-\tau}^{0} x_j(t+\theta)d\eta_{ij}(\theta)\big], \qquad i = 1,\dots,n. \tag{1.9}$$

Motivated by the above discussion, we introduce sufficient conditions for the permanence of the general delay differential system (1.5), which include both discrete and distributed delays, in Theorem 3.1.2.
Moreover, in Subsection 3.2.1, we apply our permanence result to the Lotka-Volterra

type models

$$
\begin{aligned}
x_i'(t) = x_i(t)\Bigg[& r_i(t) - \sum_{j=1}^{n} a_{ij}(t) h_{ij}\big(x_j(t)\big) - \sum_{j=1}^{n} b_{ij}(t) f_{ij}\big(x_j(t-\tau_{ij}(t))\big) \\
& - \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{0} D_{ij}(t,s)\theta_{ij}(x_j(t+s))\, ds \Bigg], \quad t \geq 0,\, i = 1, \dots, n,
\end{aligned}
\tag{1.10}
$$

where due to the biological significance of (1.10), the framework is restricted only to positive or non-negative initial conditions. We assume that $h_{ij}, f_{ij}, \theta_{ij}, \tau_{ij} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $D_{ij} : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+$ are continuous functions, and $r_i, a_{ij}, b_{ij}, c_{ij} \in C(\mathbb{R}, \mathbb{R}_0^+)$, for $i, j = 1, \dots, n$. This model is more general than (1.6), (1.7), and (1.8) in Corollary 3.2.1. Additionally, we obtain a partial result relating the dynamics of this model to the dynamic of one of its asymptotic systems in Theorem 3.2.3.

Finally, in Subsection 2.2.3 and Section 3.3, we provide some numerical examples in order to illustrate the effectiveness of some of our results.

Some results of Chapter 2 are already published in [20], and other contents of Chapter 2 are included in preprint [19].

Chapter 2

Cohen-Grossberg neural network models

In the beginning of the present chapter, we investigate the global exponential stability and the existence of a periodic solution of a general differential equation with unbounded delays. The main stability criterion depends on the dominance of the non-delay terms over the delay terms. The criterion for the existence of a periodic solution is obtained with the application of the coincidence degree theorem. We use the main results to derive criteria for the existence and global exponential stability of periodic solutions of a generalized higher-order periodic CGNN model with infinite discrete-time varying and distributed delays.
Additionally, we investigate the convergence of asymptotic systems in non-autonomous CGNN models. Moreover, we provide a comparison with the results in the literature and a numerical simulation to illustrate the effectiveness of some of our results.

2.1 A general system with unbounded delays

In the present chapter, firstly, we study the global stability of the general non-autonomous differential system with infinite delays (1.3) in a convenient phase space. For $n \in \mathbb{N}$, we consider the n-dimensional vector space $\mathbb{R}^n$ equipped with the norm

$$|x| = \max\{|x_i|,\, i = 1, \dots, n\}, \text{ for } x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

For a positive real number ε, we consider the Banach space

$$UC_\varepsilon^n = \Big\{\phi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s \le 0} \frac{|\phi(s)|}{\mathrm{e}^{-\varepsilon s}} < +\infty, \\ \frac{\phi(s)}{\mathrm{e}^{-\varepsilon s}} \text{ is uniformly continuous on } (-\infty, 0]\Big\},$$

equipped with the norm

$$\|\phi\|_\varepsilon = \sup_{s \le 0} \frac{|\phi(s)|}{\mathrm{e}^{-\varepsilon s}}.$$

In [31], a basic theory about the existence, uniqueness, and continuation of solutions is established for the general functional differential equation in the phase space UC_ε^n

$$x'(t) = f(t, x_t), \quad t \ge 0, \tag{2.1}$$

where, for an open set $D \subseteq UC_\varepsilon^n$, the function $f : [0, +\infty) \times D \to \mathbb{R}^n$ is continuous, and for $t \geq 0$, x_t denotes the function $x_t : (-\infty, 0] \to \mathbb{R}^n$ defined by $x_t(s) = x(t+s)$ for $s \leq 0$.
We denote by $x(t, t_0, \phi)$ a solution of (2.1) with initial condition $x_{t_0} = \phi$ for $t_0 \geq 0$ and $\phi \in D$.
For $x \in \mathbb{R}^n$, we also use x to denote the constant function $\phi(s) = x$ in UC_ε^n. A vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is said to be positive if $x_i > 0$ for all $i = 1, \ldots, n$ and we denote this by $x > 0$.
We introduce the Banach space BC of all continuous bounded functions $\phi : (-\infty, 0] \to \mathbb{R}^n$ equipped with the norm $\|\phi\| = \sup_{s \leq 0} |\phi(s)|$. It is clear that $BC \subseteq UC_\varepsilon^n$ and we have $\|\phi\|_\varepsilon \leq \|\phi\|$ for all $\phi \in BC$.
In the phase space UC_ε^n, for $n \in \mathbb{N}$ and $\varepsilon > 0$, we consider the differential system with infinite delays (1.3) where $a_i : [0, +\infty) \times \mathbb{R} \to (0, \infty)$, $b_i : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, and $f_i : [0, +\infty) \times UC_\varepsilon^n \to \mathbb{R}$ are continuous functions.
The goal is to apply the results to CGNN models, thus we only consider bounded initial conditions. i.e.

$$x_{t_0} \; = \phi, \text{ for } \phi \in BC \text{ and } t_0 \geq 0. \tag{2.2}$$

The continuity of a_i, b_i, and f_i functions assures that the initial value problem (1.3)-(2.2) has a solution (see [30, Theorem 2.1]).

It is generally known that the Banach space BC is not a convenient phase space for (2.1) according to [27, 30], and so the typical results about existence, uniqueness, and continuous dependency of solutions are not accessible. This is the reason why we considered the space UC_ε^n in the discussion above. Thus, more generally, we consider the space

$$\begin{aligned} UC_{\mathcal{G}}^n = \Bigg\{ &\phi \in C\big((-\infty, 0]; \mathbb{R}^n\big) : \sup_{s \leq 0} \frac{|\phi(s)|}{\mathcal{G}(s)} < +\infty, \\ &\frac{\phi(s)}{\mathcal{G}(s)} \text{ is uniformly continuous on } (-\infty, 0] \Bigg\}, \end{aligned}$$

equipped with the norm $\|\phi\|_{\mathcal{G}} = \sup_{s \leq 0} \frac{|\phi(s)|}{\mathcal{G}(s)}$, where $\mathcal{G} : (-\infty, 0] \to [1, +\infty)$ is a function satisfying the following conditions:

($\mathcal{G}_1$) $\mathcal{G}$ is a non-increasing continuous function with $\mathcal{G}(0) = 1$;

($\mathcal{G}_2$) $\lim_{u \to 0^-} \frac{\mathcal{G}(u+s)}{\mathcal{G}(s)} = 1$ uniformly on $(-\infty, 0]$;

($\mathcal{G}_3$) $\mathcal{G}(s) \to +\infty$ as $s \to -\infty$.

Note that UC^n_ε is the space $UC_\mathcal{G}$ with $\}(s) = \mathrm{e}^{-\varepsilon s}$. For more details, see [30]. As $BC \subseteq UC^n_\mathcal{G}$, then we consider the space BC with the norm $\|.\|_\mathcal{G}$. Considering the functional differential equation (2.1) in the phase space $UC^n_\mathcal{G}$, for a function $\mathcal{G}$ that satisfies $(\mathcal{G}_1)$-$(\mathcal{G}_3)$ as mentioned above, the continuity of the function f assures the existence of solutions of (2.1) with initial condition (2.2) for $t_0 \geq 0$ and $\phi \in UC^n_\mathcal{G}$ [30]. As we always consider bounded initial conditions, in this chapter, we consider the following definition of global exponential stability.

Definition 2.1.1. *The system* (1.3) *is said to be globally exponentially stable if there are* $\delta > 0$ *and* $C \geq 1$ *such that*

$$|x(t, t_0, \phi) - x(t, t_0, \psi)| \;\; \leq \;\; C\mathrm{e}^{-\delta(t-t_0)}\|\phi - \psi\|,$$

$\forall t_0 \geq 0$, $\forall t \geq t_0$, $\forall \phi, \psi \in BC$.

It should be emphasized that the preceding definition of global exponential stability is the one usually considered in the literature on neural networks (see [70, 71]).

2.1.1 Global exponential stability

We obtain sufficient conditions for the global exponential stability of (1.3). To do that, we assume the following hypotheses.
For each $i = 1, \ldots, n$:

(H1) there are $\underline{a}_i, \overline{a}_i > 0$ such that

$$\underline{a}_i < a_i(t, u) < \overline{a}_i, \qquad \forall t \geq 0, \, \forall u \in \mathbb{R};$$

(H2) the function a_i is differentiable, and there exists a continuous function D_i : $[0, +\infty) \to \mathbb{R}$ such that

$$D_i(t)a_i^2(t, u) \leq \frac{\partial a_i}{\partial t}(t, u), \qquad \forall t > 0, \, \forall u \in \mathbb{R};$$

(H3) there exists a function $\beta_i : [0, +\infty) \to (0, +\infty)$ such that

$$\frac{b_i(t, u) - b_i(t, v)}{u - v} \geq \beta_i(t), \qquad \forall t \geq 0, \, \forall u, v \in \mathbb{R}, \, u \neq v;$$

(H4) the function $f_i : [0, +\infty) \times UC^n_\varepsilon \to \mathbb{R}$ is Lipschitz on its second variable i.e., there is a continuous function $\mathcal{L}_i : [0, +\infty) \to [0, +\infty)$ such that

$$|f_i(t, \phi) - f_i(t, \psi)| \leq \mathcal{L}_i(t)||\phi - \psi||_\varepsilon, \qquad \forall t \geq 0, \, \forall \phi, \psi \in BC;$$

(H5) for all $t \geq 0$,

$$\underline{a}_i\big(\beta_i(t) + D_i(t)\big) - \overline{a}_i \mathcal{L}_i(t) > \varepsilon. \tag{2.3}$$

We state the generalized Gronwall's inequality so we will use to show that solutions of (1.3) are defined on $\mathbb{R}$.

Lemma 2.1.1. *[29, Lemma 6.2] If ϕ, α are two real valued function and continuous for $a \leq t \leq b$, $\beta(t) \geq 0$ is integrable on $[a, b]$ and*

$$\phi(t) \leq \alpha(t) + \int_a^t \beta(s)\phi(s)ds, \quad a \leq t \leq b,$$

then

$$\phi(t) \leq \alpha(t) + \int_a^t \beta(s)\alpha(s)e^{\int_s^t \beta(u)du}ds, \quad a \leq t \leq b.$$

We prove that solutions of (1.4) are defined on $\mathbb{R}$.

Lemma 2.1.2. *Assume (H1), (H3), and (H4). For $\phi \in BC$ and $t_0 \geq 0$, solution $x(t) = x(t, t_0, \phi)$ of* (1.3) *is defined on $\mathbb{R}$.*

Proof. Let $x(t) = (x_1(t), \dots, x_n(t))$ be the maximal solution of the initial value problem (1.3)-(2.2), such that $t \in (-\infty, a)$ for some $a \in (t_0, +\infty]$, and define $z(t) = (z_1(t), \dots, z_n(t)) := (|x_1(t)|, \dots, |x_n(t)|)$.
From (1.3) and (H1), we have the following

$$\begin{aligned} z_i'(t) &= \operatorname{sign}(x_i(t))x_i'(t) \\ &\leq -\operatorname{sign}(x_i(t))a_i(t, x_i(t))\big[b_i(t, x_i(t)) - f_i(t, x_t)\big] \\ &\leq -\operatorname{sign}(x_i(t))a_i(t, x_i(t))\big[b_i(t, x_i(t)) - b_i(t, 0)\big] \\ &\quad + \overline{a}_i|f_i(t, x_t) - f_i(t, 0)| + \overline{a}_i|b_i(t, 0) + f_i(t, 0)|. \end{aligned}$$

From (H3) and (H4), and by integrating over $[t_0, t]$, we obtain

$$\begin{aligned} z_i(t) \leq\ & z_i(t_0) - \int_{t_0}^t \operatorname{sign}(x_i(u))a_i(u, x_i(u))\big[b_i(u, x_i(u)) - b_i(u, 0)\big]du \\ &+ \int_{t_0}^t \overline{a}_i|f_i(u, x_u) - f_i(u, 0)|du + \int_{t_0}^t \overline{a}_i|b_i(u, 0) + f_i(u, 0)|du, \end{aligned}$$

thus

$$
\begin{aligned}
z_i(t) &\le \|\phi\| - \int_{t_0}^{t} \operatorname{sign}(x_i(u)) a_i(u, x_i(u)) x_i(u) \beta_i(u) du \\
&\quad + \int_{t_0}^{t} \overline{a}_i \mathcal{L}_i(u) \|x_u\|_\varepsilon du + \int_{t_0}^{t} \overline{a}_i |b_i(u,0) + f_i(u,0)| du \\
&\le \|\phi\| + \int_{t_0}^{t} \overline{a}_i \mathcal{L}_i(u) \|x_u\|_\varepsilon du + \int_{t_0}^{t} \overline{a}_i |b_i(u,0) + f_i(u,0)| du \\
&\le \|\phi\| + \int_{t_0}^{t} \overline{\mathcal{L}}_i(u) \|x_u\|_\varepsilon du + \int_{t_0}^{t} \big(\overline{b}_i(u,0) + \overline{f}_i(u,0)\big) du, \qquad (2.4)
\end{aligned}
$$

where

$$\overline{\mathcal{L}}_i(t) = \max_i |\overline{a}_i \mathcal{L}_i(u)|, \overline{b}_i(u,0) = \max_i |\overline{a}_i b_i(u,0)|, \text{ and } \overline{f}_i(u,0) = \max_i |\overline{a}_i f_i(u,0)|.$$

Define a continuous function $\Phi : [t_0, +\infty) \to [0, +\infty)$ by

$$\Phi(t) = ||\phi|| + \int_{t_0}^{t} \big(\overline{b}_i(u,0) + \overline{f}_i(u,0)\big) du.$$

For $t \ge t_0$, using (2.4), we obtain

$$||z_t|| \le \Phi(t) + \int_{t_0}^{t} \overline{\mathcal{L}}_i(u) ||z_u|| du,$$

and by applying the generalized Gronwall's inequality (see Lemma 2.1.1), we obtain

$$||z_t|| \le \Phi(t) + \int_{t_0}^{t} \overline{\mathcal{L}}_i(u) \Phi(u) e^{\int_u^t \overline{\mathcal{L}}_i(v) dv} du. \qquad (2.5)$$

Then, by the continuation theorem [30, Theorem 2.4],

$$\lim_{t \to a} ||x_t|| = \lim_{t \to a} ||z_t|| = +\infty. \qquad (2.6)$$

Since the functions $\overline{\mathcal{L}}_i : \mathbb{R} \to [0, +\infty)$ and $\Phi : [t_0, +\infty) \to [0, +\infty)$ are continuous functions, from (2.5) and (2.6), we conclude that $a = +\infty$, i.e., solutions of the system (1.3) are defined on $\mathbb{R}$. □

Now, we are in a position to obtain the main stability criterion for system (1.3).

Theorem 2.1.3. *If (H1)-(H5) hold, then the system* (1.3) *is globally exponentially stable.*

Proof. Let $t_0 > 0$, $\phi = (\phi_1, \ldots, \phi_n) \in BC$, $\psi = (\psi_1, \ldots, \psi_n) \in BC$, and consider two solutions, $x(t) = x(t, t_0, \phi)$ and $y(t) = x(t, t_0, \psi)$, of (1.3), note that we denote by

$x(t, t_0, \phi)$, a solution of (1.3) with initial condition $x_{t_0} = \phi$ for $t_0 \geq 0$ and $\phi \in BC$. For each $t \geq t_0$, define

$$V(t) = V(t, t_0, x(\cdot), y(\cdot)) = (V_1(t), \ldots, V_n(t)) \in \mathbb{R}^n$$

by

$$V_i(t) := \mathrm{e}^{\varepsilon(t-t_0)} sign\big(x_i(t) - y_i(t)\big) \int_{y_i(t)}^{x_i(t)} \frac{1}{a_i(t,u)} du, \qquad i = 1, \ldots, n. \tag{2.7}$$

From (H1), we conclude that

$$\mathrm{e}^{-\varepsilon(t-t_0)} \underline{a}_i V_i(t) \leq |x_i(t) - y_i(t)| \leq \mathrm{e}^{-\varepsilon(t-t_0)} \overline{a}_i V_i(t), \tag{2.8}$$

$\forall t \geq t_0$, $i = 1, \ldots, n$.

Firstly, we show that

$$|V(t)| \leq \max_i \{\underline{a}_i^{-1}\} \|\phi - \psi\|, \qquad \forall t \geq t_0. \tag{2.9}$$

Obviously, from (2.8), we have

$$|V(t_0)| \leq \max_i \left\{\underline{a}_i^{-1} |x_i(t_0) - y_i(t_0)|\right\} \leq \max_i \{\underline{a}_i^{-1}\} \|\phi - \psi\|.$$

Now, to obtain a contradiction, we assume that inequality (2.9) is false. Consequently, there exists $t_1 > t_0$ such that

$$|V(t_1)| > \max_i \{\underline{a}_i^{-1}\} \|\phi - \psi\|.$$

Define

$$T := \min \left\{ t \in [t_0, t_1] : |V(t)| = \max_{s \in [t_0, t_1]} |V(s)| \right\}.$$

Choosing $i \in \{1, \ldots, n\}$ such that $V_i(T) = |V(T)|$, we have

$$V_i(T) > 0, \qquad V_i'(T) \geq 0, \qquad \text{and} \qquad V_i(T) > |V(t)|, \ \forall t \in [t_0, T). \tag{2.10}$$

From (1.3), and hypotheses (H2), (H3), and (H4), we obtain

$$\begin{aligned} V_i'(T) &= \varepsilon V_i(T) + \mathrm{e}^{\varepsilon(T-t_0)} \operatorname{sign}\big(x_i(T) - y_i(T)\big) \Bigg[\frac{1}{a_i(T, x_i(T))} x_i'(T) \\ &\quad - \frac{1}{a_i(T, y_i(T))} y_i'(T) + \int_{y_i(T)}^{x_i(T)} -\frac{\partial_t a_i(T,u)}{a_i^2(T,u)} \, du \Bigg], \end{aligned}$$

thus,

$$\begin{aligned}
V_i'(T) &= \varepsilon V_i(T) + \mathrm{e}^{\varepsilon(T-t_0)}\,\mathrm{sign}\big(x_i(T)-y_i(T)\big)\Big[b_i(T,y_i(T)) - b_i(T,x_i(T)) \\
&\qquad + f_i(T,x_T) - f_i(T,y_T) + \int_{y_i(T)}^{x_i(T)} -\frac{\partial_t a_i(T,u)}{a_i^2(T,u)}\,du\Big] \\
&\le \varepsilon V_i(T) + \mathrm{e}^{\varepsilon(T-t_0)}\Big[-\beta_i(T)|x_i(T)-y_i(T)| + \mathcal{L}_i(T)\|x_T - y_T\|_\varepsilon \\
&\qquad - D_i(T)|x_i(T)-y_i(T)|\Big].
\end{aligned}$$

Hypothesis (H5) implies $\beta_i(T) + D_i(T) > 0$, and from (2.8), we obtain

$$\begin{aligned}
V_i'(T) &\le \varepsilon V_i(T) - \underline{a}_i\big[\beta_i(T) + D_i(T)\big]V_i(T) \\
&\qquad + \mathrm{e}^{\varepsilon(T-t_0)}\mathcal{L}_i(T)\max\Big\{\sup_{s\le t_0-T}|x(T+s)-y(T+s)|\mathrm{e}^{\varepsilon s}, \\
&\qquad \sup_{t_0-T<s\le 0}|x(T+s)-y(T+s)|\mathrm{e}^{\varepsilon s}\Big\} \\
&\le \varepsilon V_i(T) - \underline{a}_i\big[\beta_i(T) + D_i(T)\big]V_i(T) \\
&\qquad + \mathrm{e}^{\varepsilon(T-t_0)}\mathcal{L}_i(T)\max\Big\{\|\phi-\psi\|\mathrm{e}^{\varepsilon(t_0-T)}, \\
&\qquad \sup_{t_0-T<s\le 0}|x(T+s)-y(T+s)|\mathrm{e}^{\varepsilon s}\Big\}.
\end{aligned}$$

By (2.8), we obtain

$$\begin{aligned}
V_i'(T) &\le \varepsilon V_i(T) - \underline{a}_i\big[\beta_i(T) + D_i(T)\big]V_i(T) \\
&\qquad + \mathrm{e}^{\varepsilon(T-t_0)}\mathcal{L}_i(T)\max\Big\{\|\phi-\psi\|\mathrm{e}^{\varepsilon(t_0-T)}, \\
&\qquad \sup_{t_0-T<s\le 0}\mathrm{e}^{-\varepsilon(T+s-t_0)+\varepsilon s}\overline{a}_i V_i(T+s)\Big\} \\
&= \varepsilon V_i(T) - \underline{a}_i\big[\beta_i(T) + D_i(T)\big]V_i(T) \\
&\qquad + \overline{a}_i\mathcal{L}_i(T)\max\left\{\frac{\|\phi-\psi\|}{\overline{a}_i}, \sup_{t_0-T<s\le 0} V_i(T+s)\right\}.
\end{aligned}$$

By (2.10) and definitions of t_1 and T, we have

$$V_i'(T) \le \varepsilon V_i(T) - \underline{a}_i\big[\beta_i(T) + D_i(T)\big]V_i(T) + \overline{a}_i\mathcal{L}_i(T)V_i(T).$$

From (2.10) and (H5), we conclude that

$$V_i'(T) \le \big[\varepsilon - \underline{a}_i\big(\beta_i(T) + D_i(T)\big) + \overline{a}_i\mathcal{L}_i(T)\big]\,V_i(T) < 0,$$

which contradicts (2.10) and hence (2.9) holds.

From (2.8) and (2.9), we obtain

$$|x(t)-y(t)|\mathrm{e}^{\varepsilon(t-t_0)}\min_i\left\{\overline{a}_i^{-1}\right\}\leq|V(t)|\leq\max_i\{\underline{a}_i^{-1}\}\|\phi-\psi\|,$$

thus

$$|x(t)-y(t)|\leq C\mathrm{e}^{-\varepsilon(t-t_0)}\|\phi-\psi\|,\quad\forall t\geq t_0,$$

with $C=\frac{\max_i\{\underline{a}_i^{-1}\}}{\min_i\{\overline{a}_i^{-1}\}}=\frac{\max_i\{\overline{a}_i\}}{\min_i\{\underline{a}_i\}}\geq 1$, which shows that the system (1.3) is globally exponentially stable. □

We remark that hypothesis (H2) trivially holds (with $D_i(t)=0$ for all $t>0$) when all functions a_i that do not explicitly depend on time t, i.e. $a_i(t,u)=a_i(u)$ for all $i=1,\dots,n$ and $u\in\mathbb{R}$. Thus, under the assumption

(h5) For all $t\geq 0$ and $i=1,\dots,n$, we have

$$\underline{a}_i\beta_i(t)-\overline{a}_i\mathcal{L}_i(t)>\varepsilon, \tag{2.11}$$

the following stability result can be obtained for the system

$$x_i'(t)=a_i(x_i(t))\big[-b_i(t,x_i(t))+f_i(t,x_t)\big],\quad t\geq 0,\ i=1,\dots,n. \tag{2.12}$$

Corollary 2.1.4. *Assume (H1), (H3), (H4), and (h5) hold. Then, system* (2.12) *is globally exponentially stable.*

Proof. Hypothesis (H2) holds with $D(t)=0$, thus the result follows from Theorem 2.1.3. □

Now, consider the model studied in [51]

$$x_i'(t)=a_i(t,x_i(t))\left[-b_i(t,x_i(t))+\sum_{k=1}^{K}\sum_{j=1}^{n}f_{ijk}(t,x_{j_t})\right], \tag{2.13}$$

$t\geq 0$, $i=1,\dots,n$, where $n,K\in\mathbb{N}$, a_i and b_i are functions as in system (1.3) and $f_{ijk}:[0,+\infty)\times UC_\varepsilon^1\to\mathbb{R}$ are continuous functions for $i,j=1,\dots,n$ and $k=1,\dots,K$. We will also assume the following conditions:

(h4) for each $i,j=1,\dots,n$ and $k=1,\dots,K$, there exists a continuous function $\mathcal{F}_{ijk}:[0,+\infty)\to[0,+\infty)$ such that

$$|f_{ijk}(t,\phi)-f_{ijk}(t,\psi)|\leq\mathcal{F}_{ijk}(t)\|\phi-\psi\|_\varepsilon,\quad\forall t\geq 0,\ \phi,\psi\in UC_\varepsilon^1.$$

(h5') for all $t \geq 0$ and $i = 1, \ldots, n$, we have

$$\underline{a}_i\big(\beta_i(t) + D_i(t)\big) - \overline{a}_i \sum_{k=1}^{K} \sum_{j=1}^{n} \mathcal{F}_{ijk}(t) > \varepsilon.$$

As system (2.13) is a particular situation of (1.3), the following stability criterion holds.

Corollary 2.1.5. *Assume that (H1), (H2), (H3), (h4) and (h5') hold. Then system* (2.13) *is globally exponentially stable.*

Proof. System (2.13) is a particular situation of (1.3) with

$$f_i(t, \varphi) = \sum_{k=1}^{K} \sum_{j=1}^{n} f_{ijk}(t, \varphi_j), \quad \forall t \geq 0,\, \varphi = (\varphi_1, \ldots, \varphi_n) \in UC_\varepsilon^n.$$

From (h4), we know that (H4) holds with

$$\mathcal{L}_i(t) = \sum_{k=1}^{K} \sum_{j=1}^{n} \mathcal{F}_{ijk}(t), \quad \forall t \geq 0,\, i = 1, \ldots, n.$$

Moreover, (H5) reads as (h5'). Thus the result follows from Theorem 2.1.3. $\square$

Remark 2.1.6. *We remark that the exponential stability of* (2.13) *was proved in [51] under the assumptions (H1), (H2), (H3), (h4), and a condition similar to*

(h5") for all $t \geq 0$ and $i = 1, \ldots, n$, we have

$$\underline{a}_i\big(\beta_i(t) + D_i(t)\big) - \sum_{k=1}^{K} \sum_{j=1}^{n} \overline{a}_j \mathcal{F}_{ijk}(t) > \varepsilon. \tag{2.14}$$

We emphasize that conditions (h5') and (h5") are different, thus Corollary 2.1.5 presents a new exponential stability criterion for the system (2.13)*.*

2.1.2 Existence of periodic solution

In this section, we assume that (1.3) is a periodic system which means that $a_i(t, .)$, $b_i(t, .)$, and $f_i(t, .)$ are periodic functions with the same period and, we establish sufficient conditions for the existence of a periodic solution.
The existence of a periodic solution will be proved through Mawhin's Continuation Theorem [46]. Before stating the referred theorem, we need to recall some definitions and facts.

Definition 2.1.2. *Let X and Z be two Banach spaces.*
A linear mapping $L : \mathrm{Dom}\, L \subseteq X \to Z$ is called a Fredholm mapping of index zero *if* $\dim \mathrm{Ker}_L = \mathrm{codim}\, \mathrm{Im}_L < +\infty$ *and* Im_L *is closed in Z.*

Given a Fredholm mapping of index zero, $L : \text{Dom } L \subseteq X \to Z$, it is well known that there are continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\text{Im}_P = \text{Ker}_L$, $\text{Ker}_Q = \text{Im}_L = \text{Im}_{I-Q}$, $X = \text{Ker}_L \oplus \text{Ker}_P$ and $Z = \text{Im}_L \oplus \text{Im}_Q$. It follows that $L|_{\text{Dom } L \cap \ker_P} : \text{Dom } L \cap \ker_P \to \text{Im}_L$ is invertible. We denote the inverse of that map by K_P.

Definition 2.1.3. *Let U be an open bounded subset of X. We say that a continuous mapping $N : \overline{U} \subseteq X \to Z$ is L-compact on $\overline{U}$ if the set $QN(\overline{U})$ is bounded and the mapping $K_P(I-Q)N : \overline{U} \subseteq X \to X$ is compact.*

Theorem 2.1.7 (Mawhin's Continuation Theorem)**.** *Let X be a Banach space and $\Omega \subseteq X$ an open bounded set. Suppose $L : \text{Dom } L \subset X \to X$ is a Fredholm mapping of index zero and that $N : \overline{\Omega} \to X$ is L-compact on $\overline{\Omega}$. Moreover, assume that all the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx, \quad \forall x \in \partial\Omega \cap \text{Dom } L,\ \lambda \in (0,1)$;

(ii) $QNx \neq 0, \quad \forall x \in \partial\Omega \cap \text{Ker}_L$;

(iii) $\deg_B\{QN, \Omega \cap \text{Ker}_L, 0\} \neq 0$, *where* $\deg_B$ *denotes the Brouwer degree.*

Then, the equation $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

For studying the system (1.3) in case of being periodic, the following hypotheses will be considered:

(H1*) For each $i = 1, \dots, n$, there exist $\overline{a}_i\,, \underline{a}_i > 0$ such that

$$\underline{a}_i < a_i(t,u) < \overline{a}_i, \quad \forall t \geq 0,\ u \in \mathbb{R};$$

(H2*) There is $\omega > 0$ such that, for each $i = 1, \dots, n$,

$$\begin{aligned} &a_i(t,u) = a_i(t+\omega, u), \quad b_i(t,u) = b_i(t+\omega,u), \\ &f_i(t,\phi) = f_i(t+\omega,\phi), \end{aligned}$$

for all $t \geq 0$, $u \in \mathbb{R}$, and $\phi \in BC$;

(H3*) For each $i = 1, \dots, n$, there exist $\omega-$periodic continuous functions $\beta_i, \beta_i^* : [0,+\infty) \to (0,+\infty)$ such that

$$\beta_i(t) \leq \frac{b_i(t,u) - b_i(t,v)}{u-v} \leq \beta_i^*(t), \quad \forall t \in [0,\omega], \forall u,v \in \mathbb{R},\ u \neq v;$$

(H4*) For each $i = 1, \dots, n$, there exists a $\omega-$periodic continuous function $\mathcal{L}_i : [0,+\infty) \to [0,+\infty)$ such that

$$|f_i(t,\phi) - f_i(t,\psi)| \leq \mathcal{L}_i(t)\|\phi - \psi\|, \quad \forall t \in [0,\omega], \forall \phi,\psi \in BC;$$

(H5*) For each $i = 1, \ldots, n$,

$$\beta_i(t) > \mathcal{L}_i(t), \quad \forall t \in [0, \omega].$$

From (H2*), we conclude that for fixed $u \in \mathbb{R}$ and $\phi \in BC$, the continuous functions $t \mapsto b_i(t, 0)$ and $t \mapsto f_i(t, 0)$ are ω-periodic and therefore bounded. From (H3*), we also conclude that β_i are bounded away from zero and β_i^* are bounded.

Defining

$$\begin{aligned} &\underline{\beta}_i := \min_{t\in[0,\omega]} \beta_i(t), \quad \overline{\beta}_i^* := \max_{t\in[0,\omega]} \beta_i^*(t), \\ &\overline{b}_i := \max_{t\in[0,\omega]} |b_i(t, 0)|, \quad \text{and} \quad \overline{f}_i := \max_{t\in[0,\omega]} |f_i(t, 0)|, \end{aligned} \tag{2.15}$$

we have $0 < \underline{\beta}_i, \overline{\beta}_i^*$, and $0 \leq \overline{b}_i, \overline{f}_i$.

We denote by X the Banach space

$$X = \big\{\phi \in C(\mathbb{R} : \mathbb{R}^n) : \phi \text{ is } \omega - \text{periodic}\big\},$$

with the norm $\|\phi\| = \sup_{t\in[0,\omega]} |\phi(t)|$, for $\phi \in X$.

For $\mathrm{Dom}_L = \{\phi \in X : \phi' \in X\} \subseteq X$, define the linear operator $L : \mathrm{Dom}_L \to X$ by

$$L\phi = \phi' \tag{2.16}$$

i.e., for all $t \in \mathbb{R}$ and $\phi(t) = (\phi_1(t), \ldots, \phi_n(t)) \in \mathrm{Dom}_L$, we have

$$\big(L\phi\big)(t) = (\phi_1'(t), \ldots, \phi_n'(t)).$$

It is not difficult to show that $\mathrm{Ker}_L \cong \mathbb{R}^n$ and

$$\mathrm{Im}_L = \left\{\phi = (\phi_1, \ldots, \phi_n) \in X : \int_0^\omega \phi_1(t)dt = \cdots = \int_0^\omega \phi_n(t)dt = 0\right\}, \tag{2.17}$$

with Im_L closed in X and $\dim \mathrm{Ker}_L = \operatorname{codim} \mathrm{Im}_L = n$, thus L is a Fredhom operator of index zero.

Now, we consider the projection $P : X \to X$ defined by

$$P\phi = \frac{1}{\omega}\int_0^\omega \phi(t)dt = \frac{1}{\omega}\left(\int_0^\omega \phi_1(t)dt, \ldots, \int_0^\omega \phi_n(t)dt\right), \tag{2.18}$$

$\forall \phi = (\phi_1, \ldots, \phi_n) \in X$. The projection P is continuous and, considering $Q\phi = P\phi$, we have $\mathrm{Im}_P = \mathrm{Ker}_L$, $\mathrm{Ker}_Q = \mathrm{Im}_L$, and the operator $L_{|\mathrm{Dom}_L \cap \mathrm{Ker}_P} : \mathrm{Dom}_L \cap \mathrm{Ker}_P \to \mathrm{Im}_L$ is invertible. We denote the inverse by K_P. By (2.16) and (2.17), we obtain that

$K_P\phi = \big((K_P\phi)_1, \cdots, (K_P\phi)_n\big)$ with

$$(K_P\phi)_i(t) = \int_0^t \phi_i(u)du - \frac{1}{\omega}\int_0^\omega \int_0^u \phi_i(s)dsdu, \tag{2.19}$$

$\forall \phi = (\phi_1, \ldots, \phi_n) \in \mathrm{Im}_L$, for $i = 1, \ldots, n$.
For a convenient bounded open set $\Omega \subseteq X$, define the function $N : \overline{\Omega} \to X$ by $N\phi = \big((N\phi)_1, \ldots, (N\phi)_n\big)$, where

$$(N\phi)_i(t) = a_i(t, \phi_i(t))\Big[- b_i(t, \phi_i(t)) + f_i(t, \phi_t)\Big], \tag{2.20}$$

for all $t \in \mathbb{R}$, $\phi = (\phi_1, \ldots, \phi_n) \in X$, and $i = 1, \ldots, n$.
We claim that, from the continuity of a_i, b_i, and f_i, (2.19) and (2.20), we can conclude that, for any $\alpha > 0$, the mapping N is L-compact in the set $\Omega = \{\phi \in X : \|\phi\| < \alpha\}$.
In fact, for any $x \in X$, from (H1*), (H3*), (H4*), (H5*), and (2.15), we have

$$\|QNx\| \le \max_i \left\{ \overline{a}_i \left[2\overline{\beta}_i^* \|x\| + \overline{b}_i + \overline{f}_i \right] \right\}$$

from which we conclude that $QN(\overline{\Omega})$ is bounded.
Additionally, we also need to show that the mapping $K_P(I - Q)N$ is compact. To achieve this, we show that for any bounded sequence $(\phi^m)_{m\in\mathbb{N}}$ in $\overline{\Omega}$, the sequence $(K_P(I - Q)N(\phi^m))_{m\in\mathbb{N}}$ has a convergent subsequence in X.
Defining the sequence $(\Phi^m)_{m\in\mathbb{N}}$ by

$$\Phi^m := K_P(I - Q)N(\phi^m), \quad m \in \mathbb{N},$$

from the definition of Q, (2.18), and (2.19), we obtain

$$(\Phi^m)_i(t) = \int_0^t (N\phi^m)_i(u)du - \frac{t}{\omega}\int_0^\omega (N\phi^m)_i(v)dv + (c^m)_i,$$

with

$$(c^m)_i := -\frac{1}{\omega}\int_0^\omega \int_0^u (N\phi^m)_i(s)ds + \frac{1}{2}\int_0^\omega (N\phi^m)_i(v)dv,$$

for all $t \in \mathbb{R}$ and $i = 1, \ldots, n$.
As ϕ^m and Φ^m are $\omega-$periodic functions, we do not lose generality considering

$$\begin{aligned} \tilde{\phi^m} &= \phi^m{}_{|[0,\omega]} \text{ and,} \\ \tilde{\Phi^m} &= \Phi^m{}_{|[0,\omega]}. \end{aligned}$$

For simplicity, we drop the tildes.
As $(\phi^m)_{m\in\mathbb{N}}$ is a bounded sequence, then $(N\phi^m)_{m\in\mathbb{N}}$ is also a bounded sequence.

Through easy computations, we also conclude that $(\Phi^m)_{m\in\mathbb{N}}$ is a bounded sequence. Defining $\overline{N} := \sup_m \|N\phi^m\|$, for any $s_1, s_2 \in [0,\omega]$, $m \in \mathbb{N}$, and $i = 1,\ldots,n$, we have

$$\begin{aligned} |(\Phi^m(s_1)) - (\Phi^m(s_2))| &= \left| sign(s_2 - s_1) \int_{s_1}^{s_2} (N\phi^m)_i(u)du + \frac{s_1 - s_2}{\omega} \int_0^{\omega} (N\phi^m)_i(u)du \right| \\ &\leq 2\overline{N}|s_1 - s_2|. \end{aligned}$$

Consequently, $\{\Phi^m : m \in \mathbb{N}\}$ is a bounded equicontinuous set in $C([0,\omega];\mathbb{R}^n)$, thus Ascoli-Arzela theorem allows us to conclude that $(\Phi^m)_{m\in\mathbb{N}}$ has a convergent subsequence, which proves that N is L-compact in Ω.
In view of (2.20) and (2.16), for $\lambda \in (0,1)$ and $x(t) = (x_1(t),\ldots,x_n(t)) \in X$, the operator equation $Lx = \lambda Nx$ is equivalent to the following equation:

$$x_i'(t) = \lambda a_i(t, x_i(t))\Big[- b_i(t, x_i(t)) + f_i(t, x_t)\Big], \quad \forall\lambda \in (0,1),\ i = 1,\ldots,n. \tag{2.21}$$

We mention the homotopy invariance notion (see [47]) because it plays a crucial rule in proving the existence of a periodic solution of the general differential system (1.3), this property is essential in satisfying the third condition of Theorem 2.1.7.

Definition 2.1.4. *[47] If $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $\overline{\Omega}$ is an open bounded subset, $p \in \mathbb{R}^n$ is a given point such that $f(x) \neq p$ for all $x \in \partial\Omega$, then the Brouwer degree $deg_B[f,\Omega,p]$ is an algebraic count of the number of zeros of f located in Ω.*

Lemma 2.1.8 (Homotopy Invariance Property)**.** *[47]*
Assume that $f_\varepsilon : \overline{\Omega} \to \mathbb{R}^n$ is a continuous homotopy such that $f_\varepsilon(x) \neq p$ for all $x \in \partial\Omega$ and $\varepsilon \in [0,1]$. Then the Brouwer degree $\deg(f_\varepsilon,\Omega,p)$ is invariant under this homotopy, i.e.,

$$\deg(f_0,\Omega,p) = \deg(f_1,\Omega,p).$$

Now, we are in a position to prove the existence of a periodic solution of the general differential system (1.3).

Theorem 2.1.9. *Suppose that (H1*), (H2*), (H3*), (H4*), and (H5*) hold. Then, system* (1.3) *has at least one $\omega-$periodic solution.*

Proof. Our objective is to apply Theorem 2.1.7. To accomplish this, it is needed to define a bounded open set $\Omega \subseteq X$ for which the conditions 1., 2., and 3. in Theorem 2.1.7 hold.
Let $x = x(t) = (x_1(t),\ldots,x_n(t))$ be an arbitrary $\omega-$periodic solution of equation (2.21). The components $x_i(t)$ of $x(t)$ are all continuously differentiable, thus, for each

$i = 1, \dots, n$, there is $t_i \in [0, \omega]$ such that

$$|x_i(t_i)| = \max_{t \in [0,\omega]} |x_i(t)|.$$

Hence $x_i'(t_i) = 0$ for all $i = 1, \dots, n$.

Choose $i \in \{1, \dots, n\}$ such that $|x_i(t_i)| = \max_{t \in [0,\omega]} |x(t)|$. Consequently, from (2.21), we have

$$b_i(t_i, x_i(t_i)) = f_i(t_i, x_{t_i}), \tag{2.22}$$

thus

$$\big|b_i(t_i, x_i(t_i)) - b_i(t_i, 0)\big| - |b_i(t_i, 0)| \leq \big|f_i(t_i, x_{t_i}) - f_i(t_i, 0)\big| + |f_i(t_i, 0)|.$$

By (H3*), (H4*), and (2.15), we obtain

$$\beta_i(t_i)|x_i(t_i)| - \overline{b}_i \leq \mathcal{L}_i(t_i)\|x_{t_i}\| + \overline{f}_i,$$

and, as $\|x_{t_i}\| = |x(t_i)| = |x_i(t_i)|$, we get

$$|x_i(t_i)| \left(1 - \frac{\mathcal{L}_i(t_i)}{\beta_i(t_i)}\right) \leq \frac{\overline{f}_i + \overline{b}_i}{\beta_i(t_i)} \leq \frac{\overline{f}_i + \overline{b}_i}{\underline{\beta}_i}.$$

As β_j and $\mathcal{L}_j$ are $\omega-$periodic functions, from (H5*) and (2.15), we can define

$$\overline{\xi} =: \max_{j,t} \left\{ \left(1 - \frac{\mathcal{L}_j(t)}{\beta_j(t)}\right)^{-1} \frac{\overline{f} + \overline{b}}{\underline{\beta}} \right\} + 1 > 0, \tag{2.23}$$

where $\overline{b} = \max_i \overline{b}_i$, $\overline{f} = \max_i \overline{f}_i$, and $\underline{\beta} = \min_i \underline{\beta}_i$. Thus we conclude that

$$|x_i(t_i)| < \overline{\xi}. \tag{2.24}$$

Consequently, $\|x\| < \overline{\xi}$, and taking

$$\Omega = \big\{\phi \in X : \|\phi\| < \overline{\xi}\big\}, \tag{2.25}$$

we conclude that the first condition of Theorem 2.1.7 is satisfied.

Now, we prove that the second condition of Theorem 2.1.7 holds.

Let $x = x(t) = (x_1(t), \dots, x_n(t)) \in \partial\Omega \cap \mathrm{Ker}_L$. As $\mathrm{Ker}_L \cong \mathbb{R}^n$, then $x(t)$ is identified with a constant vector in $\mathbb{R}^n$, i.e. $x(t) = (x_1, \dots, x_n)$, and by (2.25), we conclude that there is $i \in \{1,\dots,n\}$ such that $|x_i| = \overline{\xi}$. By (2.18) and (2.20), we have

$$(QNx)_i(t) = (QNx)_i = \frac{1}{\omega}\int_0^\omega a_i(u,x_i)\left[-b_i(u,x_i) + f_i(u,x)\right] du. \tag{2.26}$$

We claim that

$$|(QNx)_i| > 0. \tag{2.27}$$

By contradiction, we assume that $|(QNx)_i| = 0$. Then there is $t_i^* \in [0,\omega]$ such that

$$b_i(t_i^*, x_i) = f_i(t_i^*, x).$$

Reproducing the same computations above (see how (2.22) implies (2.24)), we conclude that

$$\overline{\xi} = |x_i| < \overline{\xi},$$

which is a contradiction. Consequently, (2.27) holds and the second condition of Theorem 2.1.7 is proved.

In order to prove the last condition of Theorem 2.1.7, we consider the continuous function $\Psi : (\Omega \cap \mathrm{Ker}_L) \times [0,1] \to X$ defined by $\Psi(x,\mu) = (\Psi(x,\mu)_1, \dots, \Psi(x,\mu)_n)$ with

$$\Psi(x,\mu)_i = -\mu\overline{a}_i\overline{\beta}_i^* x_i + (1-\mu)(QNx)_i,$$

for all $x = (x_1,\dots,x_n) \in \Omega \cap \mathrm{Ker}_L \cong \Omega \cap \mathbb{R}^n$, $\mu \in [0,1]$, and $i = 1,\dots,n$.

We claim that

$$|\Psi(x,\mu)| \neq 0, \quad \forall x \in (\partial\Omega) \cap \mathrm{Ker}_L,\ \mu \in [0,1]. \tag{2.28}$$

Consequently, defining $\bar{\Phi} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\bar{\Phi}x = \left(-\overline{a}_1\overline{\beta}_1^* x_1, \dots, -\overline{a}_n\overline{\beta}_n^* x_n\right), \quad \forall x = (x_1,\dots,x_n) \in \mathbb{R}^n,$$

the homotopy invariance property (see Lemma 2.1.8) implies that

$$\deg_B\{QN, \Omega \cap \mathrm{Ker}_L, 0\} = \deg_B\left\{\bar{\Phi}, \Omega \cap \mathrm{Ker}_L, 0\right\} \neq 0.$$

Now, it remains to prove that (2.28) holds to conclude the proof.

Let $x = (x_1,\dots,x_n) \in (\partial\Omega) \cap \mathrm{Ker}_L$ and $\mu \in [0,1]$. The function x is constant because $\mathrm{Ker}_L \cong \mathbb{R}^n$ and, by (2.25), we conclude that there is $i \in \{1,\dots,n\}$ such that $|x| = |x_i| = \overline{\xi}$. We claim that

$$|\Psi(x,\mu)_i| \neq 0.$$

By contradiction assume that

$$|\Psi(x,\mu)_i| = 0. \tag{2.29}$$

From (2.20), (2.26), and (2.29), we have

$$-\mu\overline{a}_i\overline{\beta}_i^* x_i + \frac{1-\mu}{\omega}\int_0^\omega a_i(t,x_i)\big[-b_i(t,x_i)+f_i(t,x)\big]dt = 0,$$

thus there exists $t_i^{**} \in [0,\omega]$ such that

$$-\mu\overline{a}_i\overline{\beta}_i^* x_i + (1-\mu)a_i(t_i^{**},x_i)\big[-b_i(t_i^{**},x_i)+f_i(t_i^{**},x)\big] = 0. \tag{2.30}$$

Now, we assume that $|x| = x_i = \overline{\xi} > 0$ (the situation $|x| = -x_i = \overline{\xi}$ is analogous).

By condition (H1*) and (H3*), we have

$$\begin{array}{rcl} a_i(t_i^{**},x_i)b_i(t_i^{**},x_i) & = & a_i(t_i^{**},x_i)\big[b_i(t_i^{**},x_i)-b_i(t_i^{**},0)\big] + a_i(t_i^{**},x_i)b_i(t_i^{**},0) \\ & \leq & \overline{a}_i\overline{\beta}_i^* x_i + a_i(t_i^{**},x_i)b_i(t_i^{**},0), \end{array}$$

then

$$a_i(t_i^{**},x_i)b_i(t_i^{**},x_i) - \overline{a}_i\overline{\beta}_i^* x_i - a_i(t_i^{**},x_i)b_i(t_i^{**},0) \leq 0.$$

Consequently, from (2.30), we have

$$\begin{array}{rcl} -a_i(t_i^{**},x_i)b_i(t_i^{**},x_i) & + & (1-\mu)a_i(t_i^{**},x_i)f_i(t_i^{**},x) \\ & \geq & \mu\Big[a_i(t_i^{**},x_i)b_i(t_i^{**},x_i) - \overline{a}_i\overline{\beta}_i^* x_i - a_i(t_i^{**},x_i)b_i(t_i^{**},0)\Big] \\ & & \quad -a_i(t_i^{**},x_i)b_i(t_i^{**},x_i) + (1-\mu)a_i(t_i^{**},x_i)f_i(t_i^{**},x) \\ & = & -\mu\overline{a}_i\overline{\beta}_i^* x_i + (1-\mu)a_i(t_i^{**},x_i)\Big[-b_i(t_i^{**},x_i)+f_i(t_i^{**},x)\Big] \\ & & \quad -\mu a_i(t_i^{**},x_i)b_i(t_i^{**},0) \\ & = & -\mu a_i(t_i^{**},x_i)b_i(t_i^{**},0) \\ & \geq & a_i(t_i^{**},x_i)\min\big\{0,-b_i(t_i^{**},0)\big\}. \end{array}$$

and by (H1*), we obtain

$$-b_i(t_i^{**},x_i) + (1-\mu)f_i(t_i^{**},x) \geq \min\big\{0,-b_i(t_i^{**},0)\big\}.$$

Consequently,

$$b_i(t_i^{**},x_i) - b_i(t_i^{**},0) \leq |f_i(t_i^{**},x) - f_i(t_i^{**},0)| + \overline{b}_i + \overline{f}_i,$$

and recalling that $x_i > 0$, and $\|x\| = |x|$, from (H3*), (H4*), and (2.15), we have

$$x_i \leq \frac{\mathcal{L}_i(t_i^{**})}{\beta_i(t_i^{**})}|x| + \frac{\overline{b}_i + \overline{f}_i}{\underline{\beta}_i}.$$

As $|x| = x_i = \overline{\xi} > 0$, we obtain, taking into account (H5*),

$$\overline{\xi} = x_i \leq \left(1 - \frac{\mathcal{L}_i(t_i^{**})}{\beta_i(t_i^{**})}\right)^{-1} \frac{\overline{b}_i + \overline{f}_i}{\underline{\beta}_i},$$

and by (2.23) we conclude that

$$\overline{\xi} = x_i \leq \left(1 - \frac{\mathcal{L}_i(t_i^{**})}{\beta_i(t_i^{**})}\right)^{-1} \frac{\overline{b}_i + \overline{f}_i}{\underline{\beta}_i} < \overline{\xi},$$

which is a contradiction.
The case when $x_i < 0$ is very similar to the previous one and we present it briefly. From (H1*), (H3*), and (2.15), we obtain

$$a_i(t_i^{**}, x_i)b_i(t_i^{**}, x_i) - \overline{a}_i\overline{\beta}_i^* x_i - a_i(t_i^{**}, x_i)b_i(t_i^{**}, 0) \geq 0$$

and, from (H1*), (H5*), and (2.23), we obtain

$$-b_i(t_i^{**}, x_i) + (1-\mu)f_i(t_i^{**}, x) \leq \max\left\{0, -b_i(t_i^{**}, 0)\right\}.$$

Therefore,

$$b_i(t_i^{**}, x_i) - b_i(t_i^{**}, 0) \geq -|f_i(t_i^{**}, x) - f_i(t_i^{**}, 0)| - \overline{b}_i - \overline{f}_i.$$

Since $x_i < 0$ and $\|x\| = |x|$, from (H3*), (H4*), (2.15) and taking into account that $|x| = -x_i = \overline{\xi} > 0$, we obtain

$$x_i \geq \frac{\mathcal{L}_i(t_i^{**})}{\beta_i(t_i^{**})}x_i - \frac{\overline{b}_i + \overline{f}_i}{\underline{\beta}_i}.$$

Using this last equation and (2.23), we conclude that

$$-\overline{\xi} = x_i \geq -\left(1 - \frac{\mathcal{L}_i(t_i^{**})}{\beta_i(t_i^{**})}\right)^{-1} \frac{\overline{b}_i + \overline{f}_i}{\underline{\beta}_i} > -\overline{\xi},$$

and we obtain again a contradiction. □

From Theorems 2.1.3 and 2.1.9, we have the following result.

Theorem 2.1.10. *Assume (H1*), (H2*), (H2), (H3*), (H4) with $\mathcal{L}_i$ being $\omega-$periodic continuous functions, (H5*), and (H5). Then the system* (1.3) *has an $\omega-$periodic* *solution which is globally exponentially stable.*

In the case of $D_i(t) \leq 0$, for all $t \geq 0$ and $i = 1, \ldots, n$, hypothesis (H5) implies (H5*), thus the following result is an immediate consequence of Theorem 2.1.10.

Corollary 2.1.11. *If (H1*), (H2*), (H2) with $D_i(t) \leq 0$, for all $t \geq 0$ and $i = 1, \ldots, n$, (H3*), (H4) with $\mathcal{L}_i$ being $\omega-$periodic continuous functions, and (H5) hold, then the system* (1.3) *has an $\omega-$periodic solution which is globally exponentially stable.*

In the particular case of functions a_i that do not explicitly depend on time t, from the Corollary 2.1.11, we have the following result.

Corollary 2.1.12. *If (H1*), (H3*), (H4) with $\mathcal{L}_i$ being $\omega-$periodic continuous functions, and (H5) hold, then system* (2.12) *has an $\omega-$periodic solution which is globally exponentially stable.*

2.2 Cohen-Grossberg type models

In this section, we apply the results in Subsections 2.1.1 and 2.1.2 to Cohen-Grossberg type models.

2.2.1 Existence of periodic orbits and exponential stability

We consider the general CGNN model with discrete-time varying and distributed delays

$$\begin{aligned} x_i'(t) = & a_i(t, x_i(t))\Bigg[-b_i(t, x_i(t)) + F_i(\mathcal{U}_i(t, x_t)) + G_i(\mathcal{V}_i(t, x_t)) + I_i(t)\Bigg], \\ & t \geq 0,\ i = 1, \ldots, n, \end{aligned} \tag{2.31}$$

where

$$\mathcal{U}_i(t, \varphi) = \sum_{p=1}^{P} \sum_{j,l=1}^{n} c_{ijlp}(t) h_{ijlp}\big(\varphi_j(-\tau_{ijp}(t)), \varphi_l(-\tilde{\tau}_{ilp}(t))\big),$$
$$\mathcal{V}_i(t, \varphi) = \sum_{p=1}^{P} \sum_{j,l=1}^{n} d_{ijlp}(t) f_{ijlp}\bigg(\int_{-\infty}^{0} g_{ijp}(\varphi_j(s))\, d\eta_{ijp}(s), \int_{-\infty}^{0} \tilde{g}_{ilp}(\varphi_l(s))\, d\tilde{\eta}_{ilp}(s)\bigg),$$

such that $n, P, Q \in \mathbb{N}$ and $a_i : [0, +\infty) \times \mathbb{R} \to (0, +\infty)$, $b_i : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $c_{ijlp}, d_{ijlq}, I_i : [0, +\infty) \to \mathbb{R}$, $\tau_{ijp}, \tilde{\tau}_{ilp} : [0, +\infty) \to [0, +\infty)$, $h_{ijlp}, f_{ijlq} : \mathbb{R}^2 \to \mathbb{R}$, $F_i, G_i, g_{ijq}, \widetilde{g}_{ilq} : \mathbb{R} \to \mathbb{R}$ are continuous functions, and η_{ijq}, $\tilde{\eta}_{ilq} : (-\infty, 0] \to \mathbb{R}$ are non-decreasing bounded functions such that $\eta_{ijq}(0) - \eta_{ijq}(-\infty) = 1$ and $\tilde{\eta}_{ilq}(0) - \tilde{\eta}_{ilq}(-\infty) = 1$, for each $i, j, l = 1, \ldots, n$, $p = 1, \ldots, P$, and $q = 1, \ldots, Q$.

As mentioned in Section 2.1, since we intend to apply our results to neural networks, we restrict our study to solutions of (2.31) with bounded initial condition, i.e.

$$x_{t_0} = \phi, \tag{2.32}$$

for some $t_0 \geq 0$ and $\phi \in BC$. Here, we consider the space UC^n_ε for some $\varepsilon > 0$ defined in the proof of Theorem 2.2.1. as the phase space of system (2.31).

In order to apply the results obtained in Sections 2.1.1 and 2.1.2 to Cohen-Grossberg type models, we start by assuming the next Lipschitz conditions:

(H4**) For each $i, j, l = 1, \dots, n$, $p = 1, \dots, P$, and $q = 1, \dots, Q$, there are positive numbers $\gamma^{(1)}_{ijlp}$, $\gamma^{(2)}_{ijlp}$, $\mu^{(1)}_{ijlq}$, $\mu^{(2)}_{ijlq}$, ξ_{ijq}, $\tilde{\xi}_{ilq}$, ζ_i, and ς_i such that

$$\begin{aligned} |h_{ijlp}(u_1, u_2) - h_{ijlp}(v_1, v_2)| &\leq \gamma^{(1)}_{ijlp}|u_1 - v_1| + \gamma^{(2)}_{ijlp}|u_2 - v_2|, \\ |f_{ijlq}(u_1, u_2) - f_{ijlq}(v_1, v_2)| &\leq \mu^{(1)}_{ijlq}|u_1 - v_1| + \mu^{(2)}_{ijlq}|u_2 - v_2|, \end{aligned}$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$, and

$$\begin{aligned} |g_{ijq}(u) - g_{ijq}(v)| &\leq \xi_{ijq}|u - v|, \quad |\tilde{g}_{ilq}(u) - \tilde{g}_{ilq}(v)| \leq \tilde{\xi}_{ilq}|u - v|, \\ |F_i(u) - F_i(v)| &\leq \zeta_i|u - v|, \quad |G_i(u) - G_i(v)| \leq \varsigma_i|u - v|. \end{aligned}$$

for all $u, v \in \mathbb{R}$.

Now, we state our main stability criterion for model (2.31).

Theorem 2.2.1. *Assume (H1)-(H3), (H4**), that the functions $\tau_{ijp}, \widetilde{\tau}_{ijp}$ are bounded, and that there exists $\vartheta > 0$ such that*

$$\int_{-\infty}^{0} \mathrm{e}^{-\vartheta s} d\eta_{ijq}(s) < +\infty, \qquad \int_{-\infty}^{0} \mathrm{e}^{-\vartheta s} d\tilde{\eta}_{ilq}(s) < +\infty. \tag{2.33}$$

If there exist $\varepsilon > 0$ and $w = (w_1, \dots, w_n) > 0$ such that, for all $t \geq 0$ and $i = 1, \dots, n$,

$$\begin{aligned} \underline{a}_i\big(\beta_i(t) + D_i(t)\big) - \overline{a}_i \sum_{j,l=1}^{n} \Bigg[\sum_{p=1}^{P} \zeta_i |c_{ijlp}(t)| \left(\frac{w_j}{w_i} \gamma^{(1)}_{ijlp} + \frac{w_l}{w_i} \gamma^{(2)}_{ijlp} \right) \\ + \sum_{q=1}^{Q} \varsigma_i |d_{ijlq}(t)| \left(\frac{w_j}{w_i} \mu^{(1)}_{ijlq} \xi_{ijq} + \frac{w_l}{w_i} \mu^{(2)}_{ijlq} \tilde{\xi}_{ilq} \right) \Bigg] > \varepsilon, \end{aligned} \tag{2.34}$$

then the model (2.31) *is globally exponentially stable.*

Proof. With the change of variables $y_i(t) = w_i^{-1}x_i(t)$, model (2.31) is transformed into

$$\begin{aligned} y_i'(t) &= a_i(t, w_i y_i(t)) w_i^{-1}\Bigg[- b_i(t, w_i y_i(t)) + I_i(t) \\ &+F_i\Bigg(\sum_{p=1}^{P}\sum_{j,l=1}^{n} c_{ijlp}(t) h_{ijlp}\big(w_j y_j(t-\tau_{ijp}(t)), w_l y_l(t-\tilde{\tau}_{ilp}(t))\big)\Bigg) \\ &+ G_i\Bigg(\sum_{q=1}^{Q}\sum_{j,l=1}^{n} d_{ijlq}(t) f_{ijlq}\Bigg(\int_{-\infty}^{0} g_{ijq}(w_j y_j(t+s))\, d\eta_{ijq}(s), \\ &\int_{-\infty}^{0} \tilde{g}_{ilq}(w_l y_l(t+s))\, d\tilde{\eta}_{ilq}(s)\Bigg)\Bigg)\Bigg], \end{aligned} \tag{2.35}$$

for $t \geq 0$, and $i = 1, \dots, n$.

From (2.34), there exists $\nu > 0$ such that

$$\begin{aligned} \underline{a}_i\big(\beta_i(t) + D_i(t)\big) - \overline{a}_i \sum_{j,l=1}^{n}\Bigg[\sum_{p=1}^{P} \zeta_i |c_{ijlp}(t)| \left(\frac{w_j}{w_i}\gamma_{ijlp}^{(1)} + \frac{w_l}{w_i}\gamma_{ijlp}^{(2)}\right) \\ + \sum_{q=1}^{Q} \varsigma_i |d_{ijlq}(t)| \left(\frac{w_j}{w_i}\mu_{ijlq}^{(1)}\xi_{ijq} + \frac{w_l}{w_i}\mu_{ijlq}^{(2)}\tilde{\xi}_{ilq}\right)\Bigg](1+\nu) > \nu, \end{aligned} \tag{2.36}$$

for all $t \geq 0$ and $i = 1, \dots, n$.

As τ_{ijp} and $\tilde{\tau}_{ilp}$ are bounded functions, it is possible to define the non-negative real number

$$\tau := \max_{i,j,p}\left(\sup_{t\geq 0}\{\tau_{ijp}(t), \widetilde{\tau}_{ijp}(t)\}\right).$$

We claim that there exists $\alpha \in (0, \vartheta)$ such that

$$\int_{-\infty}^{0} \mathrm{e}^{-\alpha s} d\eta_{ijq}(s) < 1 + \nu \quad \text{and} \quad \int_{-\infty}^{0} \mathrm{e}^{-\alpha s} d\tilde{\eta}_{ilq}(s) < 1 + \nu, \tag{2.37}$$

for all $i, j, l = 1, \dots, n$ and $q = 1, \dots, Q$. The claim can be proved by a similar argument to the one in [24, Theorem 4.3].
Let $\varepsilon := \min\{\nu, \alpha, \frac{\log(1+\nu)}{\tau+1}\}$, and consider the system (2.35) in the phase space UC_ε^n. Defining, for each $i = 1, \dots, n$, $\widetilde{a}_i(t,u) := a_i(t, w_i u)$, $\widetilde{b}_i(t,u) = w_i^{-1} b_i(t, w_i u)$, and

$$\begin{aligned} \widetilde{f}_i(t,\phi) &:= w_i^{-1}F_i\Bigg(\sum_{p=1}^{P}\sum_{j,l=1}^{n} c_{ijlp}(t) h_{ijlp}\big(w_j\phi_j(-\tau_{ijp}(t)), w_l\phi_l(-\tilde{\tau}_{ilp}(t))\big)\Bigg) + w_i^{-1}I_i(t) \\ &+ w_i^{-1}G_i\Bigg(\sum_{q=1}^{Q}\sum_{j,l=1}^{n} d_{ijlq}(t) f_{ijlq}\Bigg(\int_{-\infty}^{0} g_{ijq}(w_j\phi_j(s))d\eta_{ijq}(s), \int_{-\infty}^{0} \widetilde{g}_{ilq}(w_l\phi_l(s))d\tilde{\eta}_{ilq}(s)\Bigg)\Bigg), \end{aligned}$$

for all $u \in \mathbb{R}$ and $\phi = (\phi_1, \dots, \phi_n) \in UC_\varepsilon^n$, the model (2.35) has the form

$$y_i'(t) = \widetilde{a}_i(t, y_i(t))\big[-\widetilde{b}_i(t, y_i(t)) + \widetilde{f}_i(t, y_t)\big], \quad t \geq 0, \, i = 1, \dots, n. \tag{2.38}$$

For the model (2.38), the hypotheses (H1), (H2), and (H3) hold with same constants $\underline{a}_i, \overline{a}_i$ and same functions $D_i(t), \beta_i(t)$.
From Theorem 2.1.3, the proof is concluded if hypotheses (H4) and (H5) hold.

$$\begin{aligned}
|\widetilde{f}_i(t,\phi) - \widetilde{f}_i(t,\psi)| \;\leq\;& w_i^{-1} \sum_{j,l=1}^{n} \Bigg[\zeta_i \sum_{p=1}^{P} |c_{ijlp}(t)| \Big| h_{ijlp}\big(w_j\phi_j(-\tau_{ijp}(t)), w_l\phi_l(-\tilde{\tau}_{ilp}(t))\big) \\
& -h_{ijlp}\big(w_j\psi_j(-\tau_{ijp}(t)), w_l\psi_l(-\tilde{\tau}_{ilp}(t))\big)\Big| \\
& +\varsigma_i \sum_{q=1}^{Q} |d_{ijlq}(t)| \Bigg| f_{ijlq}\bigg(\int_{-\infty}^{0} g_{ijq}(w_j\phi_j(s))\, d\eta_{ijq}(s), \\
& \int_{-\infty}^{0} \widetilde{g}_{ilq}(w_l\phi_l(s))\, d\widetilde{\eta}_{ilq}(s)\bigg) - f_{ijlq}\bigg(\int_{-\infty}^{0} g_{ijq}(w_j\psi_j(s))\, d\eta_{ijq}(s), \\
& \int_{-\infty}^{0} \widetilde{g}_{ilq}(w_l\psi_l(s))\, d\widetilde{\eta}_{ilq}(s)\bigg)\Bigg|\Bigg] \\
\leq\;& w_i^{-1} \sum_{j,l=1}^{n} \Bigg[\zeta_i \sum_{p=1}^{P} |c_{ijlp}(t)| \Big(\gamma_{ijlp}^{(1)} w_j \big|\phi_j(-\tau_{ijp}(t)) - \psi_j(-\tau_{ijp}(t))\big| \\
& +\gamma_{ijlp}^{(2)} w_l \big|\phi_l(-\tilde{\tau}_{ilp}(t)) - \psi_l(-\tilde{\tau}_{ilp}(t))\big|\Big) \\
& +\varsigma_i \sum_{q=1}^{Q} |d_{ijlq}(t)| \bigg(\mu_{ijlq}^{(1)} \Big| \int_{-\infty}^{0} \big(g_{ijq}(w_j\phi_j(s)) - g_{ijq}(w_j\psi_j(s))\big)\, d\eta_{ijq}(s) \Big| \\
& +\mu_{ijlq}^{(2)} \Big| \int_{-\infty}^{0} \big(\widetilde{g}_{ilq}(w_l\phi_l(s)) - \widetilde{g}_{ilq}(w_l\psi_l(s))\big)\, d\widetilde{\eta}_{ilq}(s) \Big| \bigg)\Bigg].
\end{aligned}$$

Again, from (H4**) and by the monotony of η_{ijq} and $\widetilde{\eta}_{ilq}$, we obtain,

$$\begin{aligned}
|\widetilde{f}_i(t,\phi) - \widetilde{f}_i(t,\psi)| \leq & \sum_{j,l=1}^{n} \Bigg[\zeta_i \sum_{p=1}^{P} |c_{ijlp}(t)| \bigg(\gamma_{ijlp}^{(1)} \frac{w_j}{w_i} \big|\phi_j(-\tau_{ijp}(t)) - \psi_j(-\tau_{ijp}(t))\big| \\
& + \gamma_{ijlp}^{(2)} \frac{w_l}{w_i} \big|\phi_l(-\tilde{\tau}_{ilp}(t)) - \psi_l(-\tilde{\tau}_{ilp}(t))\big| \bigg) \\
& + \varsigma_i \sum_{q=1}^{Q} |d_{ijlq}(t)| \bigg(\mu_{ijlq}^{(1)} \int_{-\infty}^{0} \xi_{ijq} \frac{w_j}{w_i} |\phi_j(s) - \psi_j(s)|\, d\eta_{ijq}(s) \\
& + \mu_{ijlq}^{(2)} \int_{-\infty}^{0} \widetilde{\xi}_{ilq} \frac{w_l}{w_i} |\phi_l(s) - \psi_l(s)|\, d\widetilde{\eta}_{ilq}(s) \bigg)\Bigg],
\end{aligned} \tag{2.39}$$

and consequently,

$$
\begin{aligned}
|\tilde{f}_i(t,\phi)-\tilde{f}_i(t,\psi)| \le \sum_{j,l=1}^{n} \Bigg[\zeta_i \sum_{p=1}^{P} |c_{ijlp}(t)| \Bigg(& \gamma_{ijlp}^{(1)} \frac{w_j}{w_i} \frac{\big|(\phi_j-\psi_j)(-\tau_{ijp}(t))\big|}{\mathrm{e}^{-\varepsilon(-\tau_{ijp}(t))}} \mathrm{e}^{\varepsilon\tau_{ijp}(t)} \\
& + \gamma_{ijlp}^{(2)} \frac{w_l}{w_i} \frac{\big|(\phi_l-\psi_l)(-\tilde{\tau}_{ilp}(t))\big|}{\mathrm{e}^{-\varepsilon(-\tilde{\tau}_{ilp}(t))}} \mathrm{e}^{\varepsilon\tilde{\tau}_{ilp}(t)} \Bigg) \\
& + \varsigma_i \sum_{q=1}^{Q} |d_{ijlq}(t)| \Bigg(\mu_{ijlq}^{(1)} \int_{-\infty}^{0} \xi_{ijq} \frac{w_j}{w_i} \frac{|(\phi_j-\psi_j)(s)|}{\mathrm{e}^{-\varepsilon s}} \mathrm{e}^{-\varepsilon s} d\eta_{ijq}(s) \\
& + \mu_{ijlq}^{(2)} \int_{-\infty}^{0} \tilde{\xi}_{ilq} \frac{w_l}{w_i} \frac{|(\phi_l-\psi_l)(s)|}{\mathrm{e}^{-\varepsilon s}} \mathrm{e}^{-\varepsilon s} d\tilde{\eta}_{ilq}(s) \Bigg) \Bigg].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\tilde{f}_i(t,\phi)-\tilde{f}_i(t,\psi)| \le \|\phi-\psi\|_\varepsilon \sum_{j,l=1}^{n} \Bigg[& \zeta_i \sum_{p=1}^{P} |c_{ijlp}(t)| \Bigg(\gamma_{ijlp}^{(1)} \frac{w_j}{w_i} + \gamma_{ijlp}^{(2)} \frac{w_l}{w_i} \Bigg) \mathrm{e}^{\varepsilon\tau} \\
& + \varsigma_i \sum_{q=1}^{Q} |d_{ijlq}(t)| \Bigg(\mu_{ijlq}^{(1)} \xi_{ijq} \frac{w_j}{w_i} \int_{-\infty}^{0} \mathrm{e}^{-\varepsilon s} d\eta_{ijq}(s) \\
& + \mu_{ijlq}^{(2)} \tilde{\xi}_{ilq} \frac{w_l}{w_i} \int_{-\infty}^{0} \mathrm{e}^{-\varepsilon s} d\tilde{\eta}_{ilq}(s) \Bigg) \Bigg].
\end{aligned}
$$

As $\varepsilon \le \alpha$, (2.37) implies

$$
\int_{-\infty}^{0} \mathrm{e}^{-\varepsilon s} d\eta_{ijq}(s) < 1+\nu \qquad \text{and} \qquad \int_{-\infty}^{0} \mathrm{e}^{-\varepsilon s} d\tilde{\eta}_{ilq}(s) < 1+\nu,
$$

for all $i,j,l = 1,\dots,n$ and $q = 1,\dots,Q$. As $\varepsilon \le \dfrac{\log(1+\nu)}{\tau+1}$, then we also have

$$
\mathrm{e}^{\varepsilon\tau} < \mathrm{e}^{\varepsilon(\tau+1)} \le 1+\nu.
$$

Consequently,

$$
\begin{aligned}
|\widetilde{f}_i(t,\phi)-\widetilde{f}_i(t,\psi)| \le \Bigg[\sum_{j,l=1}^{n} \Bigg(& \zeta_i \sum_{p=1}^{P} |c_{ijlp}(t)| \Bigg(\gamma_{ijlp}^{(1)} \frac{w_j}{w_i} + \gamma_{ijlp}^{(2)} \frac{w_l}{w_i} \Bigg) \\
& + \varsigma_i \sum_{q=1}^{Q} |d_{ijlq}(t)| \Bigg(\mu_{ijlq}^{(1)} \xi_{ijq} \frac{w_j}{w_i} + \mu_{ijlq}^{(2)} \widetilde{\xi}_{ilq} \frac{w_l}{w_i} \Bigg) \Bigg) (1+\nu) \Bigg] \|\phi-\psi\|_\varepsilon,
\end{aligned}
$$

and hypothesis (H4) holds with

$$\mathcal{L}_i(t) = \sum_{j,l=1}^{n} \left[\zeta_i \sum_{p=1}^{P} |c_{ijlp}(t)| \left(\gamma_{ijlp}^{(1)} \frac{w_j}{w_i} + \gamma_{ijlp}^{(2)} \frac{w_l}{w_i} \right) \right.$$
$$\left. + \varsigma_i \sum_{q=1}^{Q} |d_{ijlq}(t)| \left(\mu_{ijlq}^{(1)} \xi_{ijq} \frac{w_j}{w_i} + \mu_{ijlq}^{(2)} \widetilde{\xi}_{ilq} \frac{w_l}{w_i} \right) \right] (1+\nu),$$

for all $i = 1, \ldots, n$.

As $\varepsilon \leq \nu$ and (2.36) holds, the hypothesis (H5) also holds and the proof is concluded. □

Now, we assume that the model (2.31) is periodic, i.e.

(H2**) There is $\omega > 0$ such that, for each $i, j, l = 1, \ldots, n$, $p = 1, \ldots, P$, and $q = 1, \ldots, Q$,

$$\begin{aligned} a_i(t,u) &= a_i(t+\omega,u), \quad c_{ijlp}(t) = c_{ijlp}(t+\omega), \quad \tau_{ijp}(t) = \tau_{ijp}(t+\omega), \\ b_i(t,u) &= b_i(t+\omega,u), \quad d_{ijlq}(t) = d_{ijlq}(t+\omega), \quad \tilde{\tau}_{ijp}(t) = \tilde{\tau}_{ijp}(t+\omega), \text{ and} \\ I_i(t) &= I_i(t+\omega) \end{aligned}$$

for all $t \geq 0$ and $u \in \mathbb{R}$.

Theorem 2.2.2. *Assume the hypotheses (H2**), (H1), (H3*), and (H4**).*
If there exists $w = (w_1, \ldots, w_n) > 0$ such that for all $t \in [0, \omega]$, and $i = 1, \ldots, n$,

$$\beta_i(t) > \sum_{j,l=1}^{n} \left[\sum_{p=1}^{P} \zeta_i |c_{ijlp}(t)| \left(\frac{w_j}{w_i} \gamma_{ijlp}^{(1)} + \frac{w_l}{w_i} \gamma_{ijlp}^{(2)} \right) \right.$$
$$\left. + \sum_{q=1}^{Q} \varsigma_i |d_{ijlq}(t)| \left(\frac{w_j}{w_i} \mu_{ijlq}^{(1)} \xi_{ijq} + \frac{w_l}{w_i} \mu_{ijlq}^{(2)} \widetilde{\xi}_{ilq} \right) \right], \tag{2.40}$$

then the model (2.31) *has an $\omega-$periodic solution.*

Proof. As in the previous proof, the change of variables $y_i(t) = w_i^{-1} x_i(t)$ transforms the model (2.31) into (2.35).

The model (2.35) in the phase space UC_ε^n, for some $\varepsilon > 0$, has the form (2.38).

Proceeding as in the proof of Theorem 2.2.1, the functions $\tilde{f}_i$ verify (2.39) for all

$\phi = (\phi_1, \dots, \phi_n), \psi = (\psi_1, \dots, \psi_n) \in BC$ and $t \geq 0$. Consequently,

$$\begin{aligned}
|\tilde{f}_i(t,\phi) - \tilde{f}_i(t,\psi)| \leq \|\phi - \psi\| \sum_{j,l=1}^{n} \Bigg[\zeta_i \sum_{p=1}^{P} |c_{ijlp}(t)| \bigg(\gamma_{ijlp}^{(1)} \frac{w_j}{w_i} + \gamma_{ijlp}^{(2)} \frac{w_l}{w_i} \bigg) \\
+ \varsigma_i \sum_{q=1}^{Q} |d_{ijlq}(t)| \bigg(\mu_{ijlq}^{(1)} \xi_{ijq} \frac{w_j}{w_i} \int_{-\infty}^{0} d\eta_{ijq}(s) \\
+ \mu_{ijlq}^{(2)} \tilde{\xi}_{ilq} \frac{w_l}{w_i} \int_{-\infty}^{0} d\tilde{\eta}_{ilq}(s) \bigg) \Bigg].
\end{aligned}$$

Taking into account the properties of η_{ijq} and $\tilde{\eta}_{ilq}$, we obtain

$$|\tilde{f}_i(t,\phi) - \tilde{f}_i(t,\psi)| \leq \mathcal{L}_i(t) \|\phi - \psi\|,$$

with

$$\begin{aligned}
\mathcal{L}_i(t) = \sum_{j,l=1}^{n} \Bigg[\zeta_i \sum_{p=1}^{P} |c_{ijlp}(t)| \bigg(\gamma_{ijlp}^{(1)} \frac{w_j}{w_i} + \gamma_{ijlp}^{(2)} \frac{w_l}{w_i} \bigg) \\
+ \varsigma_i \sum_{q=1}^{Q} |d_{ijlq}(t)| \bigg(\mu_{ijlq}^{(1)} \xi_{ijq} \frac{w_j}{w_i} + \mu_{ijlq}^{(2)} \tilde{\xi}_{ilq} \frac{w_l}{w_i} \bigg) \Bigg].
\end{aligned}$$

Thus, hypothesis (H4*) holds for the model (2.38).
By (2.40), hypothesis (H5*) also holds and the conclusion follows from Theorem 2.1.9.

□

Immediately, from Theorems 2.2.1 and 2.2.2, we have the following result.

Corollary 2.2.3. *Assume (H1), (H2) with D_i an $\omega-$periodic continuous function, (H2**), (H3*), (H4**), and* (2.33).
If there exists $w = (w_1, \dots, w_n) > 0$ such that, for all $t \in [0, \omega]$ and $i = 1, \dots, n$, inequalities (2.40) *and*

$$\begin{aligned}
\underline{a}_i \big(\beta_i(t) + D_i(t) \big) > \overline{a}_i \sum_{j,l=1}^{n} \Bigg[\sum_{p=1}^{P} \zeta_i |c_{ijlp}(t)| \bigg(\frac{w_j}{w_i} \gamma_{ijlp}^{(1)} + \frac{w_l}{w_i} \gamma_{ijlp}^{(2)} \bigg) \\
+ \sum_{q=1}^{Q} \varsigma_i |d_{ijlq}(t)| \bigg(\frac{w_j}{w_i} \mu_{ijlq}^{(1)} \xi_{ijq} + \frac{w_l}{w_i} \mu_{ijlq}^{(2)} \tilde{\xi}_{ilq} \bigg) \Bigg]
\end{aligned} \tag{2.41}$$

hold, then the model (2.31) *has an $\omega-$periodic solution which is globally exponentially stable.*

Proof. From (H2**), the functions τ_{ijp}, $\tilde{\tau}_{ijp}$ are bounded. Moreover, from (H2**) and (H3*), we know that β_i, c_{ijlp}, and d_{ijlq} are $\omega-$periodic functions. As D_i are also

$\omega-$periodic, then there is $\varepsilon > 0$ such that inequality (2.34) holds and the conclusion follows from Theorems 2.2.1 and 2.2.2. $\square$

Now, we consider the model (2.31) with amplification functions, a_i, that do not explicitly depend on time t, i.e.

$$\begin{aligned} x_i'(t) = a_i(x_i(t))\Big[&- b_i(t, x_i(t)) \\ &+ F_i\Big(\sum_{p=1}^{P}\sum_{j,l=1}^{n} c_{ijlp}(t)h_{ijlp}\big(x_j(t-\tau_{ijp}(t)), x_l(t-\tilde{\tau}_{ilp}(t))\big)\Big) \\ &+ G_i\Big(\sum_{q=1}^{Q}\sum_{j,l=1}^{n} d_{ijlq}(t)f_{ijlq}\Big(\int_{-\infty}^{0} g_{ijq}(x_j(t+s))\,d\eta_{ijq}(s), \\ &\int_{-\infty}^{0} \tilde{g}_{ilq}(x_l(t+s))\,d\tilde{\eta}_{ilq}(s)\Big)\Big) + I_i(t)\Big], \quad t \geq 0, \quad i = 1, \dots, n. \end{aligned} \tag{2.42}$$

From Corollary 2.2.3, we have the following result.

Corollary 2.2.4. *Assume (H1), (H2**), (H3*), (H4**), and* (2.33). *If there exists $w = (w_1, \dots, w_n) > 0$ such that, for all $t \in [0, \omega]$ and $i = 1, \dots, n$,*

$$\begin{aligned} \underline{a}_i\beta_i(t) > \overline{a}_i \sum_{j,l=1}^{n} \Bigg[&\sum_{p=1}^{P} \zeta_i |c_{ijlp}(t)| \Big(\frac{w_j}{w_i}\gamma_{ijlp}^{(1)} + \frac{w_l}{w_i}\gamma_{ijlp}^{(2)} \Big) \\ &+ \sum_{q=1}^{Q} \varsigma_i |d_{ijlq}(t)| \Big(\frac{w_j}{w_i}\mu_{ijlq}^{(1)}\xi_{ijq} + \frac{w_l}{w_i}\mu_{ijlq}^{(2)}\tilde{\xi}_{ilq} \Big) \Bigg], \end{aligned} \tag{2.43}$$

then the model (2.42) *has an $\omega-$periodic solution which is globally exponentially stable.*

Proof. Noting that $a_i(t, u) = a_i(u)$ for all $t, u \in \mathbb{R}$ and $i = 1, \dots, n$, the hypothesis (H2) trivially holds with $D_i(t) = 0$. Consequently, inequality (2.43) implies (2.40), and the result follows from Corollary 2.2.3. $\square$

For the model (2.42) under the hypotheses (H2**), (H1), (H3*), and (H4**), consider the constants

$$\underline{\beta}_i := \min_{t\in[0,\omega]} \beta_i(t), \quad \overline{c}_{ijlp} := \max_{t\in[0,\omega]} |c_{ijlp}(t)|, \quad \text{and} \quad \overline{d}_{ijlq} := \max_{t\in[0,\omega]} |d_{ijlq}(t)|, \tag{2.44}$$

for each $i, j, l = 1, \dots, n$, $p = 1, \dots, P$, $q = 1, \dots, Q$, and the square real matrix $\mathcal{M}$ defined by

$$\mathcal{M} := diag\big(\underline{a}_1\underline{\beta}_1, \dots, \underline{a}_n\underline{\beta}_n\big) - \big[\mathfrak{m}_{ij}\big]_{i,j=1}^{n}, \tag{2.45}$$

where, for each $i, j = 1, \ldots, n$,

$$\mathfrak{m}_{ij} := \overline{a}_i \sum_{l=1}^{n} \left[\zeta_i \sum_{p=1}^{P} \left(\overline{c}_{ijlp} \gamma_{ijlp}^{(1)} + \overline{c}_{iljp} \gamma_{iljp}^{(2)} \right) + \varsigma_i \sum_{q=1}^{Q} \left(\overline{d}_{ijlq} \mu_{ijlq}^{(1)} \xi_{ijq} + \overline{d}_{iljq} \mu_{iljq}^{(2)} \widetilde{\xi}_{ijq} \right) \right].$$

We give the definition of M-matrix so that we use it in obtaining stability criteria for (2.42).

Definition 2.2.1. *If $\mathcal{M} = [m_{ij}]$ is a square matrix with non-positive off-diagonal entries, i.e., $m_{ij} \leq 0$ for all $i \neq j$,we say that $\mathcal{M}$ is a non-singular M-matrix if all principle minors of $\mathcal{M}$ are positive, or, equivalently, if all eigenvalues of $\mathcal{M}$ have positive real part.*

The algebraic properties of the associated M-matrix defined in (2.45) is related to the stability of (2.42). For further properties of M-matrices (see [25, Chapter 5]).

Corollary 2.2.5. *Assume (H1), (H2**), (H3*), (H4**), and* (2.33).
If $\mathcal{M}$ is a non-singular M-matrix, then the model (2.42) *has an $\omega-$periodic solution which is globally exponentially stable.*

Proof. As $\mathcal{M}$ is a non-singular M-matrix, then there exists $w = (w_1, \ldots, w_n) > 0$ such that $\mathcal{M}w^T > 0$ (see [25]), i.e.,

$$\begin{aligned} \underline{a}_i \underline{\beta}_i w_i > \sum_{j=1}^{n} w_j \Bigg[\overline{a}_i \sum_{l=1}^{n} \Bigg(\zeta_i \sum_{p=1}^{P} \left(\overline{c}_{ijlp} \gamma_{ijlp}^{(1)} + \overline{c}_{iljp} \gamma_{iljp}^{(2)} \right) \\ + \varsigma_i \sum_{q=1}^{Q} \left(\overline{d}_{ijlq} \mu_{ijlq}^{(1)} \xi_{ijq} + \overline{d}_{iljq} \mu_{iljq}^{(2)} \widetilde{\xi}_{ijq} \right) \Bigg) \Bigg], \end{aligned}$$

for all $i = 1, \ldots, n$, which is equivalent to

$$\begin{aligned} \underline{a}_i \underline{\beta}_i > \overline{a}_i \sum_{j,l=1}^{n} \Bigg[\zeta_i \sum_{p=1}^{P} \left(\overline{c}_{ijlp} \frac{w_j}{w_i} \gamma_{ijlp}^{(1)} + \overline{c}_{iljp} \gamma_{iljp}^{(2)} \frac{w_j}{w_i} \right) \\ + \varsigma_i \sum_{q=1}^{Q} \left(\overline{d}_{ijlq} \frac{w_j}{w_i} \mu_{ijlq}^{(1)} \xi_{ijq} + \overline{d}_{iljq} \mu_{iljq}^{(2)} \widetilde{\xi}_{ijq} \frac{w_j}{w_i} \right) \Bigg], \end{aligned}$$

and consequently,

$$\begin{aligned} \underline{a}_i \underline{\beta}_i > \overline{a}_i \sum_{j,l=1}^{n} \Bigg[\zeta_i \sum_{p=1}^{P} \left(\overline{c}_{ijlp} \frac{w_j}{w_i} \gamma_{ijlp}^{(1)} + \overline{c}_{ijlp} \gamma_{ijlp}^{(2)} \frac{w_l}{w_i} \right) \\ + \varsigma_i \sum_{q=1}^{Q} \left(\overline{d}_{ijlq} \frac{w_j}{w_i} \mu_{ijlq}^{(1)} \xi_{ijq} + \overline{d}_{ijlq} \mu_{ijlq}^{(2)} \widetilde{\xi}_{ilq} \frac{w_l}{w_i} \right) \Bigg]. \end{aligned} \tag{2.46}$$

From (2.44) and (2.46), we obtain (2.43). Now the result follows from Corollary 2.2.4. □

Example 2.1. Consider the following low-order CGNN model

$$x_i'(t) = a_i(x_i(t)) \left[-b_i(t, x_i(t)) + G_i\left(\sum_{j=1}^{n} c_{ij}(t) \int_0^{+\infty} x_j(t-u) K_{ij}(u) du \right) \right], \quad (2.47)$$

for $t \geq 0$ and $i = 1, \ldots, n$, where $a_i : \mathbb{R} \to (0, +\infty)$, $b_i : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $c_{ij} : [0, +\infty) \to \mathbb{R}$, $G_i : \mathbb{R} \to \mathbb{R}$, and $K_{ij} : [0, +\infty) \to [0, +\infty)$ are continuous functions such that

$$\int_0^{+\infty} K_{ij}(u) du \quad = 1, \quad (2.48)$$

for all $i, j = 1, \ldots, n$.

The model (2.47) is a generalization of the following autonomous static neural network model

$$x_i'(t) = -x_i(t) + G_i\left(\sum_{j=1}^{n} c_{ij} \int_0^{+\infty} x_j(t-u) K_{ij}(u) du \right), \; t \geq 0, \; i = 1, \ldots, n, \quad (2.49)$$

whose existence and global asymptotic stability of an equilibrium point were studied in [49].

Defining, for each $i, j = 1, \ldots, n$, $\eta_{ij} : (-\infty, 0] \to \mathbb{R}$ by

$$\eta_{ij}(s) = \int_{-\infty}^{s} K_{ij}(-v) dv, \qquad s \in (-\infty, 0], \quad (2.50)$$

we have η_{ij} non-decreasing and, from (2.48), $\eta_{ij}(0) - \eta_{ij}(-\infty) = 1$. Consequently, the model (2.47) can be written in the form

$$x_i'(t) = a_i(x_i(t)) \left[-b_i(t, x_i(t)) + G_i\left(\sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{0} x_j(t+s) d\eta_{ij}(s) \right) \right],$$

$t \geq 0$, $i = 1, \ldots, n$, which is a particular situation of (2.42). Consequently, from Corollary 2.2.4, we obtain the following result.

Corollary 2.2.6. *Assume (H1), (H3*) and that for each $i, j = 1, \ldots, n$, and $v \in \mathbb{R}$, the functions $t \mapsto b_i(t, v)$ and c_{ij} are $\omega-$periodic, the function G_i is Lipschitz with Lipschitz constant $\varsigma_i > 0$, and there is $\alpha > 0$ such that*

$$\int_0^{+\infty} K_{ij}(u) \mathrm{e}^{\alpha u} du < +\infty. \quad (2.51)$$

If there exists $w = (w_1, \dots, w_n) > 0$ such that, for all $t \in [0, \omega]$ and $i = 1, \dots, n$,

$$\underline{a}_i \beta_i(t) w_i > \overline{a}_i \sum_{j=1}^{n} \varsigma_i |c_{ij}(t)| w_j, \quad t \geq 0,\, i = 1, \dots, n, \tag{2.52}$$

then the model (2.47) *has an $\omega-$periodic solution which is globally exponentially stable.*

For the model (2.49), condition (2.52) reads as

$$w_i > \sum_{j=1}^{n} \varsigma_i |c_{ij}| w_j, \quad t \geq 0,\, i = 1, \dots, n,$$

which is equivalent to the matrix

$$\mathcal{N} = I_n - \left[\varsigma_i |c_{ij}|\right]_{i,j=1}^{n},$$

where I_n denotes the n-dimensional identity matrix, being a non-singular M-matrix (see [25]). Consequently, we also have the following result.

Corollary 2.2.7. *For each $i, j = 1, \dots, n$, assume that the function G_i is Lipschitz with a Lipschitz constant $\varsigma_i > 0$ and* (2.51)*.*
If $\mathcal{N}$ is a non-singular M-matrix, then the model (2.49) *has an equilibrium point which is globally exponentially stable.*

Remark 2.2.8. *We remark that in [49], the existence and global asymptotic stability of an equilibrium point of* (2.49) *was obtained assuming different conditions. Namely, assuming that G_i satisfy stronger conditions than being Lipschitz, that $\mathcal{N}$ is a non-singular M-matrix, and condition*

$$\int_0^{+\infty} u K_{ij}(u) du < +\infty,$$

instead of (2.51)*.*

Example 2.2. Consider the following high-order CGNN model,

$$\begin{aligned} x_i'(t) = a_i(x_i(t)) \Bigg[&- b_i(x_i(t)) + \sum_{j=1}^{n} c_{ij}(t) f_j(\rho_j x_j(t)) + I_i(t) \\ &+ \sum_{j=1}^{n} d_{ij11}(t) f_j \left(\rho_j \int_0^{+\infty} K_{ij}(u) x_j(t-u)\, du \right) \\ &+ \sum_{j,l=1}^{n} d_{ijl2}(t) f_j \left(\rho_j \int_0^{+\infty} K_{ij}(u) x_j(t-u)\, du \right) f_l \left(\rho_l \int_0^{+\infty} K_{il}(u) x_l(t-u)\, du \right) \Bigg], \end{aligned} \tag{2.53}$$

for $t \geq 0$, $i = 1, \dots, n$, where, for each $i, j, l = 1, \dots, n$ and $q = 1, 2$, $\rho_i > 0$, $a_i : \mathbb{R} \to$

$(0,+\infty)$, $b_i:\mathbb{R}\to\mathbb{R}$, $c_{ij}, d_{ijlq}, I_i:[0,+\infty)\to\mathbb{R}$, $f_j:\mathbb{R}\to\mathbb{R}$, and $K_{ij}:[0,+\infty)\to[0,+\infty)$ are continuous functions such that K_{ij} verifies (2.48).
The existence and global exponential stability of a periodic solution of (2.53) were studied in [40].
Considering the definition of the bounded variation functions η_{ij} as in (2.50), model (2.53) is a particular situation of (2.42). Thus, from Corollary 2.2.4, we obtain the following stability criterion.

Corollary 2.2.9. *Assume that (H1), (H3*) with $\beta_i(t)\equiv\beta_i$ hold, and that for each $i,j,l=1,\dots,n$ and $q=1,2$, c_{ij}, d_{ijlq}, I_i are $\omega-$periodic for some $\omega>0$, K_{ij} verifies* (2.48) *and* (2.51)*, and there are $M_j>0$ and $\mu_j>0$ such that*

$$|f_j(u)-f_j(v)|\leq\mu_j|u-v|\quad\text{and}\quad|f_j(u)|\leq M_j,\quad\forall u,v\in\mathbb{R},\ j=1,\dots,n.$$

If there exists $w=(w_1,\dots,w_n)>0$ such that, for all $t\geq 0$ and $i=1,\dots,n$,

$$\begin{aligned}\underline{a}_i\beta_i w_i>\overline{a}_i\sum_{j=1}^{n}\Bigg[&|c_{ij}(t)|\mu_j\rho_j w_j+|d_{ij11}(t)|\rho_j\mu_j w_j\\&+\sum_{l=1}^{n}|d_{ijl2}(t)|\big(w_jM_l\mu_j\rho_j+w_lM_j\mu_l\rho_l\big)\Bigg],\end{aligned}\tag{2.54}$$

then the model (2.53) *has an $\omega-$periodic solution which is globally exponentially stable.*

Proof. In the model (2.42), take $P=1$, $Q=2$ and, for each $i,j,l=1,\dots,n$, $q=1,2$, $u,u_1,u_2\in\mathbb{R}$, and $t\geq 0$, let $F_i(u)=G_i(u)=u$, $\tau_{ij1}(t)=\tilde{\tau}_{ij1}(t)=0$, $h_{ijl1}(u_1,u_2)=f_j(\rho_j u_1)$, $c_{ij11}(t)=c_{ij}(t)$, $c_{ijl1}(t)=d_{ijl1}(t)=0$ for $l\neq 1$, $f_{ijl1}(u_1,u_2)=f_j(\rho_j u_1)$, $f_{ijl2}(u_1,u_2)=f_j(\rho_j u_1)f_l(\rho_l u_2)$ and $g_{ijq}(u)=\widetilde{g}_{ijq}(u)=u$. Assuming further that $\widetilde{\eta}_{ijq}(s)=\eta_{ijq}(s)$, for $s\leq 0$, are defined by (2.50), we obtain model (2.53).
For all $u_1,u_2,v_1,v_2\in\mathbb{R}$, we have

$$\begin{aligned}|h_{ijl1}(u_1,u_2)-h_{ijl1}(v_1,v_2)| &= |f_j(\rho_j u_1)-f_j(\rho_j v_1)|\leq\rho_j\mu_j|u_1-v_1|,\\|f_{ijl1}(u_1,u_2)-f_{ijl1}(v_1,v_2)| &= |f_j(\rho_j u_1)-f_j(\rho_j v_1)|\leq\rho_j\mu_j|u_1-v_1|.\end{aligned}$$

and

$$\begin{aligned}|f_{ijl2}(u_1,u_2)-f_{ijl2}(v_1,v_2)| &= |f_j(\rho_j u_1)f_l(\rho_l u_2)-f_j(\rho_j v_1)f_l(\rho_l v_2)|\\&\leq |f_j(\rho_j u_1)-f_j(\rho_j v_1)||f_l(\rho_l u_2)|\\&\quad+|f_j(\rho_j v_1)||f_l(\rho_l u_2)-f_l(\rho_l v_2)|\\&\leq M_l\rho_j\mu_j|u_1-v_1|+M_j\rho_l\mu_l|u_2-v_2|,\end{aligned}$$

for all $i,j,l=1,\dots,n$, thus hypothesis (H4**) holds. Condition (2.33) follows from

(2.51), and the inequality (2.43) reads as (2.54). Finally, the result follows from Corollary 2.2.4. $\square$

Remark 2.2.10. *In [40], sufficient conditions for the existence and global exponential stability of a $\omega-$periodic solution of* (2.53) *were presented. However, it is important to mention that the proof of the main result is not correct. Specifically, the way inequality [40, (3.10)] is obtained is problematic. In fact, assuming the uniqueness of solution of* (2.53) *with initial condition $x_0 = \psi$ for $\psi \in BC$, denoting this solution by $x(t, 0, \psi)$, and defining $P : BC \to BC$ by $P(\psi) = x_\omega(\cdot, 0, \psi)$, we always have*

$$\begin{aligned}\|P^N(\psi_1) - P^N(\psi_2)\| &= \|x_{N\omega}(\cdot, 0, \psi_1) - x_{N\omega}(\cdot, 0, \psi_2)\| \\ &= \sup_{s\le 0} \|x(N\omega + s, 0, \psi_1) - x(N\omega + s, 0, \psi_2)\| \\ &\ge \|\psi_1 - \psi_2\|\end{aligned}$$

for all $\psi_1, \psi_2 \in BC$ and $N \in \mathbb{N}$, since model (2.53) *is $\omega-$periodic. Consequently the inequality*

$$\|P^N(\psi_1) - P^N(\psi_2)\| \le \frac{1}{2}\|\psi_1 - \psi_2\|,$$

presented in [40, (3.10)], is false.

2.2.2 Convergence of asymptotic models

In this section, we investigate the convergence of models to the Cohen-Grossberg type model (1.4) As mentioned in Section 2.1, we restrict our study to solutions of Cohen-Grossberg type models (1.4) and (2.56) below with bounded initial conditions (2.32). The following hypotheses will be considered:

(A1) For each $i, j, l = 1, \dots, n$, $p = 1, \dots, P$, and $u \in \mathbb{R}$, the functions $c_{ijlp}, d_{ijlp}, I_i : [0, +\infty) \to \mathbb{R}$, $b_i(\cdot, u) : [0, +\infty) \to \mathbb{R}$ are bounded;

(A2) For each $i = 1, \dots, n$, there exist $\underline{a}_i, \overline{a}_i > 0$ and a function $A_i : [0, +\infty) \to \mathbb{R}$ such that

$$\underline{a}_i \le a_i(t, v) \le \overline{a}_i \quad \text{and} \quad A_i(t)a_i^2(t, v) \le \frac{\partial}{\partial t}a_i(t, v), \quad t \ge 0,\ v \in \mathbb{R};$$

(A3) For each $i = 1, \dots, n$, there exists a continuous function $\beta_i : [0, +\infty) \to [0, +\infty)$ such that

$$\frac{b_i(t, u) - b_i(t, v)}{u - v} \ge \beta_i(t), \quad t \ge 0,\ u, v \in \mathbb{R} \text{ with } u \ne v;$$

(A4) For each $i, j, l = 1, \ldots, n$ and $p = 1, \ldots, P$,

$$\lim_{t \to +\infty} (t - \tau_{ijp}(t)) = \lim_{t \to +\infty} (t - \widetilde{\tau}_{ilp}(t)) = +\infty;$$

(A5) For each $i, j, l = 1, \ldots, n$ and $p = 1, \ldots, P$, there are $\gamma^{(1)}_{ijlp}, \gamma^{(2)}_{ijlp}, \mu^{(1)}_{ijlp}, \mu^{(2)}_{ijlp} > 0$ such that

$$\begin{aligned}
|h_{ijlp}(u_1, u_2) - h_{ijlp}(v_1, v_2)| &\leq \gamma^{(1)}_{ijlp}|u_1 - v_1| + \gamma^{(2)}_{ijlp}|u_2 - v_2| \\
|f_{ijlp}(u_1, u_2) - f_{ijlp}(v_1, v_2)| &\leq \mu^{(1)}_{ijlp}|u_1 - v_1| + \mu^{(2)}_{ijlp}|u_2 - v_2|
\end{aligned}$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$;

(A6) For each $i, j = 1, \ldots, n$ and $p = 1, \ldots, P$, there are $\xi_{ijp}, \widetilde{\xi}_{ijp}, \zeta_i, \varsigma_i > 0$ such that

$$|g_{ijp}(u_1) - g_{ijp}(v_1)| \leq \xi_{ijp}|u_1 - v_1|, \quad |\widetilde{g}_{ijp}(u_1) - \widetilde{g}_{ijp}(v_1)| \leq \widetilde{\xi}_{ijp}|u_1 - v_1|,$$
$$|F_i(u_1, u_2) - F_i(v_1, v_2)| \leq \zeta_i|u_1 - v_1| + \varsigma_i|u_2 - v_2|,$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$;

(A7) There exists $d = (d_1, \ldots, d_n) > 0$ such that, for each $i = 1, \ldots, n$,

$$\begin{aligned}
\limsup_{t \to +\infty} \Bigg[\; & -\underline{a}_i\big(\beta_i(t) + A_i(t)\big) + \sum_{p=1}^{P} \sum_{j,l=1}^{n} \Bigg(\zeta_i |c_{ijlp}(t)| \bigg(\overline{a}_j \frac{d_j}{d_i} \gamma^{(1)}_{ijlp} + \overline{a}_l \frac{d_l}{d_i} \gamma^{(2)}_{ijlp} \bigg) \\
+ \; & \varsigma_i |d_{ijlp}(t)| \bigg(\overline{a}_j \frac{d_j}{d_i} \xi_{ijp} \mu^{(1)}_{ijlp} + \overline{a}_l \frac{d_l}{d_i} \widetilde{\xi}_{ilp} \mu^{(2)}_{ijlp} \bigg) \Bigg) \Bigg] < 0.
\end{aligned}$$

Remark 2.2.11. *As already mentioned in Section 2.1 about condition (H2), recall that, if $a_i(t, u) \equiv a_i(u)$ for all $i = 1, \ldots, n$, $u \in \mathbb{R}$, and $t \geq 0$, then $A_i(t) \equiv 0$ fits the second condition of hypothesis (A2).*

We still denote, for all $i = 1, \ldots, n$, $t \geq 0$, and $\varphi = (\varphi_1, \ldots, \varphi_n) \in BC$,

$$\begin{aligned}
\mathcal{U}_i(t, \varphi) &= \sum_{p=1}^{P} \sum_{j,l=1}^{n} c_{ijlp}(t) h_{ijlp}\big(\varphi_j(-\tau_{ijp}(t)), \varphi_l(-\tilde{\tau}_{ilp}(t))\big), \\
\mathcal{V}_i(t, \varphi) &= \sum_{p=1}^{P} \sum_{j,l=1}^{n} d_{ijlp}(t) f_{ijlp}\bigg(\int_{-\infty}^{0} g_{ijp}(\varphi_j(s))\, d\eta_{ijp}(s), \\
&\qquad \int_{-\infty}^{0} \tilde{g}_{ilp}(\varphi_l(s))\, d\tilde{\eta}_{ilp}(s) \bigg),
\end{aligned} \tag{2.55}$$

system (1.4) can be written as

$$x_i'(t) = a_i(t, x_i(t))\big[-b_i(t, x_i(t)) + F_i\left(\mathcal{U}_i(t, x_t), \mathcal{V}_i(t, x_t)\right) + I_i(t)\big],$$

$t \geq 0$, $i = 1, \ldots, n$.

Definition 2.2.2. *The system*

$$\begin{aligned} x_i'(t) = a_i(t, x_i(t))\Bigg[& -\hat{b}_i(t, x_i(t)) + \hat{I}_i(t) \\ & + F_i\Bigg(\sum_{p=1}^{P}\sum_{j,l=1}^{n} \hat{c}_{ijlp}(t) h_{ijlp}\big(x_j(t - \hat{\tau}_{ijp}(t)), x_l(t - \hat{\tilde{\tau}}_{ilp}(t))\big), \\ & \sum_{p=1}^{P}\sum_{j,l=1}^{n} \hat{d}_{ijlp}(t) f_{ijlp}\Bigg(\int_{-\infty}^{0} g_{ijp}(x_j(t+s))\, d\eta_{ijp}(s), \\ & \int_{-\infty}^{0} \tilde{g}_{ilp}(x_l(t+s))\, d\tilde{\eta}_{ilp}(s)\Bigg)\Bigg)\Bigg], \end{aligned} \tag{2.56}$$

$t \geq 0$, $i = 1, \ldots, n$, is said to be an asymptotic system of (1.4) if, for each $i, j, l = 1, \ldots, n$, and $p = 1, \ldots, P$, the functions $\hat{b}_i : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$, $\hat{c}_{ijlp}, \hat{d}_{ijlp}, \hat{I}_i : [0, +\infty) \to \mathbb{R}$, and $\hat{\tau}_{ijp}, \hat{\tilde{\tau}}_{ilp} : [0, +\infty) \to [0, +\infty)$ are continuous such that $\hat{b}_i$ satisfies (A3), i.e there is a continuous function $\hat{\beta}_i : [0, +\infty) \to [0, +\infty)$ verifying

$$\frac{\hat{b}_i(t, u) - \hat{b}_i(t, v)}{u - v} \geq \hat{\beta}_i(t), \quad t \geq 0,\ u, v \in \mathbb{R} \text{ with } u \neq v, \tag{2.57}$$

and

$$\begin{aligned} \lim_{t\to+\infty}\big(\beta_i(t) - \hat{\beta}_i(t)\big) &= \lim_{t\to+\infty}\big(b_i(t, w(t)) - \hat{b}_i(t, w(t))\big) \\ &= \lim_{t\to+\infty}\big(c_{ijlp}(t) - \hat{c}_{ijlp}(t)\big) = \lim_{t\to+\infty}\big(d_{ijlp}(t) - \hat{d}_{ijlp}(t)\big), \\ &= \lim_{t\to+\infty}\big(\tau_{ijp}(t) - \hat{\tau}_{ijp}(t)\big) = \lim_{t\to+\infty}\big(I_i(t) - \hat{I}_i(t)\big), \\ &= \lim_{t\to+\infty}\big(\tilde{\tau}_{ilp}(t) - \hat{\tilde{\tau}}_{ilp}(t)\big) = 0, \end{aligned} \tag{2.58}$$

for every bounded continuous function $w : \mathbb{R} \to \mathbb{R}$.

Note that, in general, the behavior of a system is not necessarily determined by the behavior of one of its asymptotic systems.

Denoting, for all $i = 1, \dots, n$, $t \geq 0$, and $\varphi = (\varphi_1, \dots, \varphi_n) \in BC$,

$$\begin{aligned}
\hat{\mathcal{U}}_i(t,\varphi) &= \sum_{p=1}^{P} \sum_{j,l=1}^{n} \hat{c}_{ijlp}(t) h_{ijlp}\big(\varphi_j(-\hat{\tau}_{ijp}(t)), \varphi_l(-\hat{\tau}_{ilp}(t))\big), \\
\hat{\mathcal{V}}_i(t,\varphi) &= \sum_{p=1}^{P} \sum_{j,l=1}^{n} \hat{d}_{ijlp}(t) f_{ijlp}\bigg(\int_{-\infty}^{0} g_{ijp}(\varphi_j(s))\, d\eta_{ijp}(s), \\
&\qquad \int_{-\infty}^{0} g_{ilp}(\varphi_l(s))\, d\widetilde{\eta}_{ilp}(s) \bigg),
\end{aligned}$$

system (2.56) can be written as

$$x_i'(t) \;= a_i(t, x_i(t))\big[- \hat{b}_i(t, x_i(t)) + F_i\left(\hat{\mathcal{U}}_i(t, x_t), \hat{\mathcal{V}}_i(t, x_t)\right) + \hat{I}_i(t)\big],$$

$t \geq 0$, $i = 1, \dots, n$.
By (2.58), it is obvious that the hypothesis $(A7)$ is equivalent to the following

$$\begin{aligned}
\limsup_{t \to +\infty} \;\; \Bigg[&- \underline{a}_i\big(\hat{\beta}_i(t) + A_i(t)\big) + \sum_{p=1}^{P} \sum_{j,l=1}^{n} \Bigg(\zeta_i |\hat{c}_{ijlp}(t)| \left(\overline{a}_j \frac{d_j}{d_i} \gamma_{ijlp}^{(1)} + \overline{a}_l \frac{d_l}{d_i} \gamma_{ijlp}^{(2)} \right) \\
&+ \varsigma_i |\hat{d}_{ijlp}(t)| \left(\overline{a}_j \frac{d_j}{d_i} \xi_{ijp} \mu_{ijlp}^{(1)} + \overline{a}_l \frac{d_l}{d_i} \widetilde{\xi}_{ilp} \mu_{ijlp}^{(2)} \right) \Bigg) \Bigg] < 0.
\end{aligned}$$

To deal with the amplification functions, $a_i(t, u)$, both in Lemma 2.2.12 and in Theorem 2.2.15, it was necessary to introduce new variables, as defined in (2.64) and (2.67), respectively. These variables are adaptations of the one introduced in [51] for establishing sufficient conditions for the global exponential stability of a Cohen-Grossberg neural network model.
Before considering the global convergence of the models, we show that all solutions of (1.4), with bounded initial condition (2.32), are defined on $\mathbb{R}$.

Lemma 2.2.12. *Assume (A2)-(A3) and (A5)-(A6). Then a solution $x(t)$ of the initial value problem* (1.4)-(2.32) *is defined on* $\mathbb{R}$.

Proof. Let $x(t) = (x_1(t), \dots, x_n(t))$ be a maximal (noncontinuable) solution of the initial value problem (1.4)-(2.32). By the continuation theorem [30, Theorem 2.3], the solution $x(t)$ is defined on $(-\infty, t^*)$ with $t^* \in (t_0, +\infty]$ and there is an increasing real sequence $(t_k)_{k \in \mathbb{N}}$ such that

$$\lim_{k \to +\infty} t_k = t^* \tag{2.59}$$

and $\max\left\{ t^*, \lim\limits_{k \to +\infty} |x(t_k)| \right\} = +\infty$. To get a contradiction, we assume that $t^* \neq +\infty$.

Thus

$$\lim_{k\to+\infty} |x(t_k)| = +\infty. \tag{2.60}$$

Define $z(t) = (z_1(t), \dots, z_n(t)) = (|x_1(t)|, \dots, |x_n(t)|)$. For each $i = 1, \dots, n$, we have

$$\begin{aligned} z_i'(t) &= \operatorname{sign}(x_i(t))x_i'(t) \\ &= \operatorname{sign}(x_i(t))a_i(t, x_i(t))\Big[- b_i(t, x_i(t)) + F_i\big(\mathcal{U}_i(t, x_t), \mathcal{V}_i(t, x_t)\big) + I_i(t)\Big], \end{aligned}$$

for almost every $t \geq t_0$. For all $t \geq t_0$, and integrating over $[t_0, t]$, we obtain

$$\begin{aligned} z_i(t) &\leq z_i(t_0) + \int_{t_0}^t z_i'(v)\, dv \\ &\leq z_i(t_0) - \int_{t_0}^t a_i(v, x_i(v)) \operatorname{sign}(x_i(v)) b_i(v, x_i(v))\, dv + \int_{t_0}^t a_i(v, x_i(v))|I_i(v)|\, dv \\ &\quad + \int_{t_0}^t a_i(v, x_i(v))\big|F_i\big(\mathcal{U}_i(v, x_v), \mathcal{V}_i(v, x_v)\big)\big|\, dv. \end{aligned}$$

and by (A2), (A3), (A5), and (A6), we obtain

$$\begin{aligned} z_i(t) &\leq \|\phi\| - \int_{t_0}^t \underline{a}_i \beta_i(v) z_i(v)\, dv + \int_{t_0}^t \overline{a}_i\big(|b_i(v,0)| + |I_i(v)| + |F_i(\mathcal{U}_i(v,0), \mathcal{V}_i(v,0))|\big)\, dv \\ &\quad + \int_{t_0}^t \overline{a}_i\big|F_i(\mathcal{U}_i(v, x_v), \mathcal{V}_i(v, x_v)) - F_i(\mathcal{U}_i(v,0), \mathcal{V}_i(v,0))\big|\, dv, \\ &\leq \|\phi\| - \int_{t_0}^t \underline{a}_i \beta_i(v) z_i(v)\, dv + \int_{t_0}^t \overline{a}_i\big(|b_i(v,0)| + |I_i(v)| + |F_i(\mathcal{U}_i(v,0), \mathcal{V}_i(v,0))|\big)\, dv \\ &\quad + \int_{t_0}^t \overline{a}_i\bigg(\zeta_i|\mathcal{U}_i(v, x_v) - \mathcal{U}_i(v,0)| + \varsigma_i|\mathcal{V}_i(v, x_v) - \mathcal{V}_i(v,0)|\bigg)\, dv. \end{aligned}$$

Then,

$$\begin{aligned} z_i(t) \leq \|\phi\| &- \int_{t_0}^t \underline{a}_i \beta_i(v) z_i(v)\, dv + \int_{t_0}^t \overline{a}_i\big(|b_i(v,0)| + |I_i(v)| + |F_i(\mathcal{U}_i(v,0), \mathcal{V}_i(v,0))|\big)\, dv \\ &+ \sum_{p=1}^{P}\sum_{j,l=1}^{n} \overline{a}_i\bigg(\int_{t_0}^t \zeta_i|c_{ijlp}(v)|\big|h_{ijlp}\big(x_j(v - \tau_{ijp}(v)), x_l(v - \tilde{\tau}_{ilp}(v))\big) - h_{ijlp}(0,0)\big|\, dv \\ &+ \int_{t_0}^t \varsigma_i|d_{ijlp}(v)|\bigg|f_{ijlp}\bigg(\int_{-\infty}^0 g_{ijp}(x_j(v+s))\, d\eta_{ijp}(s), \int_{-\infty}^0 \widetilde{g}_{ilp}(x_l(v+s))\, d\tilde{\eta}_{ilp}(s)\bigg) \\ &- f_{ijlp}\big(g_{ijp}(0), \widetilde{g}_{ilp}(0)\big)\bigg|\, dv\bigg). \end{aligned}$$

Therefore,

$$
\begin{aligned}
z_i(t) \leq \|\phi\| &+ \int_{t_0}^{t} \overline{a}_i \big(|b_i(v,0)| + |I_i(v)| + |F_i(\mathcal{U}_i(v,0), \mathcal{V}_i(v,0))|\big)\, dv \\
&+ \overline{a}_i \int_{t_0}^{t} \sum_{p=1}^{P} \sum_{j,l=1}^{n} \Big(\zeta_i |c_{ijlp}(v)| \big(\gamma_{ijlp}^{(1)} + \gamma_{ijlp}^{(2)}\big) \\
&\qquad + \varsigma_i |d_{ijlp}(v)| \big(\mu_{ijlp}^{(1)} \xi_{ijp} + \mu_{ijlp}^{(2)} \widetilde{\xi}_{ilp}\big) \Big) \|z_v\|\, dv.
\end{aligned} \tag{2.61}
$$

Defining the continuous functions $\mathcal{A}, \mathcal{B} : [0, +\infty) \to (0, +\infty)$ by

$$
\mathcal{A}(t) = \max_i \left\{ \overline{a}_i \Big(|b_i(t,0)| + |I_i(t)| + \big|F_i\big(\mathcal{U}_i(t,0), \mathcal{V}_i(t,0)\big)\big| \Big) \right\},
$$

and

$$
\mathcal{B}(t) = \max_i \left\{ \overline{a}_i \sum_{p=1}^{P} \sum_{j,l=1}^{n} \Big(\zeta_i |c_{ijlp}(t)| \big(\gamma_{ijlp}^{(1)} + \gamma_{ijlp}^{(2)}\big) + \varsigma_i |d_{ijlp}(t)| \big(\mu_{ijlp}^{(1)} \xi_{ijp} + \mu_{ijlp}^{(2)} \widetilde{\xi}_{ilp}\big) \Big) \right\},
$$

respectively, from (2.61), we obtain

$$
z_i(t) \leq \|\phi\| + \int_{t_0}^{t} \big(\mathcal{A}(v) + \mathcal{B}(v)\|z_v\|\big) dv, \quad i = 1, \ldots, n, \; t \geq t_0,
$$

thus

$$
\|z_t\| \leq \|\phi\| + \int_{t_0}^{t} \mathcal{A}(v) dv + \int_{t_0}^{t} \mathcal{B}(v) \|z_v\| dv, \quad t \geq t_0.
$$

Now, by using the generalized Gronwall's inequality (see Lemma 2.1.1), we obtain

$$
\|z_t\| \leq \|\phi\| + \int_{t_0}^{t} \mathcal{A}(v) dv + \int_{t_0}^{t} \mathcal{B}(v) \left(\|\phi\| + \int_{t_0}^{v} \mathcal{A}(r) dr \right) \mathrm{e}^{\int_v^t \mathcal{B}(r) dr}\, dv,
$$

and from (2.59),

$$
\lim_k |x(t_k)| = \lim_k z(t_k) \leq \lim_k \|z_{t_k}\| < +\infty
$$

which contradicts (2.60). □

Now, we prove the boundedness of solutions of the initial value problem (1.4)-(2.32).

Lemma 2.2.13. *Assume (A1)-(A3) and (A5)-(A7). Then, solutions of the initial value problem* (1.4)-(2.32) *are bounded.*

Proof. Let $t_0 \geq 0$ and $\phi \in BC$ and consider $x(t) = (x_1(t), \ldots, x_n(t))$ a solution of the initial value problem (1.4)-(2.32).

By (A1), the functions $b_i(\cdot,0)$, c_{ijlp}, d_{ijlp}, and I_i are bounded, thus there exists $M>0$ such that

$$M \geq d_i^{-1}\Big(|F_i\left(\mathcal{U}_i(t,0),\mathcal{V}_i(t,0)\right)|+|b_i(t,0)|+|I_i(t)|\Big), \qquad t\geq 0,\, i=1,\dots,n. \tag{2.62}$$

From (A7), there exist $T>t_0$ and a negative number B such that

$$\begin{aligned}-\underline{a}_i\big(\beta_i(t)+A_i(t)\big)+\sum_{p=1}^{P}\sum_{j,l=1}^{n}\Bigg(&\zeta_i|c_{ijlp}(t)|\bigg(\overline{a}_j\frac{d_j}{d_i}\gamma_{ijlp}^{(1)}+\overline{a}_l\frac{d_l}{d_i}\gamma_{ijlp}^{(2)}\bigg)\\&+\varsigma_i|d_{ijlp}(t)|\bigg(\overline{a}_j\frac{d_j}{d_i}\xi_{ijp}\mu_{ijlp}^{(1)}+\overline{a}_l\frac{d_l}{d_i}\tilde{\xi}_{ilp}\mu_{ijlp}^{(2)}\bigg)\Bigg)<B,\end{aligned} \tag{2.63}$$

for $t\geq T$ and $i=1,\dots,n$.
Now, define

$$z(t)=(z_1(t),\dots,z_n(t))$$

by

$$z_i(t)=sign(x_i(t))d_i^{-1}\int_0^{x_i(t)}\frac{1}{a_i^*(t,v)}dv, \qquad t\in\mathbb{R},\, i=1,\dots,n, \tag{2.64}$$

where, for all $v\in\mathbb{R}$,

$$a_i^*(t,v)=\begin{cases} a_i(t,v), & \text{if } t\geq t_0\\ a_i(t_0,v), & \text{if } t<t_0\end{cases}.$$

From (1.4) and (2.55), we obtain, for $t>t_0$, and $x_i(t)\neq 0$,

$$\begin{aligned}z_i'(t)&=\operatorname{sign}(x_i(t))d_i^{-1}\bigg(\frac{x_i'(t)}{a_i(t,x_i(t))}+\int_0^{x_i(t)}-\frac{\frac{\partial}{\partial t}a_i(t,v)}{a_i(t,v)^2}\,dv\bigg)\\&=\operatorname{sign}(x_i(t))d_i^{-1}\bigg(-b_i(t,x_i(t))+F_i\big(\mathcal{U}_i(t,x_t),\mathcal{V}_i(t,x_t)\big)\\&\qquad+\int_0^{x_i(t)}-\frac{\frac{\partial}{\partial t}a_i(t,v)}{a_i(t,v)^2}\,dv+I_i(t)\bigg)\\&\leq d_i^{-1}\bigg(-\operatorname{sign}(x_i(t))\big(b_i(t,x_i(t))-b_i(t,0)\big)\\&\qquad-\operatorname{sign}(x_i(t))\int_0^{x_i(t)}\frac{\frac{\partial}{\partial t}a_i(t,v)}{a_i(t,v)^2}\,dv\\&\qquad+\big|F_i\big(\mathcal{U}_i(t,x_t),\mathcal{V}_i(t,x_t)\big)-F_i\big(\mathcal{U}_i(t,0),\mathcal{V}_i(t,0)\big)\big|\\&\qquad+|b_i(t,0)|+\big|F_i\big(\mathcal{U}_i(t,0),\mathcal{V}_i(t,0)\big)\big|+|I_i(t)|\bigg).\end{aligned}$$

By (A2), (A3), (A5), (A6), and (2.55), for all $t > t_0$ and $i = 1, \ldots, n$, we obtain

$$\begin{aligned}
z_i'(t) \;\le\; & d_i^{-1}\Big(-(\beta_i(t)+A_i(t))|x_i(t)| + |F_i(\mathcal{U}_i(t,0),\mathcal{V}_i(t,0))| + |b_i(t,0)| + |I_i(t)| \\
& +\zeta_i|\mathcal{U}_i(t,x_t)-\mathcal{U}_i(t,0)| + \varsigma_i|\mathcal{V}_i(t,x_t)-\mathcal{V}_i(t,0)|\Big) \\
\le\; & -(\beta_i(t)+A_i(t))d_i^{-1}|x_i(t)| + d_i^{-1}\Big(|F_i(\mathcal{U}_i(t,0),\mathcal{V}_i(t,0))| \\
& +|b_i(t,0)| + |I_i(t)|\Big) + d_i^{-1}\sum_{p=1}^{P}\sum_{j,l=1}^{n}\Big(\zeta_i|c_{ijlp}(t)| \\
& \cdot\big|h_{ijlp}(x_j(t-\tau_{ijp}(t)), x_l(t-\tilde{\tau}_{ilp}(t))) - h_{ijlp}(0,0)\big| \\
& +\varsigma_i|d_{ijlp}(t)|\cdot\bigg|f_{ijlp}\Big(\int_{-\infty}^{0} g_{ijp}(x_j(t+s))\,d\eta_{ijp}(s), \\
& \int_{-\infty}^{0}\widetilde{g}_{ilp}(x_l(t+s))\,d\tilde{\eta}_{ilp}(s)\Big) - f_{ijlp}(0,0)\bigg|\Big)
\end{aligned}$$

Therefore

$$\begin{aligned}
z_i'(t) \;\le\; & -(\beta_i(t)+A_i(t))d_i^{-1}|x_i(t)| + d_i^{-1}\Big(|F_i(\mathcal{U}_i(t,0),\mathcal{V}_i(t,0))| \\
& +|b_i(t,0)| + |I_i(t)|\Big) + d_i^{-1}\sum_{p=1}^{P}\sum_{j,l=1}^{n}\Big(\zeta_i|c_{ijlp}(t)| \\
& \cdot\big(\gamma_{ijlp}^{(1)}|x_j(t-\tau_{ijp}(t))| + \gamma_{ijlp}^{(2)}|x_l(t-\tilde{\tau}_{ilp}(t))|\big) \\
& +\varsigma_i|d_{ijlp}(t)|\cdot\Big(\mu_{ijlp}^{(1)}\xi_{ijp}\int_{-\infty}^{0}|x_j(t+s)|\,d\eta_{ijp}(s) \\
& +\mu_{ijlp}^{(2)}\widetilde{\xi}_{ilp}\int_{-\infty}^{0}|x_l(t+s)|\,d\tilde{\eta}_{ilp}(s)\Big)\Big).
\end{aligned} \tag{2.65}$$

Now, by contradiction, we assume that $x(t)$ is unbounded. From Lemma 2.2.12, $x(t)$ is defined on $\mathbb{R}$. From (A2) and (2.64), we conclude that there exist $i \in \{1, \ldots, n\}$ and a positive real sequence $(t_k)_{k\in\mathbb{N}}$ such that $T \le t_k \nearrow +\infty$, $0 < z_i(t_k) \nearrow +\infty$,

$$z_i(t_k) = \|z_{t_k}\| \ge \|z_t\|, \text{ and } z_i'(t_k) \ge 0, \quad k \in \mathbb{N},\ t \le t_k. \tag{2.66}$$

From(A2) and (2.64), we also have

$$(d_j\overline{a}_j)^{-1}|x_j(t)| \le z_j(t) \le (d_j\underline{a}_j)^{-1}|x_j(t)|, \quad t \in \mathbb{R},\ j = 1, \ldots, n,$$

and from (2.62) and (2.65), we obtain

$$\begin{aligned} z_i'(t_k) &\le -(\beta_i(t_k)+A_i(t_k))\underline{a}_i z_i(t_k) + \sum_{p=1}^{P}\sum_{j,l=1}^{n}\left(\zeta_i|c_{ijlp}(t_k)|\left(\overline{a}_j\frac{d_j}{d_i}\gamma_{ijlp}^{(1)}+\overline{a}_l\frac{d_l}{d_i}\gamma_{ijlp}^{(2)}\right)\|z_{t_k}\|\right. \\ &\quad \left.+\varsigma_i|d_{ijlp}(t_k)|\left(\overline{a}_j\frac{d_j}{d_i}\xi_{ijp}\mu_{ijlp}^{(1)}+\overline{a}_l\frac{d_l}{d_i}\widetilde{\xi}_{ilp}\mu_{ijlp}^{(2)}\right)\|z_k\|\right)+M, \quad k\in\mathbb{N}, \end{aligned}$$

and consequently, from (2.66),

$$\begin{aligned} z_i'(t_k) &\le \left[-(\beta_i(t_k)+A_i(t_k))\underline{a}_i + \sum_{p=1}^{P}\sum_{j,l=1}^{n}\left(\zeta_i|c_{ijlp}(t_k)|\left(\overline{a}_j\frac{d_j}{d_i}\gamma_{ijlp}^{(1)}+\overline{a}_l\frac{d_l}{d_i}\gamma_{ijlp}^{(2)}\right)\right.\right. \\ &\quad \left.\left.+\varsigma_i\,|d_{ijlp}(t_k)|\left(\overline{a}_j\frac{d_j}{d_i}\xi_{ijp}\mu_{ijlp}^{(1)}+\overline{a}_l\frac{d_l}{d_i}\widetilde{\xi}_{ilp}\mu_{ijlp}^{(2)}\right)\right)\right] z_i(t_k)+M, \quad k\in\mathbb{N}. \end{aligned}$$

Hence, by (2.63),

$$z_i'(t_k)\le Bz_i(t_k)+M \overset{k\to+\infty}{\longrightarrow} -\infty,$$

which contradicts (2.66). □

Next, we state the fluctuation lemma because it plays an important role in proving the global attractivity of solutions of system (1.4) and of its asymptotic systems (2.56).

Lemma 2.2.14. *[56, Lemma A.1] Let $f:[b,+\infty)\to\mathbb{R}$ be a bounded and differentiable function, then there exist sequences $s_k\to+\infty$ and $t_k\to\infty$ such that*

$$\begin{aligned} f(s_k) &\to f_\infty, \quad f'(s_k)\to 0 \quad \text{as } k\to\infty, \\ f(t_k) &\to f^\infty, \quad f'(t_k)\to 0 \quad \text{as } k\to\infty, \end{aligned}$$

where

$$f^\infty = \limsup_{t\to+\infty} f(t), \qquad f_\infty = \liminf_{t\to+\infty} f(t).$$

Now, we are in position to prove our main result. It presents sufficient conditions ensuring the global attractivity of solutions of system (1.4) and of its asymptotic systems (2.56).

Theorem 2.2.15. *Assume (A1)-(A7) hold. Then*

$$\lim_{t\to+\infty}|x(t)-\hat{x}(t)|=0,$$

for all $x(t)$ and $\hat{x}(t)$ solutions of systems (1.4) *and* (2.56) *respectively, with bounded initial conditions.*

Proof. Let $x(t) = (x_1(t), \dots, x_n(t))$ and $\hat{x}(t) = (\hat{x}_1(t), \dots, \hat{x}_n(t))$ be solutions of systems (1.4) and (2.56) respectively, with bounded initial conditions, and define $y(t) = (y_1(t), \dots, y_n(t))$ by

$$y_i(t) = sign(x_i(t) - \hat{x}_i(t)) d_i^{-1} \int_{\hat{x}_i(t)}^{x_i(t)} \frac{1}{a_i(t,v)} dv. \tag{2.67}$$

From Lemma 2.2.12 and Lemma 2.2.13, we know that $x(t)$ and $\hat{x}(t)$ are bounded on $\mathbb{R}$. It follows that $y(t)$ is a nonnegative bounded function on $\mathbb{R}$ and it is possible to define the limits

$$u_i = \limsup_{t \to +\infty} y_i(t), \qquad i = 1, \dots, n.$$

Taking $u = \max_i \{u_i\} \in [0, +\infty)$, the result is proved if we show that $u = 0$.

Let $i \in \{1, \dots, n\}$ be such that $u_i = u$. As y_i is a differentiable real function, by the fluctuation lemma (see Lemma 2.2.14), there exists a positive real sequence $(t_k)_{k \in \mathbb{N}}$ such that

$$t_k \nearrow +\infty, \ y_i(t_k) \to u \text{ and } y_i'(t_k) \to 0 \text{ as } k \to +\infty. \tag{2.68}$$

Assume contrarily that $u > 0$. For $\varepsilon \in (0, u)$, there is $T = T(\varepsilon) > 0$ such that $|y(t)| < u + \varepsilon$, for $t \geq T$, and

$$\max \left\{ \int_{-\infty}^{-T} d\eta_{ijp}(s), \int_{-\infty}^{-T} d\tilde{\eta}_{ilp}(s) \right\} < \frac{\varepsilon}{\mathcal{Y}}, \tag{2.69}$$

for $i, j, l = 1, \dots, n$, $p = 1, \dots, P$, where $\mathcal{Y} = \sup_{t \in \mathbb{R}} |y(t)|$.
Since

$$\lim_{t \to +\infty} (t - \tau_{ijp}(t)) = \lim_{t \to +\infty} (t - \tilde{\tau}_{ilp}(t)) = +\infty,$$
$$\lim_{t \to +\infty} (\tau_{ijp}(t) - \hat{\tau}_{ijp}(t)) = \lim_{t \to +\infty} (\widetilde{\tau}_{ilp}(t) - \hat{\widetilde{\tau}}_{ilp}(t)) = 0,$$

$t_k \nearrow +\infty$, and $\lim_k y_i(t_k) = u$, then there exists a natural number k_0 such that

$$t_k > 2T, \quad t_k - \hat{\tau}_{ijp}(t_k) > T, \quad t_k - \hat{\widetilde{\tau}}_{ilp}(t_k) > T \text{ and } y_i(t_k) > u - \varepsilon > 0, \tag{2.70}$$

for all $k \geq k_0$, $i, j, l = 1, \dots, n$, and $p = 1, \dots, P$. Consequently, for $k \geq k_0$, we have

$$|y(t_k - \hat{\tau}_{ijp}(t_k))| < u + \varepsilon \text{ and } |y(t_k - \hat{\widetilde{\tau}}_{ilp}(t_k))| < u + \varepsilon. \tag{2.71}$$

For $k \geq k_0$, from systems (1.4), (2.56) and, using the notations (2.55), (2.59), we have

$$\begin{aligned}
y_i'(t_k) &= \operatorname{sign}(x_i(t_k)-\hat{x}_i(t_k))d_i^{-1}\bigg[\frac{x_i'(t_k)}{a_i(t_k,x_i(t_k))}-\frac{\hat{x}_i'(t_k)}{a_i(t_k,\hat{x}_i(t_k))}\\
&\qquad +\int_{\hat{x}_i(t_k)}^{x_i(t_k)} -\frac{\frac{\partial}{\partial t}a_i(t_k,v)}{a_i(t_k,v)^2}\,dv\bigg]\\
&= \operatorname{sign}(x_i(t_k)-\hat{x}_i(t_k))d_i^{-1}\bigg[-b_i(t_k,x_i(t_k))+\hat{b}_i(t_k,\hat{x}_i(t_k))\\
&\qquad +\int_{\hat{x}_i(t_k)}^{x_i(t_k)} -\frac{\frac{\partial}{\partial t}a_i(t_k,v)}{a_i(t_k,v)^2}\,dv+I_i(t_k)-\hat{I}_i(t_k)\\
&\qquad +F_i(\mathcal{U}_i(t_k,x_{t_k}),\mathcal{V}_i(t_k,x_{t_k}))-F_i(\hat{\mathcal{U}}_i(t_k,\hat{x}_{t_k}),\hat{\mathcal{V}}_i(t_k,\hat{x}_{t_k}))\bigg]\\
&\leq \operatorname{sign}(x_i(t_k)-\hat{x}_i(t_k))d_i^{-1}\bigg[-b_i(t_k,x_i(t_k))+\hat{b}_i(t_k,\hat{x}_i(t_k))\\
&\qquad +\int_{\hat{x}_i(t_k)}^{x_i(t_k)} -\frac{\frac{\partial}{\partial t}a_i(t_k,v)}{a_i(t_k,v)^2}\,dv\bigg]+d_i^{-1}\big|I_i(t_k)-\hat{I}_i(t_k)\big|\\
&\qquad +d_i^{-1}\big|F_i(\mathcal{U}_i(t_k,x_{t_k}),\mathcal{V}_i(t_k,x_{t_k}))-F_i(\hat{\mathcal{U}}_i(t_k,\hat{x}_{t_k}),\hat{\mathcal{V}}_i(t_k,\hat{x}_{t_k}))\big|.
\end{aligned}$$

From (A2),(A3), and (A6), we obtain

$$\begin{aligned}
y_i'(t_k) &\leq -\operatorname{sign}(x_i(t_k)-\hat{x}_i(t_k))d_i^{-1}\bigg[\big(b_i(t_k,x_i(t_k))-b_i(t_k,\hat{x}_i(t_k))\big)\\
&\qquad +\int_{\hat{x}_i(t_k)}^{x_i(t_k)} \frac{\frac{\partial}{\partial t}a_i(t_k,v)}{a_i(t_k,v)^2}\,dv\bigg]\\
&\qquad +d_i^{-1}\big|\hat{b}_i(t_k,\hat{x}_i(t_k))-b_i(t_k,\hat{x}_i(t_k))\big|+d_i^{-1}\big|I_i(t_k)-\hat{I}_i(t_k)\big|\\
&\qquad +d_i^{-1}\Big(\zeta_i\big|\mathcal{U}_i(t_k,x_{t_k})-\hat{\mathcal{U}}_i(t_k,\hat{x}_{t_k})\big|+\varsigma_i\big|\mathcal{V}_i(t_k,x_{t_k})-\hat{\mathcal{V}}_i(t_k,\hat{x}_{t_k})\big|\Big)\\
&\leq -d_i^{-1}\big(\beta_i(t_k)+A_i(t_k)\big)|x_i(t_k)-\hat{x}_i(t_k)|+d_i^{-1}\big|\hat{b}_i(t_k,\hat{x}_i(t_k))-b_i(t_k,\hat{x}_i(t_k))\big|\\
&\qquad +d_i^{-1}\big|I_i(t_k)-\hat{I}_i(t_k)\big|+d_i^{-1}\Big(\zeta_i\big|\mathcal{U}_i(t_k,x_{t_k})-\hat{\mathcal{U}}_i(t_k,\hat{x}_{t_k})\big|\\
&\qquad +\varsigma_i\big|\mathcal{V}_i(t_k,x_{t_k})-\hat{\mathcal{V}}_i(t_k,\hat{x}_{t_k})\big|\Big).
\end{aligned}$$

From (2.55) and (2.59), we have

$$
\begin{aligned}
y_i'(t_k) &\le -d_i^{-1}(\beta_i(t_k)+A_i(t_k))|x_i(t_k)-\hat{x}_i(t_k)| + d_i^{-1}\left|\hat{b}_i(t_k,\hat{x}_i(t_k)) - b_i(t_k,\hat{x}_i(t_k))\right| \\
&\quad + d_i^{-1}\left|I_i(t_k)-\hat{I}_i(t_k)\right| + d_i^{-1}\sum_{p=1}^{P}\sum_{j,l=1}^{n}\Bigg[\zeta_i\Bigg(|c_{ijlp}(t_k)-\hat{c}_{ijlp}(t_k)| \\
&\quad \cdot\left|h_{ijlp}(\hat{x}_j(t_k-\hat{\tau}_{ijp}(t_k)),\hat{x}_l(t_k-\hat{\tilde{\tau}}_{ilp}(t_k)))\right| \\
&\quad + |c_{ijlp}(t_k)|\Bigg(\left|h_{ijlp}(x_j(t_k-\hat{\tau}_{ijp}(t_k)),x_l(t_k-\hat{\tilde{\tau}}_{ilp}(t_k))) - h_{ijlp}(\hat{x}_j(t_k-\hat{\tau}_{ijp}(t_k)),\hat{x}_l(t_k-\hat{\tilde{\tau}}_{ilp}(t_k)))\right| \\
&\quad + \left|h_{ijlp}(x_j(t_k-\tau_{ijp}(t_k)),x_l(t_k-\tilde{\tau}_{ilp}(t_k))) - h_{ijlp}(x_j(t_k-\hat{\tau}_{ijp}(t_k)),x_l(t_k-\hat{\tilde{\tau}}_{ilp}(t_k)))\right|\Bigg)\Bigg) \\
&\quad + \varsigma_i\Bigg(\left|d_{ijlp}(t_k)-\hat{d}_{ijlp}(t_k)\right|\cdot\left|f_{ijlp}\left(\int_{-\infty}^{0} g_{ijp}(\hat{x}_j(t_k+s))\,d\eta_{ijp}(s),\int_{-\infty}^{0}\widetilde{g}_{ilp}(\hat{x}_l(t_k+s))\,d\tilde{\eta}_{ilp}(s)\right)\right| \\
&\quad + |d_{ijlp}(t_k)|\Bigg|f_{ijlp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t_k+s))\,d\eta_{ijp}(s),\int_{-\infty}^{0}\widetilde{g}_{ilp}(x_l(t_k+s))\,d\tilde{\eta}_{ilp}(s)\right) \\
&\quad - f_{ijlp}\left(\int_{-\infty}^{0} g_{ijp}(\hat{x}_j(t_k+s))\,d\eta_{ijp}(s),\int_{-\infty}^{0}\widetilde{g}_{ilp}(\hat{x}_l(t_k+s))\,d\tilde{\eta}_{ilp}(s)\right)\Bigg|\Bigg)\Bigg].
\end{aligned}
$$

and by hypotheses (A5) and (A6), we obtain

$$
\begin{aligned}
y_i'(t_k) \ \le\ & -d_i^{-1}(\beta_i(t_k)+A_i(t_k))|x_i(t_k)-\hat{x}_i(t_k)| \\
&+d_i^{-1}\left|\hat{b}_i(t_k,\hat{x}_i(t_k)) - b_i(t_k,\hat{x}_i(t_k))\right| + d_i^{-1}\left|I_i(t_k)-\hat{I}_i(t_k)\right| \\
&+d_i^{-1}\sum_{p=1}^{P}\sum_{j,l=1}^{n}\Bigg[\zeta_i\Bigg(\left|c_{ijlp}(t_k)-\hat{c}_{ijlp}(t_k)\right|\left|h_{ijlp}(\hat{x}_j(t_k-\hat{\tau}_{ijp}(t_k)),\hat{x}_l(t_k-\hat{\tilde{\tau}}_{ilp}(t_k)))\right| \\
&+|c_{ijlp}(t_k)|\Big(\gamma_{ijlp}^{(1)}|x_j(t_k-\hat{\tau}_{ijp}(t_k))-\hat{x}_j(t_k-\hat{\tau}_{ijp}(t_k))| \\
&+\gamma_{ijlp}^{(2)}|x_l(t_k-\hat{\tilde{\tau}}_{ilp}(t_k))-\hat{x}_l(t_k-\hat{\tilde{\tau}}_{ilp}(t_k))|\Big) \\
&+|c_{ijlp}(t_k)|\Big(\gamma_{ijlp}^{(1)}|x_j(t_k-\tau_{ijp}(t_k))-x_j(t_k-\hat{\tau}_{ijp}(t_k))| \\
&+\gamma_{ijlp}^{(2)}|x_l(t_k-\tilde{\tau}_{ilp}(t_k))-x_l(t_k-\hat{\tilde{\tau}}_{ilp}(t_k))|\Big)\Bigg) \\
&+\varsigma_i\Bigg(\left|d_{ijlp}(t_k)-\hat{d}_{ijlp}(t_k)\right|\left|f_{ijlp}\left(\int_{-\infty}^{0} g_{ijp}(\hat{x}_j(t_k+s)d\eta_{ijp}(s),\int_{-\infty}^{0}\tilde{g}_{ilp}(\hat{x}_l(t_k+s))d\tilde{\eta}_{ilp}(s)\right)\right| \\
&+|d_{ijlp}(t_k)|\left(\mu_{ijlp}^{(1)}\xi_{ijp}\int_{-\infty}^{0}|x_j(t_k+s)-\hat{x}_j(t_k+s)|d\eta_{ijp}(s)\right. \\
&\left.+\mu_{ijlp}^{(2)}\widetilde{\xi}_{ilp}\int_{-\infty}^{0}|x_l(t_k+s)-\hat{x}_l(t_k+s)|d\tilde{\eta}_{ilp}(s)\right)\Bigg)\Bigg].
\end{aligned}
$$

From (A2) and (2.67), we have

$$
(d_i\overline{a}_i)^{-1}|x_i(t)-\hat{x}_i(t)| \le y_i(t) \le (d_i\underline{a}_i)^{-1}|x_i(t)-\hat{x}_i(t)|, \quad t\ge 0,\ i=1,\ldots,n,
$$

and consequently,

$$
\begin{aligned}
y_i'(t_k) \leq & -(\beta_i(t_k) + A_i(t_k))\underline{a}_i y_i(t_k) + \delta_i(t_k) \\
& + \sum_{p=1}^{P} \sum_{j,l=1}^{n} \Bigg[\zeta_i |c_{ijlp}(t_k)| \Bigg(\gamma_{ijlp}^{(1)} \overline{a}_j \frac{d_j}{d_i} y_j(t_k - \hat{\tau}_{ijp}(t_k)) \\
& \quad + \gamma_{ijlp}^{(2)} \overline{a}_l \frac{d_l}{d_i} y_l(t_k - \hat{\tilde{\tau}}_{ilp}(t_k)) \Bigg) \\
& + \varsigma_i |d_{ijlp}(t_k)| \Bigg(\mu_{ijlp}^{(1)} \xi_{ijp} \overline{a}_j \frac{d_j}{d_i} \int_{-\infty}^{0} y_j(t_k + s)\, d\eta_{ijp}(s) \\
& \quad + \mu_{ijlp}^{(2)} \widetilde{\xi}_{ilp} \overline{a}_l \frac{d_l}{d_i} \int_{-\infty}^{0} y_l(t_k + s)\, d\tilde{\eta}_{ilp}(s) \Bigg) \Bigg].
\end{aligned} \tag{2.72}
$$

where

$$
\begin{aligned}
\delta_i(t) = & \, d_i^{-1} \big| \hat{b}_i(t, \hat{x}_i(t)) - b_i(t, \hat{x}_i(t)) \big| + d_i^{-1} \big| I_i(t) - \hat{I}_i(t) \big| \\
& + d_i^{-1} \sum_{p=1}^{P} \sum_{j,l=1}^{n} \Bigg[\zeta_i \Bigg(\big| c_{ijlp}(t) - \hat{c}_{ijlp}(t) \big| \cdot \big| h_{ijlp}(\hat{x}_j(t - \hat{\tau}_{ijp}(t)), \hat{x}_l(t - \hat{\tilde{\tau}}_{ilp}(t))) \big| \\
& \quad + |c_{ijlp}(t)| \big(\gamma_{ijlp}^{(1)} |x_j(t - \tau_{ijp}(t)) - x_j(t - \hat{\tau}_{ijp}(t))| \\
& \qquad + \gamma_{ijlp}^{(2)} |x_l(t - \tilde{\tau}_{ilp}(t)) - x_l(t - \hat{\tilde{\tau}}_{ilp}(t))| \big) \Bigg) \\
& \quad + \varsigma_i \big| d_{ijlp}(t) - \hat{d}_{ijlp}(t) \big| \cdot \Bigg| f_{ijlp} \Bigg(\int_{-\infty}^{0} g_{ijp}(\hat{x}_j(t+s))\, d\eta_{ijp}(s), \\
& \quad \int_{-\infty}^{0} \widetilde{g}_{ilp}(\hat{x}_l(t+s))\, d\tilde{\eta}_{ilp}(s) \Bigg) \Bigg| \Bigg].
\end{aligned}
$$

By (A1), $t \mapsto b_j(t, w)$ is bounded for all $w \in \mathbb{R}$ and $j = 1, \ldots, n$, and by (A3), $w \mapsto b_j(t, w)$ is a non-decreasing function for all $t \in [0, +\infty)$. As $x_j(t)$ is bounded, then $t \mapsto b_j(t, x_j(t))$ is a bounded function. Moreover from (A1), (1.4), and the continuity of F_i, h_{ijlp}, f_{ijlp}, g_{ijp}, $\widetilde{g}_{ilp}$, we obtain that $x_j'(t)$ is a bounded function on $(0, +\infty)$, thus $x_j(t)$ is uniformly continuous for all $j = 1, \ldots, n$. Finally, as $\hat{x}(t)$ is bounded, by (A5) and from (2.58), we conclude that

$$
\lim_{t \to +\infty} \delta_i(t) = 0. \tag{2.73}
$$

From (2.70), (2.71) and (2.72), for all $k \geq k_0$, we have

$$\begin{aligned} y_i'(t_k) &\leq -(\beta_i(t_k) + A_i(t_k))\underline{a}_i(u-\varepsilon) + \delta_i(t_k), \\ &+ \sum_{p=1}^{P}\sum_{j,l=1}^{n}\Bigg[\zeta_i|c_{ijlp}(t_k)|\left(\gamma_{ijlp}^{(1)}\overline{a}_j\frac{d_j}{d_i} + \gamma_{ijlp}^{(2)}\overline{a}_l\frac{d_l}{d_i}\right)(u+\varepsilon) \\ &+ \varsigma_i|d_{ijlp}(t_k)|\mu_{ijlp}^{(1)}\xi_{ijp}\overline{a}_j\frac{d_j}{d_i}\left(\int_{-\infty}^{-T} y_j(t_k+s)d\eta_{ijp}(s) + \int_{-T}^{0} y_j(t_k+s)d\eta_{ijp}(s)\right) \\ &+ \varsigma_i|d_{ijlp}(t_k)|\mu_{ijlp}^{(2)}\widetilde{\xi}_{ilp}\overline{a}_l\frac{d_l}{d_i}\left(\int_{-\infty}^{-T} y_l(t_k+s)d\tilde{\eta}_{ilp}(s) + \int_{-T}^{0} y_l(t_k+s)d\tilde{\eta}_{ilp}(s)\right)\Bigg], \end{aligned}$$

and from (2.69), we obtain

$$\begin{aligned} y_i'(t_k) &\leq -(\beta_i(t_k) + A_i(t_k))\underline{a}_i(u-\varepsilon) + \delta_i(t_k) \\ &+ \sum_{p=1}^{P}\sum_{j,l=1}^{n}\Bigg[\zeta_i|c_{ijlp}(t_k)|\left(\gamma_{ijlp}^{(1)}\overline{a}_j\frac{d_j}{d_i} + \gamma_{ijlp}^{(2)}\overline{a}_l\frac{d_l}{d_i}\right)(u+\varepsilon) \\ &+ \varsigma_i|d_{ijlp}(t_k)|\mu_{ijlp}^{(1)}\xi_{ijp}\overline{a}_j\frac{d_j}{d_i}\left(\varepsilon + \int_{-T}^{0} y_j(t_k+s)d\eta_{ijp}(s)\right) \\ &+ \varsigma_i|d_{ijlp}(t_k)|\mu_{ijlp}^{(2)}\widetilde{\xi}_{ilp}\overline{a}_l\frac{d_l}{d_i}\left(\varepsilon + \int_{-T}^{0} y_l(t_k+s)d\tilde{\eta}_{ilp}(s)\right)\Bigg]. \end{aligned}$$

Finally, as $t_k > 2T$, $|y(t)| \leq u+\varepsilon$ for $t \geq T$, and $\int_{-\infty}^{0} d\eta_{ijp}(s) = \int_{-\infty}^{0} d\tilde{\eta}_{ilp}(s) = 1$, we obtain

$$\begin{aligned} y_i'(t_k) \leq{}& -(\beta_i(t_k) + A_i(t_k))\underline{a}_i(u-\varepsilon) + \delta_i(t_k) \\ &+ \sum_{p=1}^{P}\sum_{j,l=1}^{n}\Bigg[\zeta_i|c_{ijlp}(t_k)|\left(\gamma_{ijlp}^{(1)}\overline{a}_j\frac{d_j}{d_i} + \gamma_{ijlp}^{(2)}\overline{a}_l\frac{d_l}{d_i}\right)(u+\varepsilon) \\ &+ \varsigma_i|d_{ijlp}(t_k)|\left(\mu_{ijlp}^{(1)}\xi_{ijp}\overline{a}_j\frac{d_j}{d_i} + \mu_{ijlp}^{(2)}\widetilde{\xi}_{ilp}\overline{a}_l\frac{d_l}{d_i}\right)(u+2\varepsilon)\Bigg]. \end{aligned} \tag{2.74}$$

Since $\lim\limits_{k\to+\infty} y_i'(t_k) = 0$, then by letting $\varepsilon \to 0^+$ and $k \to +\infty$, it follows from (2.68), (2.73), (2.74), and hypothesis (A7) that

$$\begin{aligned} 0 \leq \Bigg[&\limsup_{k\to+\infty}\Bigg(-(\beta_i(t_k) + A_i(t_k))\underline{a}_i \\ &+ \sum_{p=1}^{P}\sum_{j,l=1}^{n}\Bigg(\zeta_i|c_{ijlp}(t_k)|\left(\gamma_{ijlp}^{(1)}\overline{a}_j\frac{d_j}{d_i} + \gamma_{ijlp}^{(2)}\overline{a}_l\frac{d_l}{d_i}\right) \\ &+ \varsigma_i|d_{ijlp}(t_k)|\left(\mu_{ijlp}^{(1)}\xi_{ijp}\overline{a}_j\frac{d_j}{d_i} + \mu_{ijlp}^{(2)}\widetilde{\xi}_{ilp}\overline{a}_l\frac{d_l}{d_i}\right)\Bigg)\Bigg)\Bigg]u < 0, \end{aligned}$$

which is a contradiction. Thus $u = 0$. $\square$

Obviously, the system (1.4) can be considered as an asymptotic system of itself. Thus, we have the following result

Corollary 2.2.16. *Assume (A1)-(A7) hold.*
If $x(t)$ and $\hat{x}(t)$ are solutions of systems (1.4) *with bounded initial conditions, then*

$$\lim_{t\to+\infty} |x(t) - \hat{x}(t)| = 0.$$

Next, we apply our results to the following examples to demonstrate their efficiency, and we provide a comparison with various stability criteria from the literature.

Example 2.3. Consider the low-order CGNN model (2.47) which can be written in the form

$$x_i'(t) = a_i(x_i(t)) \left[-b_i(t, x_i(t)) + G_i \left(\sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{0} x_j(t+s) d\eta_{ij}(s) \right) \right],$$

which is a particular situation of (1.4). Consequently, from Corollary 2.2.16, we obtain the following global attractivity criterion for model (2.47).

Corollary 2.2.17. *Assume that (A3) and* (2.48) *hold and, for each $i, j = 1, \dots, n$, c_{ij} is a bounded function, G_i is a Lipshcitz function with Lipschitz constant $\varsigma_i > 0$, and there exist $\underline{a}_i, \overline{a}_i > 0$ such that*

$$\underline{a}_i \le a_i(u) \le \overline{a}_i, \qquad \forall u \in \mathbb{R}. \tag{2.75}$$

If there exists $d = (d_1, \dots, d_n) > 0$ such that,

$$\limsup_{t\to+\infty} \left(\underline{a}_i \beta_i(t) d_i - \overline{a}_i \sum_{j=1}^{n} \varsigma_i |c_{ij}(t)| d_j \right) > 0, \qquad i = 1, \dots, n,$$

then any two solutions $x(t)$ and $\hat{x}(t)$ of (2.47)*, with bounded initial condition, verify*

$$\lim_{t\to+\infty} |x(t) - \hat{x}(t)| = 0.$$

In case of (2.47) being an $\omega-$periodic model, for some $\omega > 0$, from Theorem 2.2.2, we obtain sufficient conditions for the existence of an $\omega-$periodic solution of (2.47) and the following result holds.

Corollary 2.2.18. *Assume that* (2.48) *holds and that, for each $i, j = 1, \dots, n$, G_i is a Lipshcitz function with Lipschitz constant $\varsigma_i > 0$, a_i verifies* (2.75)*, the functions $t \to b_i(t, u)$ and c_{ij} are $\omega-$periodic for all $u \in \mathbb{R}$, and there exist $\omega-$periodic continuous* *functions* $\beta_i, \beta_i^* : [0, +\infty) \to (0, +\infty)$ *such that*

$$\beta_i(t) \leq \frac{b_i(t,u) - b_i(t,v)}{u - v} \leq \beta_i^*(t), \qquad \forall t \in [0, \omega],\ u, v \in \mathbb{R}, u \neq v. \tag{2.76}$$

If there exists $d = (d_1, \ldots, d_n) > 0$ *such that,*

$$\underline{a}_i \beta_i(t) d_i > \overline{a}_i \sum_{j=1}^{n} \varsigma_i |c_{ij}(t)| d_j, \qquad \forall t \in [0, \omega],\ i = 1, \ldots, n, \tag{2.77}$$

then there exists an $\omega-$*periodic solution of* (2.47), $\tilde{x}(t)$, *which is globally attractive in the set of the solutions of* (2.47) *with bounded initial condition, i.e., for each solution of* (2.47) *with bounded initial conditions,* $x(t)$, *we have:*

$$\lim_{t \to +\infty} |x(t) - \tilde{x}(t)| = 0.$$

Remark 2.2.19. *In [49], sufficient conditions for the existence and global asymptotic stability of an equilibrium point of the autonomous static neural network model* (2.49) *were established. We stress that model* (2.49) *is a particular situation of* (2.47), *and assumptions in Corollary 2.2.18 are weaker than the ones in [49, Theorem 3.2]. In fact, conditions* (2.75) *and* (2.76) *trivially hold in model* (2.49), *for each* $i, j = 1, \ldots, n$, *they assumed that* G_i *satisfies stronger conditions than being Lipschitz and the kernel function,* K_{ij}, *verify* (2.48) *with the additional assumptions*

$$\int_0^{+\infty} u K_{ij}(u) du < +\infty \quad \text{and} \quad K_{ij} = K_{ji}.$$

For model (2.49), *inequality* (2.77) *reads as*

$$d_i > \sum_{j=1}^{n} \varsigma_i |c_{ij}| d_j, \qquad i = 1, \ldots, n,$$

which is equivalent to the matrix

$$\mathcal{N} = I_n - \left[\varsigma_i |c_{ij}|\right]_{i,j=1}^{n},$$

where I_n *denotes the n-dimension identity matrix, being a non-singular M-matrix (see [25]), which is assumed in [49, Theorem 3.2].*

Example 2.4. Considering in (1.4) and (2.56)

$$F_i(u, v) = G_i(u) + H_i(v), \qquad \text{for all } i = 1, \ldots, n, \text{ and } u, v \in \mathbb{R},$$

where $G_i, H_i : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions, we have the following high-order

CGNN models,

$$
\begin{aligned}
x_i'(t) = a_i(t, x_i(t))\Big[& -b_i(t, x_i(t)) + I_i(t) \\
& + G_i\Big(\sum_{p=1}^{P}\sum_{j,l=1}^{n} c_{ijlp}(t) h_{ijlp}\big(x_j(t-\tau_{ijp}(t)), x_l(t-\tilde{\tau}_{ilp}(t))\big)\Big) \\
& + H_i\Big(\sum_{q=1}^{Q}\sum_{j,l=1}^{n} d_{ijlq}(t) f_{ijlq}\Big(\int_{-\infty}^{0} g_{ijq}(x_j(t+s))\, d\eta_{ijq}(s), \\
& \int_{-\infty}^{0} \widetilde{g}_{ilq}(x_l(t+s))\, d\tilde{\eta}_{ilq}(s)\Big)\Big)\Big], \quad t \geq 0, \quad i = 1, \ldots, n,
\end{aligned}
\tag{2.78}
$$

and

$$
\begin{aligned}
x_i'(t) = a_i(t, x_i(t))\Big[& -\hat{b}_i(t, x_i(t)) + \hat{I}_i(t) \\
& + G_i\Big(\sum_{p=1}^{P}\sum_{j,l=1}^{n} \hat{c}_{ijlp}(t) h_{ijlp}\big(x_j(t-\hat{\tau}_{ijp}(t)), x_l(t-\hat{\tilde{\tau}}_{ilp}(t))\big)\Big) \\
& + H_i\Big(\sum_{q=1}^{Q}\sum_{j,l=1}^{n} \hat{d}_{ijlq}(t) f_{ijlq}\Big(\int_{-\infty}^{0} g_{ijq}(x_j(t+s))\, d\eta_{ijq}(s), \\
& \int_{-\infty}^{0} \widetilde{g}_{ilq}(x_l(t+s))\, d\tilde{\eta}_{ilq}(s)\Big)\Big)\Big], \quad t \geq 0, \quad i = 1, \ldots, n,
\end{aligned}
\tag{2.79}
$$

respectively.

If (A3), (2.57), and (2.58) hold, then the system (2.79) is an asymptotic system of (2.78). In Theorem 2.2.2, the following result for the existence of a periodic solution of (2.79) was established.

Theorem 2.2.20. *Assume (A2), (A5), (A6), and the following hypotheses:*

(i) For each $i = 1, \ldots, n$, there exist ζ_i, $\varsigma_i > 0$ such that

$$|G_i(u) - G_i(v)| \leq \zeta_i |u - v|, \quad |H_i(u) - H_i(v)| \leq \varsigma_i |u - v|, \quad u, v \in \mathbb{R};$$

(ii) There is $\omega > 0$ such that, for each $i, j, l = 1, \ldots, n$, $p = 1, \ldots, P$, and $q = 1, \ldots, Q$,

$$
\begin{aligned}
& a_i(t,u) = a_i(t+\omega, u), && \hat{c}_{ijlp}(t) = \hat{c}_{ijlp}(t+\omega), && \hat{\tau}_{ijp}(t) = \hat{\tau}_{ijp}(t+\omega), \\
& \hat{b}_i(t,u) = \hat{b}_i(t+\omega, u), && \hat{d}_{ijlq}(t) = \hat{d}_{ijlq}(t+\omega), && \hat{\tilde{\tau}}_{ijp}(t) = \hat{\tilde{\tau}}_{ijp}(t+\omega), \ \textit{and} \\
& \hat{I}_i(t) = \hat{I}_i(t+\omega), && \textit{for all } t \geq 0 \textit{ and } u \in \mathbb{R};
\end{aligned}
$$

(iii) For each $i = 1, \ldots, n$, there exist $\omega-$periodic continuous functions $\hat{\beta}_i, \hat{\beta}_i^*$:

$[0, +\infty) \to (0, +\infty)$ *such that*

$$\hat{\beta}_i(t) \le \frac{\hat{b}_i(t,u) - \hat{b}_i(t,v)}{u - v} \le \hat{\beta}_i^*(t), \quad \forall t \in [0,\omega],\, u, v \in \mathbb{R},\, u \neq v;$$

(iv) There exists $d = (d_1, \dots, d_n) > 0$ *such that for all* $t \in [0, \omega]$*, and* $i = 1, \dots, n$*,*

$$\begin{aligned} \hat{\beta}_i(t) > \sum_{j,l=1}^{n} \Bigg[&\sum_{p=1}^{P} \zeta_i |\hat{c}_{ijlp}(t)| \left(\frac{d_j}{d_i} \gamma_{ijlp}^{(1)} + \frac{d_l}{d_i} \gamma_{ijlp}^{(2)} \right) \\ &+ \sum_{q=1}^{Q} \varsigma_i |\hat{d}_{ijlq}(t)| \left(\frac{d_j}{d_i} \mu_{ijlq}^{(1)} \xi_{ijq} + \frac{d_l}{d_i} \mu_{ijlq}^{(2)} \widetilde{\xi}_{ilq} \right) \Bigg]. \end{aligned} \tag{2.80}$$

Then, the system (2.79) *has an* $\omega-$*periodic solution.*

From Theorems 2.2.15 and 2.2.20, we obtain the following result:

Corollary 2.2.21. *Assume (A3) and the hypotheses in Theorem 2.2.20. If* (2.58) *holds and there is* $d = (d_1, \dots, d_n) > 0$ *such that*

$$\begin{aligned} -(\hat{\beta}_i(t) + A_i(t))\underline{a}_i + \sum_{j,l=1}^{n} \Bigg(&\sum_{p=1}^{P} \zeta_i |\hat{c}_{ijlp}(t)| \left(\gamma_{ijlp}^{(1)} \overline{a}_j \frac{d_j}{d_i} + \gamma_{ijlp}^{(2)} \overline{a}_l \frac{d_l}{d_i} \right) \\ &+ \sum_{q=1}^{Q} \varsigma_i |\hat{d}_{ijlq}(t)| \left(\mu_{ijlq}^{(1)} \xi_{ijq} \overline{a}_j \frac{d_j}{d_i} + \mu_{ijlq}^{(2)} \widetilde{\xi}_{ilq} \overline{a}_l \frac{d_l}{d_i} \right) \Bigg) < 0, \end{aligned} \tag{2.81}$$

$i = 1, \dots, n, \quad t \in [0, \omega]$, *then every solution of the system* (2.78) *with bounded initial condition,* $x(t)$*, satisfies*

$$\lim_{t \to +\infty} |x(t) - \hat{x}(t)| = 0,$$

where $\hat{x}(t)$ *is the* $\omega-$*periodic solution of* (2.79).

Remark 2.2.22. *We note that Corollary 2.2.21 establishes sufficient conditions for all solutions of* (2.78)*, which is not necessarily a periodic model, to converge to a periodic function as time goes to infinity. In the case of the CGNN model* (2.78) *not being a periodic system, then the criterion in Theorem 2.2.2 cannot be applied to prove the existence of a periodic solution of* (2.78).

As a particular example of model (2.78), we recall the high-order CGNN model (2.53). The existence and global exponential stability of a periodic solution of (2.53) were studied in [40].

Considering $\hat{c}_{ij}, \hat{d}_{ijlq} : [0, +\infty) \to \mathbb{R}$ continuous functions such that

$$\begin{aligned} \lim_{t\to+\infty} \big(c_{ij}(t) - \hat{c}_{ij}(t)\big) &= \lim_{t\to+\infty} \big(d_{ijlq}(t) - \hat{d}_{ijlq}(t)\big) \\ &= \lim_{t\to+\infty} \big(I_i(t) - \hat{I}_i(t)\big) = 0, \end{aligned} \tag{2.82}$$

for each $i, j, l = 1, \dots, n$ and $q = 1, 2$, then (2.53) is an asymptotic system of

$$\begin{aligned} x_i'(t) = a_i(x_i(t))\Bigg[&- b_i(x_i(t)) + \sum_{j=1}^{n} \hat{c}_{ij}(t) f_j(\rho_j x_j(t)) \\ &+ \sum_{j=1}^{n} \hat{d}_{ij11}(t) f_j\left(\rho_j \int_0^{+\infty} K_{ij}(u) x_j(t-u)\, du\right) \\ &+ \sum_{j,l=1}^{n} \hat{d}_{ijl2}(t) f_j\left(\rho_j \int_0^{+\infty} K_{ij}(u) x_j(t-u)\, du\right) \\ &\cdot f_l\left(\rho_l \int_0^{+\infty} K_{il}(u) x_l(t-u)\, du\right) + \hat{I}_i(t)\Bigg]. \end{aligned} \tag{2.83}$$

Assume the following hypotheses for (2.53). For each $i = 1, \dots, n$, and $q = 1, 2$:

(B1) the functions c_{ij}, d_{ijlq}, and I_i are $\omega-$periodic for some $\omega > 0$;

(B2) there are $\underline{a}_i, \overline{a}_i > 0$ such that

$$\underline{a}_i < a_i(u) < \overline{a}_i;$$

(B3) there are $\beta_i, \beta_i^* > 0$ such that

$$\beta_i \le \frac{b_i(u) - b_i(v)}{u - v} \le \beta_i^*, \quad \forall u, v \in \mathbb{R},\ u \ne v;$$

(B4) there exist $M_i, \mu_i > 0$ such that

$$|f_i(u) - f_i(v)| \le \mu_i |u - v| \quad \text{and} \quad |f_i(u)| \le M_i, \quad \forall u, v \in \mathbb{R}.$$

Consequently, from Corollary 2.2.21, we obtain the following result.

Corollary 2.2.23. *Assume (B1)-(B4). If there exists $d = (d_1, \dots, d_n) > 0$ such that, for all $t \in [0, \omega]$ and $i = 1, \dots, n$,*

$$\begin{aligned} \beta_i > \sum_{j=1}^{n} \Bigg[&\big(|c_{ij}(t)| + |d_{ij11}(t)|\big)\mu_j \rho_j \frac{d_j}{d_i} \\ &+ \sum_{l=1}^{n} |d_{ijl2}(t)| \left(\frac{d_j}{d_i} M_l \mu_j \rho_j + \frac{d_l}{d_i} M_j \mu_l \rho_l \right) \Bigg], \end{aligned} \tag{2.84}$$

then the model (2.53) *has an* $\omega-$*periodic solution.*
If, in additional to condition (2.84)*, there exists* $d^* = (d_1^*, \dots, d_n^*) > 0$ *such that, for all* $t \in [0, \omega]$ *and* $i = 1, \dots, n$*,*

$$\begin{aligned} \underline{a}_i \beta_i > \sum_{j=1}^{n} \Bigg[& \big(|c_{ij}(t)| + |d_{ij11}(t)|\big) \mu_j \rho_j \overline{a}_j \frac{d_j^*}{d_i^*} \\ & + \sum_{l=1}^{n} |d_{ijl2}(t)| \left(\frac{d_j^*}{d_i^*} \overline{a}_j M_l \mu_j \rho_j + \frac{d_l^*}{d_i^*} \overline{a}_l M_j \mu_l \rho_l \right) \Bigg], \end{aligned} \tag{2.85}$$

then every solution of the system (2.83) *with bounded initial condition,* $\hat{x}(t)$*, satisfies*

$$\lim_{t \to +\infty} |\hat{x}(t) - \tilde{x}(t)| = 0,$$

where $\tilde{x}$ *is the* $\omega-$*periodic solution of* (2.53).

Proof. In the model (2.78), taking $P = 1$, $Q = 2$ and, for each $i, j, l = 1, \dots, n$, $q = 1, 2$, $u, u_1, u_2 \in \mathbb{R}$, and $t \geq 0$, letting $a_i(t, u) = a_i(u)$, $b_i(t, u) = b_i(u)$, $F_i(u) = H_i(u) = u$, $\tau_{ij1}(t) = \tilde{\tau}_{ij1}(t) = 0$, $h_{ijl1}(u_1, u_2) = f_j(\rho_j u_1)$, $c_{ij11}(t) = c_{ij}(t)$, $c_{ijl1}(t) = d_{ijl1}(t) = 0$ for $l \neq 1$, $f_{ijl1}(u_1, u_2) = f_j(\rho_j u_1)$, $f_{ijl2}(u_1, u_2) = f_j(\rho_j u_1) f_l(\rho_l u_2)$, $g_{ijq}(u) = \widetilde{g}_{ijq}(u) = u$, and $\widetilde{\eta}_{ijq}(s) = \eta_{ijq}(s) = \int_{-\infty}^{0} K_{ij}(-v)dv$, for $s \leq 0$, we obtain model (2.53).
Trivially hypotheses (A6) and (i) in Theorem 2.2.20 hold. From (B1), (B2), and (B3), hypotheses (ii) and (iii) in Theorem 2.2.20 also hold.
For all $u_1, u_2, v_1, v_2 \in \mathbb{R}$, we have

$$\begin{aligned} |h_{ijl1}(u_1, u_2) - h_{ijl1}(v_1, v_2)| &= |f_j(\rho_j u_1) - f_j(\rho_j v_1)| \leq \rho_j \mu_j |u_1 - v_1|, \\ |f_{ijl1}(u_1, u_2) - f_{ijl1}(v_1, v_2)| &= |f_j(\rho_j u_1) - f_j(\rho_j v_1)| \leq \rho_j \mu_j |u_1 - v_1|, \end{aligned}$$

and

$$\begin{aligned} |f_{ijl2}(u_1, u_2) - f_{ijl2}(v_1, v_2)| &= |f_j(\rho_j u_1) f_l(\rho_l u_2) - f_j(\rho_j v_1) f_l(\rho_l v_2)| \\ &\leq |f_j(\rho_j u_1) - f_j(\rho_j v_1)||f_l(\rho_l u_2)| \\ &\quad + |f_j(\rho_j v_1)||f_l(\rho_l u_2) - f_l(\rho_l v_2)| \\ &\leq M_l \rho_j \mu_j |u_1 - v_1| + M_j \rho_l \mu_l |u_2 - v_2|. \end{aligned}$$

for all $i, j, l = 1, \dots, n$, thus hypothesis (A5) holds. Condition (2.80) follows from (2.84), and the inequality (2.81) reads as (2.85). Finally, the result follows from Corollary 2.2.21. □

Remark 2.2.24. *We emphasize that since* (2.53) *is an asymptotic system of itself, its periodic solution,* $\tilde{x}(t)$*, serves as a global attractor for all solutions with bounded initial conditions. The global exponential stability of* $\tilde{x}(t)$ *is proved in Corollary 2.2.9, given* *the additional condition* $\int_0^{+\infty} K_{ij}(u)\mathrm{e}^{\alpha u}du < +\infty$, *for some* $\alpha > 0$.

Example 2.5. Consider the following low-order CGNN model

$$\begin{aligned} x_i'(t) = a_i(x_i(t))\Big[& - b_i(x_i(t)) + \sum_{j=1}^n c_{ij}(t) f_j(x_j(t)) \\ & + \sum_{j=1}^n d_{ij}(t) g_j\left(x_j(t-\tau_{ij}(t))\right) \\ & + \sum_{j=1}^n p_{ij}(t) \int_0^{+\infty} K_{ij}(u) h_j(x_j(t-u))\,du + I_i(t)\Big]. \end{aligned} \tag{2.86}$$

for $t \geq 0$, $i = 1, \dots, n$, where, for each $i, j = 1, \dots, n$, $a_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$, $c_{ij}, d_{ij}, p_{ij}, I_i : [0, +\infty) \to \mathbb{R}$, $\tau_{ij}, K_{ij} : [0, +\infty) \to [0, +\infty)$, and $f_j, g_j, h_j : \mathbb{R} \to \mathbb{R}$ are continuous functions.
The existence and global exponential stability of an almost periodic solution of (2.86) is studied in [62].
For model (2.86), the following hypotheses will be considered:

(C1) There is $\mu > 0$ such that for each $i, j = 1, \dots, n$,

$$\int_0^{+\infty} K_{ij}(t)dt = 1 \quad \text{and} \quad \int_0^{+\infty} \mathrm{e}^{\mu t} K_{ij}(t)dt < +\infty;$$

(C2) For each $i = 1, \dots, n$, there are $\underline{a}_i, \overline{a}_i > 0$ such that

$$\underline{a}_i < a_i(u) < \overline{a}_i, \quad \forall u \in \mathbb{R};$$

(C3) For each $i = 1, \dots, n$, there is $\beta_i > 0$ such that

$$\beta_i \leq \frac{b_i(u) - b_i(v)}{u - v}, \quad u, v \in \mathbb{R},\ u \neq v;$$

(C4) For each $i, j = 1, \dots, n$, the function τ_{ij} is almost periodic;

(C5) For each $i = 1, \dots, n$, $f_i(0) = g_i(0) = h_i(0) = 0$ and there exist $L_i^f, L_i^g, L_i^h > 0$ such that

$$|f_i(u)-f_i(v)| \leq L_i^f|u-v|,\ |g_i(u)-g_i(v)| \leq L_i^g|u-v|,\ \text{and } |h_i(u)-h_i(v)| \leq L_i^h|u-v|,$$

for all $u, v \in \mathbb{R}$.

Theorem 2.2.25. *Assume (C1)-(C5).*
If there are $\tilde{c}_{ij}, \tilde{d}_{ij}, \tilde{p}_{ij}, \tilde{I}_i : \mathbb{R} \to \mathbb{R}$ *continuous and almost periodic functions on* $\mathbb{R}$ *such that:*

(i) $\lim_{t\to+\infty}(c_{ij}(t)-\tilde{c}_{ij}(t)) = \lim_{t\to+\infty}(d_{ij}(t)-\tilde{d}_{ij}(t)) = \lim_{t\to+\infty}(p_{ij}(t)-\tilde{p}_{ij}(t))$ $= \lim_{t\to+\infty}(I_i(t)-\tilde{I}_i(t)) = 0$*;*

(ii) there exists $d = (d_1, \dots, d_n) > 0$ *verifying*

$$d_i \underline{a}_i \beta_i > \sum_{j=1}^{n} \overline{a}_j d_j \left(\tilde{c}_{ij}^{+} L_j^f + \tilde{d}_{ij}^{+} L_j^g + \tilde{p}_{ij}^{+} L_j^h \right), \quad i = 1, \dots, n,$$

where $\tilde{c}_{ij}^{+} = \sup_{t\in\mathbb{R}} |\tilde{c}_{ij}(t)|$, $\tilde{d}_{ij}^{+} = \sup_{t\in\mathbb{R}} |\tilde{d}_{ij}(t)|$, *and* $\tilde{p}_{ij}^{+} = \sup_{t\in\mathbb{R}} |\tilde{p}_{ij}(t)|$,

then there exists an almost periodic function $\tilde{x} : \mathbb{R} \to \mathbb{R}$ *such that*

$$\lim_{t\to+\infty} |x(t) - \tilde{x}(t)| = 0,$$

for all solution $x(t)$ *of model* (2.86) *with bounded initial condition.*

Proof. Clearly, system (2.86) is a particular situation of model (1.4).
By condition (i) and Definition 2.56, the system

$$\begin{aligned} x_i'(t) = a_i(x_i(t))\Bigg[-b_i(x_i(t)) &+ \sum_{j=1}^{n} \tilde{c}_{ij}(t) f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} \tilde{d}_{ij}(t) g_j\left(x_j(t-\tau_{ij}(t))\right) \\ &+ \sum_{j=1}^{n} \tilde{p}_{ij}(t) \int_0^{+\infty} K_{ij}(u) h_j(x_j(t-u))\,du + \tilde{I}_i(t) \Bigg]. \end{aligned} \tag{2.87}$$

is an asymptotic system of (2.86).
From [62, Theorem 1], system (2.87) has a unique almost periodic solution. Denote it by $\tilde{x}(t)$.
From (C1)-(C5), hypotheses (A1)-(A6) hold. From condition (ii), we conclude that

$$\limsup_{t\to+\infty} \left[-\underline{a}_i \beta_i + \sum_{j=1}^{n} \overline{a}_j \frac{d_j}{d_i} \left(c_{ij}(t) L_j^f + d_{ij}(t) L_j^g + p_{ij}(t) L_j^h \right) \right] < 0,$$

and (A7) holds. Now the conclusion follows from Theorem 2.2.15. □

Remark 2.2.26. *We note that the function* $\tilde{x}(t)$ *may not be a solution of the model* (2.86)*. Moreover, the coefficient functions* c_{ij}, d_{ij}, p_{ij} *and the inputs* I_i *are not necessarily almost periodic functions, thus [62, Theorem 2] can not be used to prove the global attractivity of* (2.86)*.*

2.2.3 Numerical examples

Firstly, we present numerical examples to illustrate the applicability of some new results given in Section 2.2.1.

Example 2.6. Taking $c, d, \hat{c}, \hat{d} \in \mathbb{R}$, the system

$$\begin{aligned} x_1'(t) &= \left(\frac{1}{48}\sin\big(x_1(t)\big) + \frac{7}{48}\right)\Bigg[-(9+\sin(t))x_1(t) \\ &\quad + c\cos(t)\arctan\big(x_1(t-|\sin(t)|)\big)\arctan\big(x_2(t-|\cos(t)|)\big) \\ &\quad + d\sin(t)\int_0^{+\infty} \mathrm{e}^{-u}x_2(t-u)\,du + \cos(t)\Bigg], \\ x_2'(t) &= \big(2+\cos\big(x_2(t)\big)\big)\Bigg[-(2+\cos(t))x_2(t) \\ &\quad + \hat{c}\sin(t)\arctan\big(x_1(t-|\cos(t)|)\big) + \mathrm{e}^{\sin(t)} \\ &\quad + \hat{d}\cos(t)\tanh\left(\int_0^{+\infty}\mathrm{e}^{-u}x_1(t-u)\,du\right)\tanh\left(\int_0^{+\infty}\mathrm{e}^{-u}x_2(t-u)\,du\right)\Bigg], \end{aligned} \tag{2.88}$$

for $t \geq 0$, is a $2\pi-$periodic example of a high-order CGNN model.
Defining $\eta_{ij1}, \widetilde{\eta}_{ij1} : (-\infty, 0] \to \mathbb{R}$ by

$$\eta_{ij1}(s) = \widetilde{\eta}_{ij1}(s) = \int_{-\infty}^{s} \mathrm{e}^{v} dv, \qquad s \in (-\infty, 0],\ i, j = 1, 2,$$

system (2.88) is a particular situation of (2.42) with $n = 2$, $P = Q = 1$, and for $u, v \in \mathbb{R}$, $F_1(u) = F_2(u) = G_1(u) = G_2(u) = u$, $g_{111}(u) = 1$, $g_{121}(u) = g_{211}(u) = g_{221}(u) = u$, $h_{1121}(u,v) = \arctan(u)\arctan(v)$, $h_{2111}(u,v) = \arctan(u)$, $f_{1211}(u,v) = v$, $f_{2121}(u,v) = \tanh(u)\tanh(v)$, and all other $h_{ijl1}(u,v) = f_{ijl1}(u,v) = 0$, for $i, j, l = 1, 2$. However, (2.88) is not a particular case of (2.53), thus the model studied in [40] is not general enough to include (2.88) as a particular example.
Following the notations in (2.42) and (2.44), we have $\underline{a}_1 = \frac{1}{8}$, $\overline{a}_1 = \frac{1}{6}$, $\underline{a}_2 = 1$, $\overline{a}_2 = 3$, $\underline{\beta}_1 = 8$, $\underline{\beta}_2 = 1$, $\zeta_i = \varsigma_i = 1$, $\gamma_{1121}^{(1)} = \gamma_{1121}^{(2)} = \frac{\pi}{2}$, $\gamma_{2111}^{(1)} = 1$, $\gamma_{2111}^{(2)} = \mu_{1211}^{(1)} = 0$, $\mu_{1211}^{(2)} = \mu_{2121}^{(1)} = \mu_{2121}^{(2)} = 1$, $\xi_{ij1} = \tilde{\xi}_{ij1} = 1$, $I_1(t) = \cos(t)$, $I_2(t) = \mathrm{e}^{\sin(t)}$, $c_{1121}(t) = c\cos(t)$, $c_{2111}(t) = \hat{c}\sin(t)$, $d_{1211}(t) = d\sin(t)$, $d_{2121}(t) = \hat{d}\cos(t)$, and all other $c_{ijl1}(t) = d_{ijl1}(t) = 0$, for $i, j, l = 1, 2$.
Consequently, example (2.88) is $2\pi-$periodic and the matrix $\mathcal{M}$, defined in (2.45), has the form

$$\mathcal{M} = \begin{bmatrix} 1 - \frac{\pi}{12}|c| & -\frac{1}{2}\left(\frac{\pi}{2}|c| + |d|\right) \\ \\ -3\left(|\hat{c}| + |\hat{d}|\right) & 1 - 3|\hat{d}| \end{bmatrix}.$$

Condition (2.33) trivially holds with $\vartheta \in (0, 1)$. Consequently, Corollary 2.2.5 assures the existence and exponential stability of a $2\pi-$periodic solution of (2.88) in case $\mathcal{M}$ being a non-singular M-matrix. For example, if we consider $c = \frac{1}{\pi}$, $d = \frac{1}{100}$, $\hat{c} = \frac{1}{15}$, $\hat{d} = \frac{1}{30}$, we have

$$\mathcal{M} = \begin{bmatrix} \frac{11}{12} & -\frac{51}{200} \\ \\ -\frac{3}{10} & \frac{9}{10} \end{bmatrix},$$

which is a non-singular M-matrix.

Example 2.7. The following $2\pi-$periodic delay differential system is a particular situation of (1.3),

$$\begin{aligned} x_1'(t) &= \big(\sin\big(x_1(t)\big) + 2\big)\Bigg[- 4x_1(t)\mathrm{e}^{\frac{\sin(x_1(t))}{1+x_1(t)^2}} \\ &\qquad + \frac{1}{3}\cos(t)\tanh\big(x_2(t - |\sin(t)|)\big) + \mathrm{e}^{\sin(t)}\Bigg], \\ x_2'(t) &= \big(\cos\big(x_2(t)\big) + 2\big)\Bigg[- (5+\cos(t))x_2(t) \\ &\qquad + \frac{2}{3}\sin(t)\tanh\big(x_1(t - |\cos(t)|)\big) + \cos(t)\Bigg], \end{aligned} \tag{2.89}$$

where $n = 2$, $a_1(t,u) = \sin(u){+}2$, $b_1(t,u) = 4u\mathrm{e}^{\frac{\sin(u)}{1+u^2}}$, $f_1(t,\varphi) = \frac{1}{3}\cos(t)\tanh(\varphi_2(-|\sin(t)|)){+}$ $\mathrm{e}^{\sin(t)}$, $a_2(t,u) = \cos(u){+}2$, $b_2(t,u) = (5{+}\cos(t))u$, and $f_2(t,\varphi) = \frac{2}{3}\sin(t)\tanh(\varphi_1(-|\cos(t)|)){+}$ $\cos(t)$.

The hypothesis (H1)-(H5) and (H1*)-(H5*) holds with $\underline{a}_1 = \underline{a}_2 = 1$, $\overline{a}_1 = \overline{a}_2 = 3$, $\beta_1(t) = 2$, $\beta_1^*(t) = 8$, $\beta_2(t) = \beta_2^*(t) = 5 + \cos(t)$, $\mathcal{L}_1(t) = \frac{1}{3}\cos(t)$, and $\mathcal{L}_2(t) = \frac{2}{3}\sin(t)$, thus Theorem 2.1.10 assures the existence and exponential stability of a $2\pi-$ periodic solution of (2.89).

Remark 2.2.27. *Comparing with the hypotheses assumed in [38], the functions* a_1 *and* b_1 *do not satisfy the required properties. Furthermore, considering the system* (2.89), *the matrix* E*, as defined on page 6 of [38], satisfies* $E^* \geq E$*, where*

$$E^* = \begin{bmatrix} \frac{12\pi}{11} - 88\pi^2 & -\frac{10\pi}{11} - 4\pi^2 \\ \\ -\pi - 8\pi^2 & \pi - 80\pi^2 \end{bmatrix}.$$

Since E^* *does not constitute a non-singular M-matrix, it follows that* E *is not a non-singular matrix (see [25, Theorem 5.7]). Consequently, the criterion in [38] cannot be used to conclude the existence of a periodic solution of* (2.89). *In figures 2.1 and 2.2, three solutions* $(x_1(t), x_2(t))$ *of the system* (2.89), *with initial condition* $\varphi(s) = (\sin(s), -\cos(s))$, $\varphi(s) = (1/2, -\mathrm{e}^s/2)$, $\varphi(s) = (cos(s)/2, -\mathrm{e}^s)$ *for* $s \leq 0$*, at* $t_0 = 0$*, are* *plotted.*

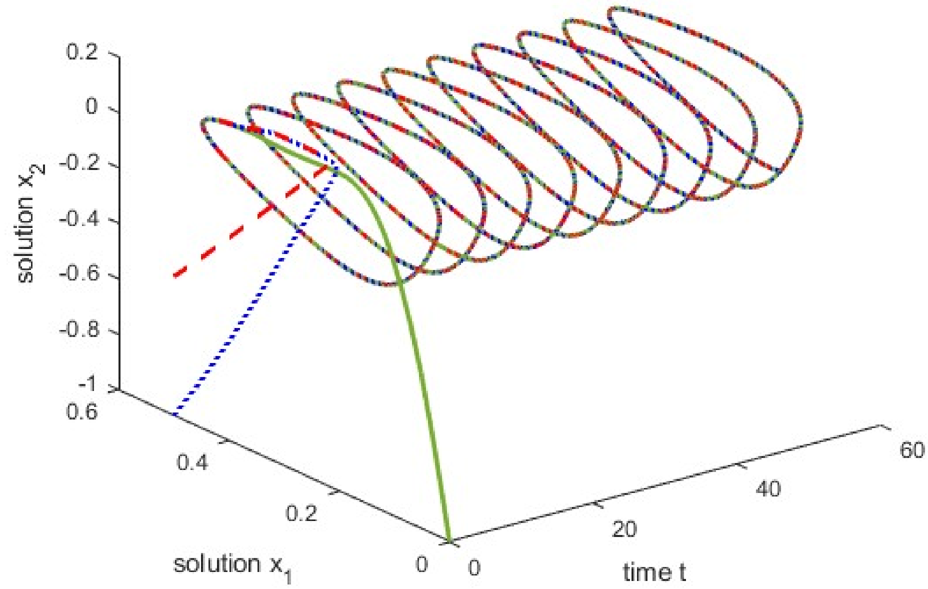

Figure 2.1: Numerical simulations of three solutions $(x_1(t), x_2(t))$ of system (2.89), with initial condition $\varphi(s) = (\sin(s), -\cos(s))$, $\varphi(s) = (1/2, -e^s/2)$, $\varphi(s) = (cos(s)/2, -e^s)$ for $s \leq 0$, at $t_0 = 0$.

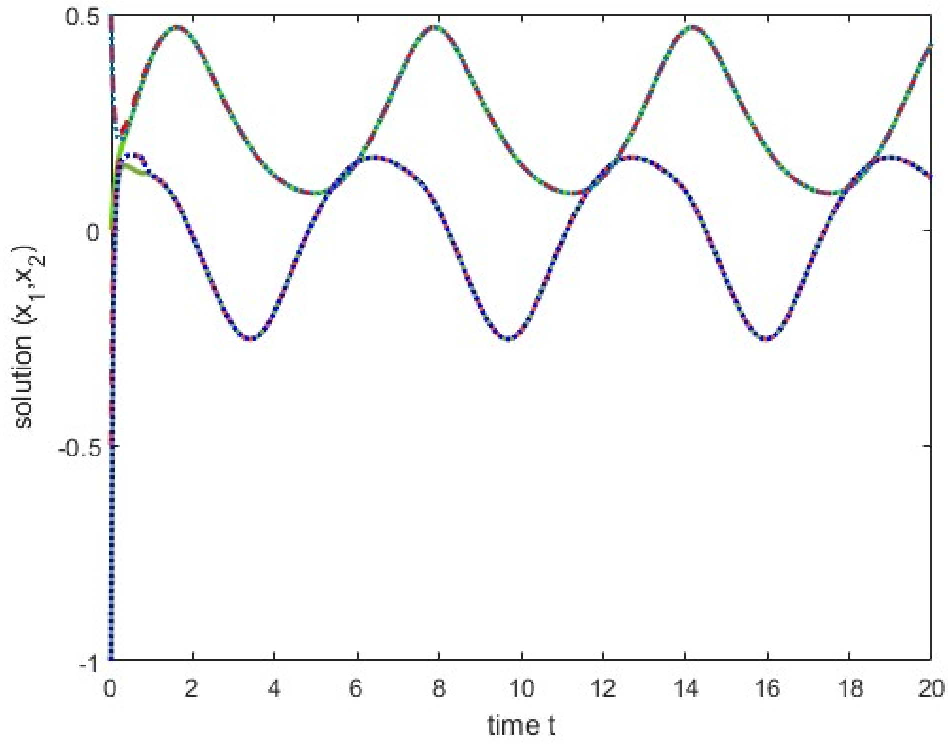

Figure 2.2: Numerical simulations of three solutions $(x_1(t), x_2(t))$ of system (2.89), with initial condition $\varphi(s) = (\sin(s), -\cos(s))$, $\varphi(s) = (1/2, -e^s/2)$, $\varphi(s) = (cos(s)/2, -e^s)$ for $s \leq 0$, at $t_0 = 0$.

Furthermore, we present a numerical example to illustrate the effectiveness of new results presented in Section 2.2.2.

Example 2.8. Consider the following numerical example which is a particular situation of (1.4).

$$\begin{aligned} x_1'(t) &= \big(\sin\big(x_1(t)\big)+2\big)\Bigg[-(4+\mathrm{e}^{-t})x_1(t)\mathrm{e}^{\frac{\sin(x_1(t))}{1+x_1(t)^2}} \\ &\quad + \left(\frac{1}{3}\cos(t)+\mathrm{e}^{-t}\right)\tanh\big(x_2(t-|\sin(t)|)\big)+\mathrm{e}^{\sin(t)}+\mathrm{e}^{-t}\Bigg] \\ x_2'(t) &= \big(\cos\big(x_2(t)\big)+2\big)\Bigg[-(5+\cos(t)+\mathrm{e}^{-t})x_2(t) \\ &\quad + \left(\frac{2}{3}\sin(t)+\mathrm{e}^{-t}\right)\tanh\big(x_1(t-|\cos(t)|)\big)+\cos(t)+\mathrm{e}^{-t}\Bigg]. \end{aligned} \tag{2.90}$$

Here, $n=2$, $P=1$, $a_1(t,u)=\sin(u)+2$, $a_2(t,u)=\cos(u)+2$, $b_2(t,u)=(5+\cos(t)+\mathrm{e}^{-t})u$, $b_1(t,u)=(4+\mathrm{e}^{-t})u\mathrm{e}^{\frac{\sin(u)}{1+u^2}}$, $F_1(u)=F_2(u)=u$, $c_{1211}(t)=\frac{1}{3}\cos(t)+\mathrm{e}^{-t}$, $c_{2111}(t)=\frac{2}{3}\sin(t)+\mathrm{e}^{-t}$, $h_{1211}(u,v)=h_{2111}(u,v)=\tanh(u)$, $\tau_{121}(t)=|\sin(t)|$, $\tau_{211}(t)=|\cos(t)|$, $I_1(t)=\mathrm{e}^{\sin(t)}+\mathrm{e}^{-t}$, $I_2(t)=\cos(t)+\mathrm{e}^{-t}$, and $c_{11j1}(t)=c_{22j1}(t)=d_{ijl1}(t)=0$.

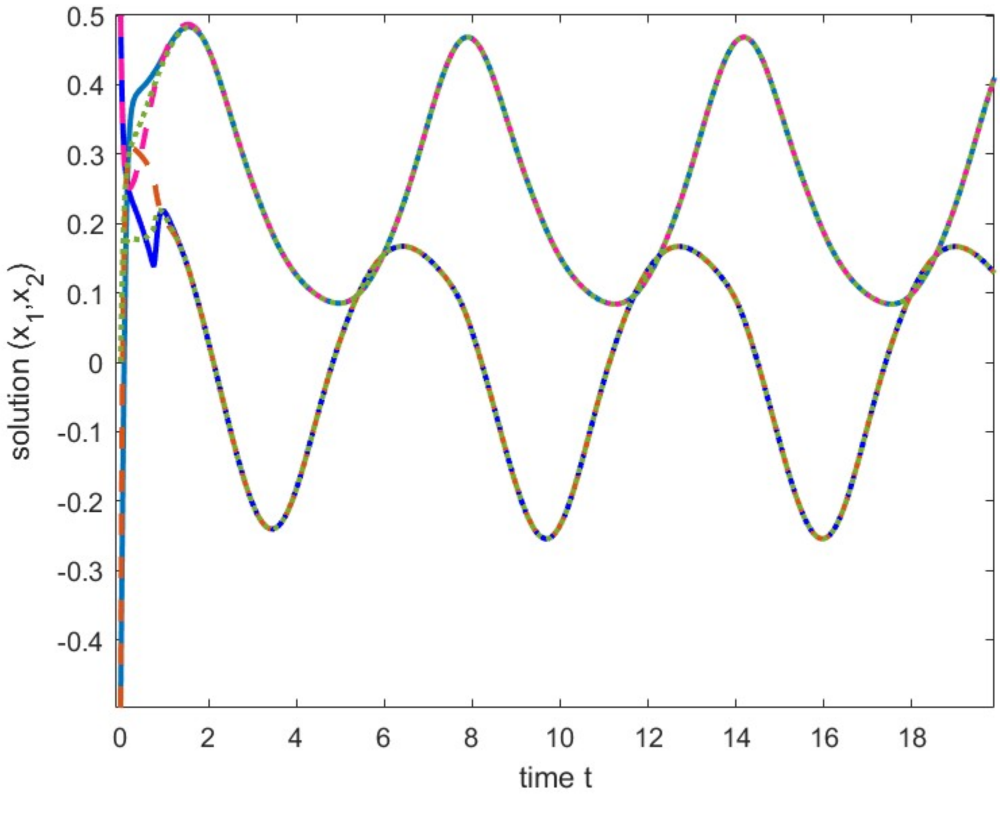

Figure 2.3: Numerical simulation of three solutions $(x_1(t), x_2(t))$ of system (2.90), with initial condition $\varphi(s)=(-\mathrm{e}^s/2, \cos(s)/2)$, $\varphi(s)=(\cos(s)/2, -\mathrm{e}^s/2)$, $\varphi(s)=(\sin(s), \mathrm{e}^s-1)$ for $s\le 0$, at $t_0=0$.

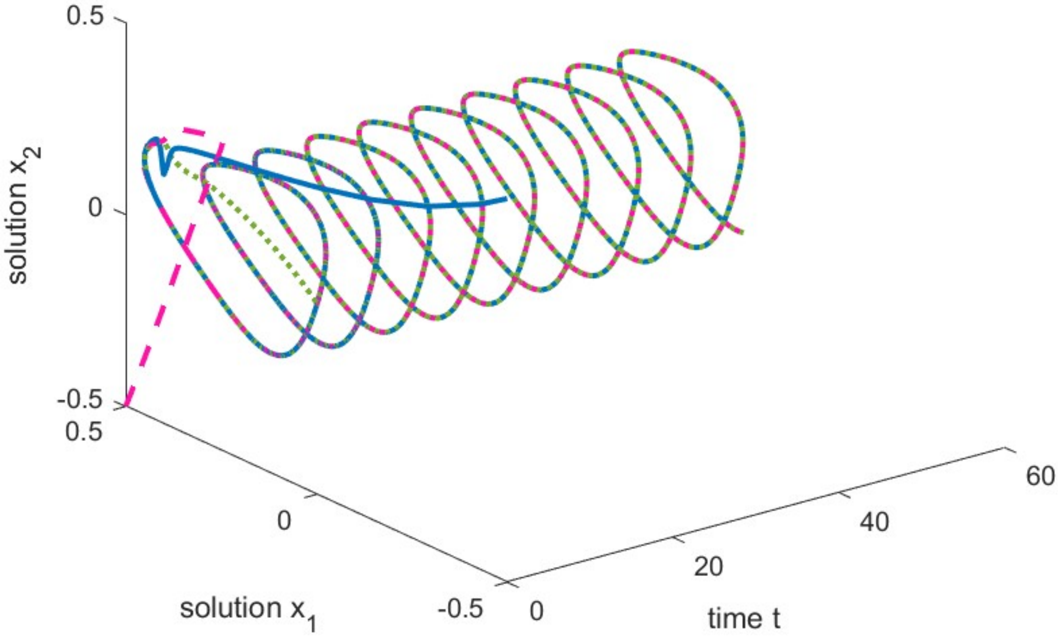

Figure 2.4: Numerical simulation of three solutions $(x_1(t), x_2(t))$ of system (2.90), with initial condition $\varphi(s) = (-\mathrm{e}^s/2, \cos(s)/2)$, $\varphi(s) = (\cos(s)/2, -\mathrm{e}^s/2)$, $\varphi(s) = (\sin(s), \mathrm{e}^s - 1)$ for $s \leq 0$, at $t_0 = 0$.

System (2.90) has the following system

$$\begin{aligned}
\hat{x}_1'(t) &= \big(\sin\big(\hat{x}_1(t)\big) + 2\big)\bigg[-4x_1(t)\mathrm{e}^{\frac{\sin(\hat{x}_1(t))}{1+\hat{x}_1(t)^2}} \\
&\qquad + \frac{1}{3}\cos(t)\tanh\big(\hat{x}_2(t - |\sin(t)|)\big) + \mathrm{e}^{\sin(t)}\bigg], \\
\hat{x}_2'(t) &= \big(\cos\big(\hat{x}_2(t)\big) + 2\big)\bigg[-(5 + \cos(t))\hat{x}_2(t) \\
&\qquad + \frac{2}{3}\sin(t)\tanh\big(\hat{x}_1(t - |\cos(t)|)\big) + \cos(t)\bigg],
\end{aligned} \tag{2.91}$$

as one of its asymptotic systems. As hypotheses (A1)-(A7) hold, Theorem 2.2.15 allows us to conclude that

$$\lim_{t\to+\infty} |x(t) - \hat{x}(t)| = 0,$$

for all solutions $x(t) = (x_1(t), x_2(t))$ and $\hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t))$, of (2.90) and (2.91) respectively, with bounded initial conditions. Noting that (2.91) is a $2\pi-$periodic system, Theorem 2.2.2 assures that (2.91) has a $2\pi-$periodic solution $\tilde{x}(t) = (\tilde{x}_1(t), \tilde{x}_2(t))$. Thus

$$\lim_{t\to+\infty} |x(t) - \tilde{x}(t)| = 0,$$

for all solutions $x(t)$ of (2.90) with bounded initial conditions. We stress that all solutions of (2.90) converge asymptotically to $\tilde{x}(t)$ which is not a solution of (2.90).
Figures 2.3 and 2.4 show the plot of three solutions $(x_1(t), x_2(t))$ of system (2.90), with initial condition $\varphi(s) = (-\mathrm{e}^s/2, \cos(s)/2)$, $\varphi(s) = (\cos(s)/2, -\mathrm{e}^s/2)$, $\varphi(s) = (\sin(s), \mathrm{e}^s - 1)$ for $s \leq 0$, at $t_0 = 0$.

2.2.4 Conclusion

In this chapter, we separately present criteria for the existence of a periodic solution (Theorem 2.1.9) and exponential stability (Theorem 2.1.3) of a general family of functional differential equations (1.3) with unbounded delays. This represents a significant extension of the previous results in [24].
The global stability of neural network models holds a significant role in the solution of optimization problems [14] and in computational image processing [1]. In particular, Cohen-Grossberg-type models are an important tool for pattern formation and partial memory storage [15]. As the interest for this models is increasing (see [8] and reference therein), here we derived criteria for the existence (Theorem 2.2.2) and exponential stability (Theorem 2.2.1) of a periodic solution for a general Cohen-Grossberg type model (2.31). Model (2.31) is general enough to encompass both low-order and high-order models. In particular, the model we studied is more general than the one explored in [3]. In Example 2.1., we established a new criterion for the existence and exponential stability of a periodic solution (Corollary 2.2.6) for the low-order Cohen-Grossberg model (2.47). Similarly, in Example 2.2., a criterion (Corollary 2.2.9) is established for the high-order Cohen-Grossberg model (2.53), result particularly significant because the previous proof presented in [40] had issues (Remark 2.2.10).
Furthermore, we have presented sufficient conditions for the global convergence of asymptotic systems in high-order CGNN models with infinite discrete time-varying and distributed delays, as shown in Theorem 2.2.15. This represents a significant extension of the previous results found in [50, 63, 66, 67, 72].
A global attractivity criterion is obtained for the CGNN model (2.47), as stated in Corollary 2.2.17. This new result enhances the previous criterion found in [49] for the static neural network model (2.49). In Example 2.4., we derived sufficient conditions for all solutions of the high-order CGNN model (2.78), with bounded initial conditions, to converge to a periodic function as time goes to infinity, as detailed in Corollary 2.2.21. It is relevant to note that Theorem 2.2.2 cannot be applied to prove the existence of a periodic solution of (2.78) because (2.78) is not necessarily a periodic model.
Finally, in Example 2.5., we obtain sufficient conditions for all solutions of CGNN model (2.86), presented in [62], with bounded initial conditions, to converge an almost periodic function as times goes to infinity, as stated in Theorem 2.2.25. It is worth noting that the coefficients in (2.86) are not necessarily almost periodic.

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Chapter 3

Lotka-Volterra model

In this chapter, we investigate the concept of permanence for a general delay differential system and discuss the novality of our result with literature. Additionally, we study the permanence and provide a partial result regarding convergence of asymptotic systems of Lotka-Volterra type model. Moreover, we provide a comparison with the results in the literature.

3.1 Permanence of a general system with unbounded delay

In this chapter, firstly, we study the permanence of the general differential system (1.5) in a convenient phase space. We refer to the notation mentioned in Chapter 2.
We consider the set

$$BC^n_+ = \big\{\phi \in BC^n : \phi(s) \geq 0 \text{ for } s \in (-\infty, 0), \text{ and } \phi(0) > 0\big\},$$

where $BC^n = \{\phi \in C((-\infty, 0]; \mathbb{R}^n), \phi \text{ is bounded } \}$ is subset of the space $UC_{\mathcal{G}}$ defined before in Section 2.1. Thus, we consider the functional differential equation (1.5) in the phase space $UC_{\mathcal{G}}$, for a function $\mathcal{G}$ that satisfies $(\mathcal{G}_1)$-$(\mathcal{G}_3)$ as mentioned in Section 2.1, the continuity of the functions f_i and g_i assures the existence of solutions of (1.5) with initial condition

$$x_{t_0} = \phi, \text{ for } \phi \in BC^n_+ \text{ and } t_0 \geq 0, \tag{3.1}$$

for more details, see [30]. The goal is to apply the results to Lotka-Volterra models, thus we restrict our study only to solutions of (1.5) with non-negative bounded initial conditions (3.1).

We obtain sufficient conditions for the permanence of solutions to (1.5)- (3.1). In order to do that, the following set of hypotheses is assumed.

(P1) For $i = 1, \dots, n$, the functions $g_i : \mathbb{R}^+_0 \times \mathbb{R} \to \mathbb{R}$ and $f_i : \mathbb{R}^+_0 \times BC^n_+ \to \mathbb{R}^+_0$ are continuous and locally Lipschitz in the second variable;

(P2) There are continuous functions $\alpha_{ij} : \mathbb{R}^+_0 \times BC^1_+ \to \mathbb{R}^+_0$, $\tau_{ij} : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, and

$c_{ij} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and, a constant $\gamma_{ij} > 0$ satisfying

$$f_i(t, \phi) \leq \sum_{j=1}^{n} \alpha_{ij}(t, \phi_j) + \sum_{j=1}^{n} \gamma_{ij} c_{ij}(t) \int_{-\infty}^{0} \phi_j(s) d\eta_{ij}(s), \text{ for all } t \geq 0,$$

and $\phi = (\phi_1, \ldots, \phi_n) \in BC_+^n$. Moreover the function α_{ij} satisfy the following,

(P2.1) if ϕ, $\psi \in BC_+^1$ and $t \geq 0$ verify $\phi(s) \leq \psi(s)$ for all $s \geq -\tau_{ij}(t)$, then

$$\alpha_{ij}(t, \phi) \leq \alpha_{ij}(t, \psi);$$

(P2.2) for each $\alpha > 0$, the function α_{ij} is bounded on $\mathbb{R}_0^+ \times \mathcal{B}_\alpha(0)$ where

$$\mathcal{B}_\alpha(0) = \{\phi \in BC_+^1 : \|\phi\| \leq \alpha\};$$

(P3) For $i, j = 1, \ldots, n$, $\lim_{t \to +\infty} (t - \tau_{ij}(t)) = +\infty$;

(P4) For $i, j = 1, \ldots, n$, the function $\eta_{ij} : \mathbb{R}_0^- \to \mathbb{R}$ is bounded and non-decreasing with

$$\int_{-\infty}^{0} d\eta_{ij}(s) = \eta_{ij}(0) - \eta_{ij}(-\infty) = 1;$$

(P5) For each $i = 1, \ldots, n$, the function $g_i : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ satisfy the following

(P5.1) for any constant $\sigma > 1$, $g_i(t, u)$ is bounded on $\mathbb{R}_0^+ \times [\sigma^{-1}, \sigma]$, and there is a positive constant ω_i such that the function

$$u \to \liminf_{t \to +\infty} \int_{t}^{t+\omega_i} g_i(s, u) ds$$

is continuous at $u = 0$;

(P5.2) there are positive constants $k_1, k_2, \varpi_1, \varpi_2$, with $k_2 > k_1$ such that

$$\liminf_{t \to +\infty} \int_{t}^{t+\varpi_1} g_i(s, k_1) ds > 0, \quad \limsup_{t \to +\infty} \int_{t}^{t+\varpi_2} g_i(s, k_2) ds < 0;$$

(P5.3) the partial derivative $\frac{\partial}{\partial u} g_i(t, u)$ exists for all $(t, u) \in \mathbb{R}_0^+ \times \mathbb{R}^+$, and there are a nonnegative continuous function $q(t)$ and a constant $\omega > 0$, satisfying $\liminf_{t \to +\infty} \int_{t}^{t+\omega} q(s) ds > 0$, and a continuous function $p(u) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\frac{\partial}{\partial u} g_i(t, u) \leq -q(t) p(u), \quad (t, u) \in \mathbb{R}_0^+ \times \mathbb{R}^+. \tag{3.2}$$

Now, we consider the following $n-$species Kolmogorov system

$$u_i'(t) = u_i(t) g_i(t, u_i(t)), \quad t \geq 0\,, i = 1, \ldots, n, \tag{3.3}$$

where g_i is the function in system (1.5). As g_i is continuous for any $(t_0, u_0) \in \mathbb{R}_0^+ \times \mathbb{R}^+$, the system (3.3) has a solution $u_i(t)$ satisfying $u_i(t_0) = u_0$. If $u_i(t) > 0$ on the interval of existence, then the solution $u_i(t)$ is said to be a positive solution. In order to establish our main result regarding permanence, we introduce a lemma which will be a crucial tool in our proofs.

Lemma 3.1.1. *[57, Lemma 1] Assuming (P5.1)-(P5.3) hold, we have the following*

(1) there are constants $m, M > 0$ such that

$$m \leq \liminf_{t \to +\infty} u_i(t) \leq \limsup_{t \to +\infty} u_i(t) \leq M$$

for any solution $u(t) = (u_1(t), \ldots, u_n(t)) > 0$ of system (3.3).

(2) each fixed solution $u^*(t) = (u_1^*(t), \ldots, u_n^*(t)) > 0$ of system (3.3) *satisfies*

$$\lim_{t \to +\infty} |u(t) - u^*(t)| = 0$$

for every positive solution $u(t)$ of (3.3).

We noticed that the assumptions (P5.1)-(P5.3) are quite weak and applicable for wide classes of ecologically reasonable functions [57]. Now, we are in a position to state our main result.

Theorem 3.1.2. *Assume that the system* (1.5) *satisfies (P1)-(P5) and that there is $\varepsilon_1 > 0$, such that*

$$\begin{aligned} \liminf_{t \to +\infty} \int_t^{t+\omega_i} \Bigg[& g_i(s, 0) - \sum_{j=1}^n \alpha_{ij}(s, (x_j^*)_s + \varepsilon_1) \\ & - \sum_{j=1}^n c_{ij}(s) \gamma_{ij} \int_{-s}^0 x_j^*(s+r)\, d\eta_{ij}(r) \Bigg] ds > 0, \end{aligned} \tag{3.4}$$

where $x^*(t) = (x_1^*(t), \ldots, x_n^*(t))$ is a fixed positive solution of system (3.3), *and $x^*(t) = x^*(t_0)$ if $t \leq t_0$. Then system* (1.5) *is permanent, that is, for $i = 1, \ldots, n$, there are constants $m_i, M_i > 0$ such that*

$$m_i \leq \liminf_{t \to +\infty} x_i(t) \leq \limsup_{t \to +\infty} x_i(t) \leq M_i,$$

for any positive solution $x(t) = (x_1(t), \ldots, x_n(t))$ of system (1.5)-(3.1).

Proof. From Lemma 3.1.1, we can conclude that there are positive constants p_i and P_i such that

$$p_i \le x_i^*(t) \le P_i, \qquad \text{for all } t \ge 0,\ i = 1, \dots, n,$$

and x^* is a globally attractive solution for system (3.3). We can obtain that condition (3.4) is independent of the choice of x^*. As g_i and α_{ij} are continuous functions, from condition (3.4), (P1), (P2), and (P5), one can conclude that there are constants $\varepsilon_0 \in (0, \varepsilon_1)$, and $T_0 > 0$ such that

$$\begin{aligned}\int_t^{t+\omega_i} \Bigg[& g_i(s, \varepsilon_0) - \sum_{j=1}^n \alpha_{ij}(s, (x_j^*)_s + \varepsilon_0) \\ & - \sum_{j=1}^n c_{ij}(s) \left(\varepsilon_0 + \gamma_{ij} \int_{-s}^0 (x_j^*(s+r) + \varepsilon_0)\, d\eta_{ij}(r) \right) \Bigg] ds > \varepsilon_0,\end{aligned} \tag{3.5}$$

for all $t \ge T_0$ and $i = 1, \dots, n$.

Let $x(t, t_0, \phi) = x(t) = (x_1(t), \dots, x_n(t))$ be a solution of (1.5)-(3.1). Without loss of generality, we may assume $t_0 = 0$. Note that

$$x_i(t) = \phi_i(0) \exp\left(\int_0^t (g_i(s, x_i(s)) - f_i(s, x_s)) ds \right) > 0,$$

$t \ge 0$, $i = 1, \dots, n$.

Since

$$x_i'(t) \le x_i(t) g_i(t, x_i(t)), \qquad t \ge 0, i = 1, \dots, n,$$

we obtain $x_i(t) \le u_i(t)$ for all $t \ge 0$ and $i = 1, \dots, n$, where $u(t) = (u_1(t), \dots, u_n(t)) > 0$ is a solution of system (3.3) with initial condition $u(0) = x(0)$. Consequently, by Lemma 3.1.1, we conclude that $x_i(t)$ is defined on $\mathbb{R}_0^+$, and there exists a constant $T_1 \ge T_0$ such that for each $i = 1, \dots, n$,

$$x_i(t) \le x_i^*(t) + \varepsilon_0, \qquad \text{for } t \ge T_1. \tag{3.6}$$

As $x(t) > 0$ for all $t \ge 0$, we conclude that $x(t)$ is bounded.
From (P4), there is $\tau_1 > 0$ such that

$$\int_{-\infty}^{-\tau_1} d\eta_{ij}(s) \le \frac{\varepsilon_0}{N}, \tag{3.7}$$

where $N = \max_{i,j} \sup_{t \in \mathbb{R}} \{\gamma_{ij} x_j(t)\}$. Moreover, from $(P3)$, there exists $T_2 \ge T_1$ such that

$t-\tau_{ij}(t) \geq T_1$ for $t \geq T_2$. Thus, we conclude that

$$(x_j)_t(s) = x_j(t+s) \leq x_j^*(t+s) + \varepsilon_0 = (x_j^*)_t(s) + \varepsilon_0, \tag{3.8}$$

for $t \geq T_2$, for all $s \geq -\tau_{ij}(t)$, and $i, j = 1, \dots, n$. For each $i = 1, \dots, n$, we define the constant

$$M_i = \sup\{x_i^*(t) + \varepsilon_0 : t \geq 0\}.$$

We have $0 < M_i < +\infty$ and M_i is independent of any positive solution of system (1.5). Obviously, we have $x_i(t) \leq M_i$ for all $t \geq T_1$ and $i = 1, \dots, n$, thus

$$\limsup_{t \to +\infty} x_i(t) \leq M_i, \quad i = 1, \dots, n.$$

Now, we need to show that $\liminf_{t \to +\infty} x_i(t) \geq m_i$ for some $m_i > 0$ independent of solution $x(t)$. First, we prove that

$$\limsup_{t \to +\infty} x_i(t) \geq \varepsilon_0, \quad i = 1, \dots, n. \tag{3.9}$$

In fact, if (3.9) is not true, then there exists an $i \in \{1, \dots, n\}$, and $T_3 \geq T_2$ such that

$$x_i(t) < \varepsilon_0 \text{ for all } t \geq T_3. \tag{3.10}$$

From (P2), for each $i = 1, \dots, n$, and all $t \geq T_1$, we obtain

$$f_i(t, x_t) \leq \sum_{j=1}^{n} \alpha_{ij}(t, (x_j)_t) + \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{0} \gamma_{ij} x_j(t+s) d\eta_{ij}(s). \tag{3.11}$$

By (3.10), and (P5.3), we have

$$g_i(t, x_i(t)) \geq g_i(t, \varepsilon_0), \quad \text{for } t \geq T_3,\ i = 1, \dots, n.$$

Consequently, from (3.11), for $t > T_3$, we have

$$x_i'(t) \;\geq\; x_i(t) \left[g_i(t, \varepsilon_0) - \sum_{j=1}^{n} \alpha_{ij}(t, (x_j)_t) - \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{0} \gamma_{ij} x_j(t+s)\, d\eta_{ij}(s) \right].$$

From (P2.1), (3.8), for $t > T_3$,

$$\begin{aligned} x_i'(t) &\geq x_i(t)\Bigg[g_i(t,\varepsilon_0) - \sum_{j=1}^n \alpha_{ij}(t,(x_j^*)_t + \varepsilon_0) \\ &\quad - \sum_{j=1}^n c_{ij}(t)\int_{-\infty}^{-\tau_1} \gamma_{ij}x_j(t+s)d\eta_{ij}(s) \\ &\quad - \sum_{j=1}^n c_{ij}(t)\int_{-\tau_1}^{0} \gamma_{ij}x_j(t+s)d\eta_{ij}(s)\Bigg], \end{aligned}$$

From (3.6), and (3.7), we obtain for $t > T_3 + \tau_1$,

$$\begin{aligned} x_i'(t) &\geq x_i(t)\Bigg[g_i(t,\varepsilon_0) - \sum_{j=1}^n \alpha_{ij}(t,(x_j^*)_t + \varepsilon_0) \\ &\quad - \sum_{j=1}^n c_{ij}(t)\left(\varepsilon_0 + \int_{-\tau_1}^{0} \gamma_{ij}(x_j^*(t+s)+\varepsilon_0)d\eta_{ij}(s)\right)\Bigg]. \end{aligned}$$

Additionally, for any $t \geq T_3 + \tau_1$, there are constants $l \in \mathbb{N}_0$ and $v \in [0,\omega_i)$ such that $t = T_3 + \tau_1 + l\omega_i + v$. Integrating the above inequality from $T_3 + \tau_1$ to t, we obtain

$$\begin{aligned} x_i(t) &\geq x_i(T_3+\tau_1)\exp\Bigg[\int_{T_3+\tau_1}^{t}\Bigg(g_i(s,\varepsilon_0) - \sum_{j=1}^n \alpha_{ij}(s,(x_j^*)_s + \varepsilon_0) \\ &\quad - \sum_{j=1}^n c_{ij}(s)\left(\varepsilon_0 + \gamma_{ij}\int_{-\tau_1}^{0}(x_j^*(s+r)+\varepsilon_0)d\eta_{ij}(r)\right)\Bigg)ds\Bigg], \end{aligned}$$

thus,

$$\begin{aligned} x_i(t) &\geq x_i(T_3+\tau_1)\Bigg[\exp\left(\int_{T_3+\tau_1}^{T_3+\tau_1+\omega_i} + \ldots + \int_{T_3+\tau_1+(l-1)\omega_i}^{T_3+\tau_1+l\omega_i} + \int_{T_3+\tau_1+l\omega_i}^{t}\right) \\ &\quad \times\Bigg(g_i(s,\varepsilon_0) - \sum_{j=1}^n \alpha_{ij}(s,(x_j^*)_s + \varepsilon_0) - \\ &\quad - \sum_{j=1}^n c_{ij}(s)\left(\varepsilon_0 + \gamma_{ij}\int_{-s}^{0}(x_j^*(s+r)+\varepsilon_0)\,d\eta_{ij}(r)\right)\Bigg)\Bigg]ds, \end{aligned}$$

hence, from (3.5), we obtain

$$x_i(t) \geq \; x_i(T_3+\tau_1)\exp(l\varepsilon_0 - \beta_i\omega_i),$$

where

$$\begin{aligned}\beta_i = \sup_{t\geq 0} \Bigg\{ & |g_i(t,\varepsilon_0)| + \sum_{j=1}^{n} \alpha_{ij}(t,(x_j^*)_t + \varepsilon_0) \\ & + \sum_{j=1}^{n} c_{ij}(t) \left(\varepsilon_0 + \gamma_{ij} \int_{-t}^{0} (x_j^*(t+r) + \varepsilon_0)\, d\eta_{ij}(r) \right) \Bigg\}.\end{aligned} \tag{3.12}$$

Therefore, we have $x_i(t) \to \infty$ as $t \to +\infty$, which is a contradiction. We obtain (3.9).

Now, we prove that there is positive constants m_i such that

$$\liminf_{t\to+\infty} x_i(t) \geq m_i, \quad i = 1,\dots,n.$$

Assuming that this is not true, then there exists an $i \in \{1,\dots,n\}$, and a sequence $(\phi_k)_{k\in\mathbb{N}} \in BC_+^n$ such that

$$\liminf_{t\to+\infty} x_i(t,\phi_k) < \frac{\varepsilon_0}{k^2}, \qquad \text{for all } k = 1,2,\dots, \tag{3.13}$$

where $x(t,\phi_k)$ is a solution of system (1.5) satisfying the initial condition $x(s) = \phi_k(s)$ for all $s \leq 0$. From (3.9) and (3.13), we conclude that for each $k \in \mathbb{N}$, there are two sequences $\left(s_q^{(k)}\right)_{q\in\mathbb{N}}$ and $\left(t_q^{(k)}\right)_{q\in\mathbb{N}}$ such that

$(D1)$ $\quad 0 < s_1^{(k)} < t_1^{(k)} < s_2^{(k)} < t_2^{(k)} < \ldots < s_q^{(k)} < t_q^{(k)} \ldots$

$(D2)$ $\quad s_q^{(k)} \to +\infty, \quad t_q^{(k)} \to +\infty \quad$ as $q \to +\infty$

$(D3)$ $\quad x_i\left(s_q^{(k)},\phi_k\right) = \dfrac{\varepsilon_0}{k}, \quad x_i\left(t_q^{(k)},\phi_k\right) = \dfrac{\varepsilon_0}{k^2}$

$(D4)$ $\quad \dfrac{\varepsilon_0}{k^2} \leqslant x_i(t,\phi_k) \leqslant \dfrac{\varepsilon_0}{k}$ for all $t \in \left(s_q^{(k)}, t_q^{(k)}\right)$.

Let $G^{(k)} = \sup_{t\geq 0} \|x_t(.,\phi_k)\|$. By (P4), for each $k \in \mathbb{N}$, we can choose a constant $\tau_1^{(k)} > 0$ such that

$$\int_{-\infty}^{-\tau_1^{(k)}} x_j(t+s,\phi_k)\, d\eta_{ij}(s) \leq G^{(k)} \int_{-\infty}^{-\tau_1^{(k)}} d\eta_{ij}(s) \leq \varepsilon_0 \tag{3.14}$$

and from (P5.3) and (D4), we have

$$g_i(t, x_i(t,\phi_k)) \geq g_i(t,\varepsilon_0), \quad \text{for } t \in [s_q^{(k)}, t_q^{(k)}],\ i = 1,\dots,n. \tag{3.15}$$

As in (3.6), there exists a $T_1^{(k)} \geq T_0$ such that, for all $j = 1,\dots,n$,

$$x_j(t,\phi_k) \leq x_j^*(t) + \varepsilon_0, \tag{3.16}$$

As in (3.8), and from (P3), for each $k \in \mathbb{N}$, there is $T_2^{(k)} > T_1^{(k)}$ such that $t - \tau_{ij}(t) \geq T_1^{(k)}$ for $t \geq T_2^{(k)}$, thus

$$\begin{aligned}(x_j)_t(s, \phi_k) &= x_j(t+s, \phi_k) \\ &\leq x_j^*(t+s) + \varepsilon_0 \\ &= (x_j^*)_t(s) + \varepsilon_0, \quad \text{for } t \geq T_2^{(k)} \text{ and } -\tau_{ij}(t) \leq s \leq 0.\end{aligned} \tag{3.17}$$

By (D1) and (D2), for each $k \in \mathbb{N}$, there exists $N_1^{(k)} \in \mathbb{N}$ such that $s_q^{(k)} > T_2^{(k)} + \tau_1^{(k)}$ for all $q \geq N_1^{(k)}$. Hence, for any $t \in \left[s_q^{(k)}, t_q^{(k)}\right]$ with $q \geq N_1^{(k)}$, by (P2) and (3.15), we have

$$\begin{aligned} x_i'(t, \phi_k) \;\; &\geq \;\; x_i(t, \phi_k)\Bigg[g_i(t, \varepsilon_0) - \sum_{j=1}^{n} \alpha_{ij}(t, (x_j)_t(., \phi_k)) \\ &\quad - \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{0} \gamma_{ij} x_j(t+s, \phi_k)\, d\eta_{ij}(s)\Bigg], \end{aligned}$$

and, from (P2.1) and (3.17), we have

$$\begin{aligned} x_i'(t, \phi_k) \;\; &\geq \;\; x_i(t, \phi_k)\Bigg[g_i(t, \varepsilon_0) - \sum_{j=1}^{n} \alpha_{ij}(t, (x_j^*)_t + \varepsilon_0) \\ &\quad - \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{-\tau_1^{(k)}} \gamma_{ij} x_j(t+s, \phi_k) d\eta_{ij}(s) \\ &\quad - \sum_{j=1}^{n} c_{ij}(t) \int_{-\tau_1^{(k)}}^{0} \gamma_{ij} x_j(t+s, \phi_k) d\eta_{ij}(s)\Bigg]. \end{aligned}$$

From (3.14) and (3.16), for $t \in \left[s_q^{(k)}, t_q^{(k)}\right]$ with $q \geq N_1^{(k)}$, we obtain

$$\begin{aligned} x_i'(t, \phi_k) \;\; &\geq \;\; x_i(t, \phi_k)\Bigg[g_i(t, \varepsilon_0) - \sum_{j=1}^{n} \alpha_{ij}(t, (x_j^*)_t + \varepsilon_0) \\ &\quad - \sum_{j=1}^{n} \gamma_{ij} c_{ij}(t) \left(\varepsilon_0 + \int_{-t}^{0} (x_j^*(t+s) + \varepsilon_0) d\eta_{ij}(s)\right)\Bigg]. \end{aligned}$$

We can choose $l_q^{(k)} \in \mathbb{N}_0$ such $t_q^{(k)} = s_q^{(k)} + l_q^{(k)} w_i + v_q^{(k)}$, where $v_q^{(k)} \in [0, \omega_i)$.
For each $k \in \mathbb{N}$ and $q \geq N_1^{(k)}$, from (3.5), (D3), and integrating the above inequality

from $s_q^{(k)}$ to $t_q^{(k)}$, we obtain

$$
\begin{aligned}
\frac{\varepsilon_0}{k^2} &= x_i(t_q^{(k)}, \phi_k) \\
&\geq x_i(s_q^{(k)}, \phi_k) \exp\left[\int_{s_q^{(k)}}^{t_q^{(k)}} \left(g_i(s, \varepsilon_0) - \sum_{j=1}^{n} \alpha_{ij}(s, (x_j^*)_s + \varepsilon_0)\right.\right. \\
&\quad \left.\left. - \sum_{j=1}^{n} \gamma_{ij} c_{ij}(s) \left(\varepsilon_0 + \int_{-s}^{0} (x_j^*(s+r) + \varepsilon_0)\, d\eta_{ij}(r)\right)\right) ds\right] \\
&\geq x_i(s_q^{(k)}, \phi_k) \exp\left[-\int_{s_q^{(k)}}^{t_q^{(k)}} |g_i(s, \varepsilon_0)| ds + \int_{s_q^{(k)} + l_q^{(k)} w_i}^{t_q^{(k)}} \left(g_i(s, \varepsilon_0)\right.\right. \\
&\quad \left.\left. - \sum_{j=1}^{n} \alpha_{ij}(s, (x_j^*)_s + \varepsilon_0) - \sum_{j=1}^{n} \gamma_{ij} c_{ij}(s) \left(\varepsilon_0 + \int_{-s}^{0} (x_j^*(s+r) + \varepsilon_0)\, d\eta_{ij}(r)\right)\right) ds\right] \\
&\geq \frac{\varepsilon_0}{k} \exp\left(-\zeta_i(t_q^{(k)} - s_q^{(k)}) - \beta_i \omega_i\right),
\end{aligned}
$$

where β_i is given by (3.12) and $\zeta_i = \sup\{|g_i(t, \varepsilon_0)| : t \geq 0\}$. We note that (P5.1) and (3.5) imply that $\zeta_i \in \mathbb{R}^+$. Thus, we can conclude that

$$t_q^{(k)} - s_q^{(k)} \geq \frac{\ln k - \beta_i w_i}{\zeta_i}, \tag{3.18}$$

for all $q \geq N_1^{(k)}$ and $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$ and $t > s_q^{(k)}$ with $q \geq N_1^{(k)}$, we have $t > s_q^{(k)} > T_1^{(k)} + \tau_1^{(k)}$ and

$$\int_{-\infty}^{T_1^{(k)} - t} x_j(t+s, \phi_k) d\eta_{ij}(s) \leq G^{(k)} \int_{-\infty}^{T_1^{(k)} - t} d\eta_{ij}(s), \tag{3.19}$$

and

$$\int_{T_1^{(k)} - t}^{s_q^{(k)} - t} x_j(t+s, \phi_k) d\eta_{ij}(s) \leq M_j \int_{-\infty}^{s_q^{(k)} - t} d\eta_{ij}(s), \tag{3.20}$$

for all $i, j = 1, \ldots, n$.

Moreover, for each $k \in \mathbb{N}$, by (P4) and $(D2)$, there exist $N_2^{(k)} \geq N_1^{(k)}$ and a constant $L > 0$ such that

$$\gamma_{ij} G^{(k)} \int_{-\infty}^{-T_1^{(k)} - s_q^{(k)}} d\eta_{ij}(s) \leq \frac{\varepsilon_0}{2}, \tag{3.21}$$

and

$$\gamma_{ij} M_j \int_{-\infty}^{-L} d\eta_{ij}(s) \leq \frac{\varepsilon_0}{2} \tag{3.22}$$

for all $q \geq N_2^{(k)}$, $i, j = 1, \ldots, n$. From (3.18), there is $K \in \mathbb{N}$ such that $t_q^{(k)} \geq s_q^{(k)} + L$ for all $k \geq K$ and $q \geq N_2^{(k)}$, thus

$$t_q^{(k)} = s_q^{(k)} + L + r_q^{(k)}\omega_i + \omega_q^{(k)}$$

where $r_q^{(k)} \in \mathbb{N}_0$ and $\omega_q^{(k)} \in [0, \omega_i)$.
By (3.18), there exists a large enough $K_1 \geq K$ such that

$$r_q^{(k)}\varepsilon_0 - \beta_i\omega_i \geq \varepsilon_0 \tag{3.23}$$

for all $k \geq K_1$ and $q \geq N_2^{(k)}$. Hence, for any $k \geq K_1$, $q \geq N_2^{(k)}$, and $t \in [s_q^{(k)} + L, t_q^{(k)}]$, by (P2), (P5.3), $(D4)$, (3.16), (3.17), (3.19), and (3.20), we obtain

$$\begin{aligned} x_i'(t, \phi_k) \;\; \geq \;\; & x_i(t, \phi_k)\Bigg[g_i(t, x_i(t, \phi_k)) - \sum_{j=1}^{n} \alpha_{ij}(t, (x_j)_t(., \phi_k)) \\ & - \sum_{j=1}^{n} c_{ij}(t)\bigg(\int_{-\infty}^{T_1^{(k)}-t} + \int_{T_1^{(k)}-t}^{s_q^{(k)}-t} + \int_{s_q^{(k)}-t}^{0}\bigg)\gamma_{ij}x_j(t+s, \phi_k)\, d\eta_{ij}(s)\Bigg], \end{aligned}$$

thus

$$\begin{aligned} x_i'(t, \phi_k) \;\; \geq \;\; & x_i(t, \phi_k)\Bigg[g_i(t, x_i(t, \phi_k)) - \sum_{j=1}^{n} \alpha_{ij}(t, (x_j)_t(., \phi_k)) \\ & - \sum_{j=1}^{n} c_{ij}(t)\bigg(\gamma_{ij}G^{(k)} \int_{-\infty}^{T_1^{(k)}-t} d\eta_{ij}(s) + \gamma_{ij}M_j \int_{T_1^{(k)}-t}^{s_q^{(k)}-t} d\eta_{ij}(s) \\ & + \int_{s_q^{(k)}-t}^{0} \gamma_{ij}x_j(t+s, \phi_k)\, d\eta_{ij}(s)\bigg)\Bigg] \\ \geq \;\; & x_i(t, \phi_k)\Bigg[g_i(t, \frac{\varepsilon_0}{k}) - \sum_{j=1}^{n} \alpha_{ij}(t, (x_j^*)_t + \varepsilon_0) \\ & - \sum_{j=1}^{n} c_{ij}(t)\bigg(\gamma_{ij}G^{(k)} \int_{-\infty}^{T_1^{(k)}-t} d\eta_{ij}(s) + \gamma_{ij}M_j \int_{-\infty}^{-L} d\eta_{ij}(s) \\ & + \int_{s_q^{(k)}-t}^{0} \gamma_{ij}\left(x_j^*(t+s) + \varepsilon_0\right)\, d\eta_{ij}(s)\bigg)\Bigg], \end{aligned}$$

and from (P5.3), (3.21), and (3.22), we obtain

$$\begin{aligned} x_i'(t, \phi_k) \;\; \geq \;\; & x_i(t, \phi_k)\Bigg[g_i(t, \varepsilon_0) - \sum_{j=1}^{n} \alpha_{ij}(t, (x_j^*)_t + \varepsilon_0) \\ & - \sum_{j=1}^{n} c_{ij}(t)\bigg(\frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} + \gamma_{ij} \int_{-t}^{0} \left(x_j^*(t+s) + \varepsilon_0\right) d\eta_{ij}(s)\bigg)\Bigg]. \end{aligned}$$

Integrating the above inequality from $s_q^{(k)} + L$ to $t_q^{(k)}$, then by $(D3)$, $(D4)$, (3.5), and (3.23), we conclude that

$$
\begin{aligned}
\frac{\varepsilon_0}{k^2} &= x_i(t_q^{(k)}, \phi_k) \\
&\geq x_i(s_q^{(k)} + L, \phi_k) \exp\Bigg[\Bigg(\int_{s_q^{(k)}+L}^{s_q^{(k)}+L+\omega_i} + \cdots + \int_{s_q^{(k)}+L+(r_q^{(k)}-1)\omega_i}^{s_q^{(k)}+L+r_q^{(k)}\omega_i} + \int_{s_q^{(k)}+L+r_q^{(k)}\omega_i}^{t_q^{(k)}} \Bigg) \\
&\quad \times \Bigg(g_i(s, \varepsilon_0) - \sum_{i=1}^{n} \alpha_{ij}(s, (x_i^*)_s + \varepsilon_0) \\
&\quad - \sum_{j=1}^{n} c_{ij}(s) \left(\varepsilon_0 + \gamma_{ij} \int_{-s}^{0} (x_i^*(s+r) + \varepsilon_0)\, d\eta_{ij}(r) \right) \Bigg) ds \Bigg] \\
&\geq \frac{\varepsilon_0}{k^2} \exp(r_q^{(k)} \varepsilon_0 - \beta_i \omega_i) \\
&\geq \frac{\varepsilon_0}{k^2} \exp(\varepsilon_0) > \frac{\varepsilon_0}{k^2}
\end{aligned}
$$

which is a contradiction. This contradiction shows that there exist constants $m_i > 0$ such that

$$\liminf_{t \to +\infty} x_i(t) \geq m_i, \quad , i = 1, \dots, n,$$

for any positive solution $x(t) = (x_1(t), \dots, x_n(t))$ of system (1.5), this completes the proof. $\square$

3.2 Lotka-Volterra type models

In this section, we apply our permanence result to the general Lotka-Volterra type model (1.10), and we derive a convergence result for asymptotic systems of Lotka-Volterra type models. Here, we restrict our study to solutions of (1.10) with bounded initial condition (3.1). For the system (1.10), the following set of hypotheses will be considered:

(H1) For $i = 1, \dots, n$, function $D_{ij} : \mathbb{R}_0^+ \times \mathbb{R}_0^- \to \mathbb{R}_0^+$, $r_i, a_{ij}, b_{ij}, c_{ij} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous and bounded, and $\tau_{ij} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous;

(H2) For each $i = 1, \dots, n$, function h_{ii} is continuous, and there exists $\alpha_i > 0$ such that
$$\frac{h_{ii}(u) - h_{ii}(v)}{u - v} \geq \alpha_i, \quad \forall u, v \in \mathbb{R}_0^+, u \neq v;$$

(H3) For each $i, j = 1, \dots, n$, there are $\eta_{ij}, \beta_{ij}, \gamma_{ij} > 0$ such that

$$|h_{ij}(u) - h_{ij}(v)| \leq \eta_{ij}|u - v| \text{ for } i \neq j, \quad |f_{ij}(u) - f_{ij}(v)| \leq \beta_{ij}|u - v|,$$
$$\text{and } |\theta_{ij}(u) - \theta_{ij}(v)| \leq \gamma_{ij}|u - v|, \quad \text{for all } u, v \in \mathbb{R}_0^+;$$

(H4) For each $i, j = 1, \ldots, n$, we have $\lim_{t \to +\infty} \big(t - \tau_{ij}(t)\big) = +\infty$;

(H5) For each $i, j = 1, \ldots, n$, there exists $D_{ij}^* : \mathbb{R}_0^- \to \mathbb{R}_0^+$ such that $D_{ij}(t,s) \le D_{ij}^*(s)$ for all $(t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^-$, and $\displaystyle\int_{-\infty}^{0} D_{ij}^*(s)ds = 1$;

(H6) For each $i = 1, \ldots, n$, there is $d = (d_1, \ldots, d_n) > 0$ such that

$$\limsup_{t \to +\infty} \Bigg[-\alpha_i a_{ii}(t) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{d_j}{d_i} \eta_{ij} a_{ij}(t) + \sum_{j=1}^{n} \frac{d_j}{d_i} \beta_{ij} b_{ij}(t) + \sum_{j=1}^{n} \frac{d_j}{d_i} \gamma_{ij} c_{ij}(t) \Bigg] < 0;$$

(H7) There are positive constants $\omega_i > 0$, $i = 1, \ldots, n$, and $\varepsilon_1 > 0$ such that

$$\begin{aligned} \liminf_{t \to +\infty} \int_t^{t+\omega_i} \Bigg[\quad & r_i(s) - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(s) \widetilde{h}_{ij}((x_j^*)_s + \varepsilon_1) \\ & - \sum_{j=1}^{n} b_{ij}(s) \widetilde{f}_{ij}(s, (x_j^*)_s + \varepsilon_1) - \sum_{j=1}^{n} \gamma_{ij} c_{ij}(s) \theta_{ij}(0) \\ & - \sum_{j=1}^{n} \gamma_{ij} c_{ij}(s) \int_{-s}^{0} D_{ij}^*(r) x_j^*(s+r)\, dr \Bigg] ds > 0 \end{aligned}$$

where $x^* = (x_1^*, \ldots, x_n^*)$ is a solution for $x_i'(t) = x_i(t)\left(r_i(t) - a_{ii}(t) h_{ii}(t)\right)$, for $i, j = 1, \ldots, n$, $i \neq j$, and $\phi \in BC_+^1$

$$\widetilde{h}_{ij}(\phi) = \sup\{h_{ij}(x) : 0 \le x \le \phi(0)\}$$

and, for $i, j = 1, \ldots, n$, $t \ge 0$,

$$\widetilde{f}_{ij}(t, \phi) = \sup\{f_{ij}(x) : 0 \le x \le \phi(-\tau_{ij}(t))\}.$$

3.2.1 Permanence of Lotka-Volterra type models

We have proved the permanence for the general differential system (1.5) in Section 3.1, and in this subsection, we apply our permanence result to the Lotka-Volterra type model (1.10).

Note that in comparison with model (1.5), model (1.10) is a particular case and can be written in the general form of (1.5) if we express the functions g_i and f_i as follows:

$$\begin{aligned} g_i(t,u) &= r_i(t) - a_{ii}(t)h_{ii}(u), \quad \text{and} \\ f_i(t,\phi) &= \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)h_{ij}(\phi_j(0)) + \sum_{j=1}^{n} b_{ij}(t)f_{ij}\big(\phi_j(-\tau_{ij}(t))\big) \\ &\quad + \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{0} D_{ij}(t,s)\theta_{ij}(\phi_j(s))\,ds, \end{aligned}$$

for all $t \geq 0$, $u \in \mathbb{R}_0^+$, $\phi = (\phi_1, \ldots, \phi_n) \in BC_+^n$, and recall that the functions are defined in (1.10).

Theorem 3.2.1. *Assume (H1)-(H5) and (H7). If the function h_{ii} is differentiable, satisfies $h_{ii}(0) = 0$, and there exists $\omega_0 > 0$ such that*

$$\liminf_{t\to+\infty} \int_{t}^{t+\omega_0} a_{ii}(s)ds > 0, \tag{3.24}$$

then system (1.10) *is permanent.*

Proof. We prove the conditions (P1)-(P5) as follows:

(P1) By (H1), (H3), and differentiability of h_{ii}, the functions involved in g_i and f_i are continuous, so g_i and f_i are.

(P2) We define the following

(i) for $i \neq j$, we have, for $t \geq 0$, and $\phi \in BC_+^1$,

$$\begin{aligned} \alpha_{ij}(t,\phi) &= a_{ij}(t)\sup\{h_{ij}(x) : 0 \leq x \leq \phi(0)\} \\ &\quad + b_{ij}(t)\sup\{f_{ij}(x) : 0 \leq x \leq \phi(-\tau_{ij}(t))\} \\ &\quad + c_{ij}(t)\theta_{ij}(0), \end{aligned}$$

(ii) for $i = j$, we have, for $t \geq 0$, and $\phi \in BC_+^1$,

$$\begin{aligned} \alpha_{ii}(t,\phi) &= \sup\{b_{ii}(t)f_{ii}(x) : 0 \leq x \leq \phi(-\tau_{ii}(0))\} \\ &\quad + c_{ii}(t)\theta_{ii}(0), \quad \text{and} \end{aligned}$$

(iii)

$$\eta_{ij}(s) = \int_{-\infty}^{s} D_{ij}^{*}(r)dr, \qquad \text{for } s \leq 0,\ i,j = 1,\ldots,n.$$

From (H3) and (H5), we obtain

$$\begin{aligned}\int_{-\infty}^{0} D_{ij}(t,s)\theta_{ij}(\phi_j(s))ds &\le \int_{-\infty}^{0} D_{ij}^*(s)|\theta_{ij}(\phi_j(s)) - \theta_{ij}(0)|ds \\ &\quad + \int_{-\infty}^{0} D_{ij}^*(s)\theta_{ij}(0)ds \\ &\le \int_{-\infty}^{0} D_{ij}^*(s)\gamma_{ij}\phi_j(s)ds + \theta_{ij}(0) \\ &\le \gamma_{ij}\int_{-\infty}^{0} \phi_j(s)d\eta_{ij}(s) + \theta_{ij}(0).\end{aligned}$$

Notice that, by definition of η_{ij}, we have

$$\int_{-\infty}^{0} D_{ij}^*(s)\phi_j(s)ds = \int_{-\infty}^{0} \phi_j(s)d\eta_{ij}(s).$$

Consequently, by the definition of α_{ij}, we have

$$f_i(t,\phi) \le \sum_{j=1}^{n} \alpha_{ij}(t,\phi_j) + \sum_{j=1}^{n} \gamma_{ij}c_{ij}(t)\int_{-\infty}^{0} \phi_j(s)d\eta_{ij}(s).$$

From (H1) and (H3), α_{ij} is continuous. From the definition of α_{ij}, (P2.1) is trivial. Moreover, by (H1), (H3), and (H5), we conclude that (P2.2) holds;

(P3) property (P3) is identical to (H4);

(P4) by definition of η_{ij}, (H5) implies (P4);

(P5) From (H1) and (H2), function $g_i(t,u) = r_i(t) - a_{ii}(t)h_{ii}(u)$ is bounded on $\mathbb{R}_0^+ \times [\sigma^{-1}, \sigma]$ for any constant $\sigma > 1$ and we conclude (P5.1).

By (H7), we conclude that

$$\liminf_{t\to+\infty} \int_{t}^{t+\omega_i} r_i(s)ds > 0.$$

Thus, there are $\varepsilon > 0$ and $T > 0$ such that

$$\int_{t}^{t+\varpi_1} r_i(s)ds > \varepsilon, \quad \text{for all } t \ge T,\ i = 1,\dots,n, \tag{3.25}$$

where $\varpi_1 = \max_i\{\omega_i\}$.
By (H1), $a_{ii}(t) < \overline{a}_{ii}$ for some $\overline{a}_{ii} > 0$. Since h_{ii} is continuous and $h_{ii}(0) = 0$, thus, there is $k_1 > 0$ such that

$$h_{ii}(k_1) < \frac{\varepsilon}{2\overline{a}_{ii}\varpi_1}.$$

Consequently, for $t \geq T$,

$$\int_t^{t+\varpi_1} a_{ii}(s)h_{ii}(k_1)ds \leq \varpi_1 \overline{a}_{ii} h_{ii}(k_1) < \frac{\varepsilon}{2}.$$

Thus, from (3.25), we have

$$\begin{aligned} \int_t^{t+\varpi_1} \left(r_i(s) - a_{ii}(s)h_{ii}(k_1)\right) ds &> \int_t^{t+\varpi_1} r_i(s)ds - \frac{\varepsilon}{2} \\ &> \frac{\varepsilon}{2}, \text{ for all } t \geq T, \end{aligned}$$

and we conclude that

$$\liminf_{t \to +\infty} \int_t^{t+\varpi_1} g_i(s, k_1)ds > 0.$$

From (H2), and that h_{ii} is differentiable with $h_{ii}(0) = 0$, we can conclude that

$$g_i(t, u) \leq r_i(t) - a_{ii}(t)\alpha_i u.$$

By (H1), r_i is bounded, thus $r_i(t) < \overline{r}_i$ for some $\overline{r}_i > 0$. By (3.24) and $\alpha_i > 0$, there are $\varepsilon > 0$ and $T > 0$ such that

$$\int_t^{t+\varpi_2} a_{ii}(s)ds > \frac{\varepsilon}{\alpha_i}, \text{ for } t \geq T, \ \varpi_2 > 0, \text{ and } i = 1, \ldots, n.$$

As $\varepsilon > 0$, there is $k_2 > k_1 > 0$ such that $k_2\varepsilon > \varpi_2 \overline{r}_i$. Thus, for $t \geq T$ and $i = 1 \ldots, n$, we have

$$\begin{aligned} \int_t^{t+\varpi_2} g_i(s, k_2)ds &\leq \int_t^{t+\varpi_2} \left(r_i(s) - a_{ii}(s)\alpha_i k_2\right) ds \\ &\leq \varpi_2 \overline{r}_i - \alpha_i k_2 \int_t^{t+\varpi_2} a_{ii}(s)ds \\ &< \varpi_2 \overline{r}_i - k_2\varepsilon < 0. \end{aligned}$$

Consequently,

$$\limsup_{t \to +\infty} \int_t^{t+\varpi_2} g_i(s, k_2)ds < 0,$$

hence, we conclude (P5.2).
From (H1) and (H2), we obtain

$$\frac{\partial}{\partial u} g_i(t, u) \leq -a_{ii}(t)\alpha_i, \quad (t, u) \in \mathbb{R}_0^+ \times \mathbb{R}.$$

Taking into account (3.24) and choosing $q(t) = a_{ii}(t)$ and $p(u) = \alpha_i$, we obtain (P5.3). Finally, in this context, condition (3.4) translates to (H7).

Now, by Theorem 3.1.2, we conclude that the system (1.10) is permanent. $\square$

The next result shows that, under stronger assumptions, hypothesis (H7) can be simplified.

Corollary 3.2.2. *Assume (H1)-(H5) and that*

$$\liminf_{t\to+\infty}\int_t^{t+\omega_i}\Bigg[r_i(s)-\sum_{\substack{j=1\\j\neq i}}^n a_{ij}(s)\eta_{ij}x_j^*(s)-\sum_{j=1}^n b_{ij}(s)\beta_{ij}x_j^*(s-\tau_{ij}(s)) \\ -\gamma_{ij}c_{ij}(s)\theta_{ij}(0)-\sum_{j=1}^n\gamma_{ij}c_{ij}(s)\int_{-s}^0 D_{ij}^*(r)x_j^*(s+r)dr\Bigg]ds>0. \tag{3.26}$$

If the function h_{ii} is differentiable, $h_{ij}(0) = f_{ij}(0) = 0$, and there exists $\omega_0 > 0$ such that (3.24) *holds, then system* (1.10) *is permanent.*

Proof. From (H3) and $h_{ij}(0) = f_{ij}(0) = 0$, we have $h_{ij}(u) \leq \eta_{ij}u$ for $i \neq j$, and $f_{ij}(u) \leq \beta_{ij}u$ for $u \in \mathbb{R}$. Consequently,

$$\begin{aligned}-\sum_{j=1}^n\alpha_{ij}(s,(x_j^*)_s+\varepsilon_1) \;&\geq\; -\sum_{\substack{j=1\\j\neq i}}^n a_{ij}(s)\eta_{ij}(x_j^*(s)+\varepsilon_1)-\sum_{j=1}^n c_{ij}(s)\theta_{ij}(0)\\ &\quad -\sum_{j=1}^n b_{ij}(s)\beta_{ij}(x_j^*(s-\tau_{ij}(s))+\varepsilon_1).\end{aligned}$$

Therefore, for any $\varepsilon_1 > 0$,

$$\begin{aligned}g_i(s,0) \;&-\; \sum_{j=1}^n\alpha_{ij}(s,(x_j^*)_s+\varepsilon_1)-\sum_{j=1}^n c_{ij}(s)\gamma_{ij}\int_{-s}^0 x_j^*(s+r)\,d\eta_{ij}(r)\\ &\geq\; r_i(s)-\sum_{\substack{j=1\\j\neq i}}^n a_{ij}(s)\eta_{ij}(x_j^*(s)+\varepsilon_1)-\sum_{j=1}^n b_{ij}(s)\beta_{ij}(x_j^*(s-\tau_{ij}(s))+\varepsilon_1)\\ &\quad -\sum_{j=1}^n\gamma_{ij}c_{ij}(s)\theta_{ij}(0)-\sum_{j=1}^n\gamma_{ij}c_{ij}(s)\int_{-s}^0 D_{ij}^*(r)x_j^*(s+r)dr,\end{aligned}$$

and thus

$$\liminf_{t\to+\infty}\int_t^{t+\omega_i}\left[g_i(s,0)-\sum_{j=1}^n \alpha_{ij}(s,(x_j^*)_s+\varepsilon_1)-\sum_{j=1}^n c_{ij}(s)\gamma_{ij}\int_{-s}^0 x_j^*(s+r)\ d\eta_{ij}(r)\right]ds$$

$$\begin{aligned}\geq \liminf_{t\to+\infty}\int_t^{t+\omega_i}&\Bigg[r_i(s)-\sum_{\substack{j=1\\ j\neq i}}^n a_{ij}(s)\eta_{ij}(x_j^*(s)+\varepsilon_1)-\sum_{j=1}^n b_{ij}(s)\beta_{ij}(x_j^*(s-\tau_{ij}(s))+\varepsilon_1)\\ &-\sum_{j=1}^n \gamma_{ij}c_{ij}(s)\theta_{ij}(0)-\sum_{j=1}^n\gamma_{ij}c_{ij}(s)\int_{-s}^0 x_j^*(s+r)\,D_{ij}^*(r)dr\Bigg]ds-\omega(\eta_{ij}\overline{a}_{ij}+\beta_{ij}\overline{b}_{ij})\varepsilon_1,\end{aligned}\tag{3.27}$$

where

$$\overline{a}_{ij}=\sup\{a_{ij}(t):\ t\geq 0\}\text{ and }\overline{b}_{ij}=\sup\{b_{ij}(t):\ t\geq 0\}.$$

Now, note that the condition (3.26) implies that the right-hand side of the equation (3.27) is positive for sufficiently small $\varepsilon_1>0$. The result follows from Theorem 3.1.2. □

3.2.2 Convergence of Lotka-Volterra type models

In this section, we introduce a partial result regarding the convergence of solutions of system (1.10) to solutions of its asymptotic system (3.28). Namely, assuming conditions (H1)-(H6) and permanence of the models, we proof that any solution of (1.10) whose distance to a solution of (3.28) converge to a real number, must be asymptotic to the solution of (3.28).

Definition 3.2.1. *The system*

$$\begin{aligned}\hat{x}_i'(t) &= \hat{x}_i(t)\Bigg[\hat{r}_i(t)-\sum_{j=1}^n \hat{a}_{ij}(t)h_{ij}\big(\hat{x}_j(t)\big)-\sum_{j=1}^n \hat{b}_{ij}(t)f_{ij}\big(\hat{x}_j(t-\hat{\tau}_{ij}(t))\big)\\ &\quad -\sum_{j=1}^n \hat{c}_{ij}(t)\int_{-\infty}^0 D_{ij}(t,s)\theta_{ij}(\hat{x}_j(t+s))ds\Bigg],\end{aligned}\tag{3.28}$$

$t\geq 0$, $i=1,\dots,n$,

is said to be an asymptotic system of (1.10),

$$\begin{aligned}x_i'(t)=x_i(t)\Bigg[&r_i(t)-\sum_{j=1}^n a_{ij}(t)h_{ij}\big(x_j(t)\big)-\sum_{j=1}^n b_{ij}(t)f_{ij}\big(x_j(t-\tau_{ij}(t))\big)\\ &-\sum_{j=1}^n c_{ij}(t)\int_{-\infty}^0 D_{ij}(t,s)\theta_{ij}(x_j(t+s))\,ds\Bigg],\quad t\geq 0,\ i=1,\dots,n,\end{aligned}$$

if, for each $i,j=1,\dots,n$, *the functions* $\hat{r}_i,\hat{a}_{ij},\hat{b}_{ij},\hat{c}_{ij},\hat{\tau}_{ij}:\mathbb{R}_0^+\to\mathbb{R}_0^+$ *are continuous* *such that*

$$\begin{aligned}\lim_{t\to+\infty}\big(r_i(t)-\hat{r}_i(t)\big) &= \lim_{t\to+\infty}\big(a_{ij}(t)-\hat{a}_{ij}(t)\big)=\lim_{t\to+\infty}\big(b_{ij}(t)-\hat{b}_{ij}(t)\big)\\ &= \lim_{t\to+\infty}\big(c_{ij}(t)-\hat{c}_{ij}(t)\big)=\lim_{t\to+\infty}\big(\tau_{ij}(t)-\hat{\tau}_{ij}(t)\big)\big)=0.\end{aligned} \tag{3.29}$$

By (3.29), it is obvious that the hypothesis ($H6$) is equivalent to the following

$$\limsup_{t\to+\infty}\Bigg[-\alpha_i\hat{a}_{ii}(t)+\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{d_j}{d_i}\eta_{ij}\hat{a}_{ij}(t)+\sum_{j=1}^{n}\frac{d_j}{d_i}\beta_{ij}\hat{b}_{ij}(t)+\sum_{j=1}^{n}\frac{d_j}{d_i}\gamma_{ij}\hat{c}_{ij}(t)\Bigg]<0.$$

Theorem 3.2.3. *Assume (H1)-(H6) and that systems* (1.10) *and* (3.28) *are permanent. Then, for all* $x(t)$ *and* $\hat{x}(t)$ *solutions of systems* (1.10) *and* (3.28) *respectively, with bounded initial conditions* (3.1)*, such that*

$$\lim_{t\to+\infty}|x(t)-\hat{x}(t)| \ \textit{exists,}$$

we have

$$\lim_{t\to+\infty}|x(t)-\hat{x}(t)|=0.$$

Proof. Let $x(t)=(x_1(t),\dots,x_n(t))$ and $\hat{x}(t)=(\hat{x}_1(t),\dots,\hat{x}_n(t))$ be solutions of systems (1.10) and (3.28) respectively, with bounded initial conditions (3.1), such that

$$\lim_{t\to+\infty}|x(t)-\hat{x}(t)|$$

exists.
Without loss of generality, and up to the change of variables $y_i(t)=x_i(t)d_i$ and $\hat{y}_i(t)=\hat{x}_i(t)d_i$, one may assume $d_i=1$ for all $i=1,\dots,n$. By hypotheses, there are positive constants m and M such that

$$\begin{aligned} m &\le \min_{i=1,\dots,n}\{\liminf_{t\to+\infty} x_i(t),\liminf_{t\to+\infty}\hat{x}_i(t)\}\\ &\le \max_{i=1,\dots,n}\{\limsup_{t\to+\infty} x_i(t),\limsup_{t\to+\infty}\hat{x}_i(t)\}\le M.\end{aligned} \tag{3.30}$$

For each $i=1,\dots,n$, define the function

$$\begin{aligned} V_i(t) &:= |\log(x_i(t))-\log(\hat{x}_i(t))|\\ &= \operatorname{sign}(x_i(t)-\hat{x}_i(t))(\log(x_i(t))-\log(\hat{x}_i(t))).\end{aligned}$$

For $t \geq t_0$, and $x_i(t) \neq \hat{x}_i(t)$, we have the following:

$$V_i'(t) \quad = \quad sign(x_i(t) - \hat{x}_i(t)) \left[\frac{x_i'(t)}{x_i(t)} - \frac{\hat{x}_i'(t)}{\hat{x}_i(t)} \right],$$

then

$$\begin{aligned} V_i'(t) \quad = \quad & sign(x_i(t) - \hat{x}_i(t)) \Big[r_i(t) - \hat{r}_i(t) + \hat{a}_{ii}(t) h_{ii}(\hat{x}_i(t)) - a_{ii}(t) h_{ii}(x_i(t)) \\ & + \sum_{\substack{j=1 \\ j \neq i}}^{n} \big(\hat{a}_{ij}(t) h_{ij}(\hat{x}_j(t)) - a_{ij}(t) h_{ij}(x_j(t)) \big) \\ & + \sum_{j=1}^{n} b_{ij}(t) \big(f_{ij}(\hat{x}_j(t - \hat{\tau}_{ij}(t))) - f_{ij}(x_j(t - \tau_{ij}(t))) \big) \\ & + \sum_{j=1}^{n} \big(\hat{b}_{ij}(t) - b_{ij}(t) \big) f_{ij}(\hat{x}_j(t - \hat{\tau}_{ij}(t))) \\ & + \sum_{j=1}^{n} c_{ij}(t) \bigg(\int_{-\infty}^{0} D_{ij}(t,s) \theta_{ij}(\hat{x}_j(t+s)) ds - \int_{-\infty}^{0} D_{ij}(t,s) \theta_{ij}(x_j(t+s)) ds \bigg) \\ & + \sum_{j=1}^{n} \big(\hat{c}_{ij}(t) - c_{ij}(t) \big) \int_{-\infty}^{0} D_{ij}(t,s) \theta_{ij}(\hat{x}_j(t+s)) d\rho. \end{aligned}$$

From (H2), we have

$$\begin{aligned} V_i'(t) \quad \leq \quad & |r_i(t) - \hat{r}_i(t)| - \alpha_i a_{ii}(t) |x_i(t) - \hat{x}_i(t)| \\ & + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(t) |h_{ij}(\hat{x}_j(t)) - h_{ij}(x_j(t))| + \sum_{j=1}^{n} |\hat{a}_{ij}(t) - a_{ij}(t)| |h_{ij}(\hat{x}_j(t))| \\ & + \sum_{j=1}^{n} b_{ij}(t) |f_{ij}(\hat{x}_j(t - \hat{\tau}_{ij}(t))) - f_{ij}(x_j(t - \tau_{ij}(t)))| \\ & + \sum_{j=1}^{n} |\hat{b}_{ij}(t) - b_{ij}(t)| |f_{ij}(\hat{x}_j(t - \hat{\tau}_{ij}(t)))| \\ & + \sum_{j=1}^{n} c_{ij}(t) \bigg| \int_{-\infty}^{0} D_{ij}(t,s) \big(\theta_{ij}(\hat{x}_j(t+s)) - \theta_{ij}(x_j(t+s)) \big) ds \bigg| \\ & + \sum_{j=1}^{n} \big| \hat{c}_{ij}(t) - c_{ij}(t) \big| \bigg| \int_{-\infty}^{0} D_{ij}(t,s) \theta_{ij}(\hat{x}_j(t+s)) ds \bigg| \end{aligned}$$

From (H3) and (H5), we obtain

$$\begin{aligned} V_i'(t) \leq \quad & -\alpha_i a_{ii}(t)|x_i(t) - \hat{x}_i(t)| + \sum_{\substack{j=1 \\ j\neq i}}^{n} \eta_{ij} a_{ij}(t)|\hat{x}_j(t)) - x_j(t)| \\ & + \sum_{j=1}^{n} \beta_{ij} b_{ij}(t)|\hat{x}_j(t - \hat{\tau}_{ij}(t)) - x_j(t - \hat{\tau}_{ij}(t))| + \delta_i(t) \\ & + \sum_{j=1}^{n} \gamma_{ij} c_{ij}(t) \int_{-\infty}^{0} D^*_{ij}(s)\big|\hat{x}_j(t+s) - x_j(t+s)\big| ds, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} \delta_i(t) \quad = \quad & |r_i(t) - \hat{r}_i(t)| + \sum_{i=1}^{n} b_{ij}(t)\beta_{ij}\big|x_j(t - \hat{\tau}_{ij}(t)) - x_j(t - \tau_{ij}(t))\big| \\ & + \sum_{j=1}^{n} |\hat{a}_{ij}(t) - a_{ij}(t)||h_{ij}(\hat{x}_i(t))| + \sum_{j=1}^{n} |\hat{b}_{ij}(t) - b_{ij}(t)||f_{ij}(\hat{x}_j(t - \hat{\tau}_{ij}(t)))| \\ & + \sum_{i=1}^{n} \big|\hat{c}_{ij}(t) - c_{ij}(t)\big| \left| \int_{-\infty}^{0} D_{ij}(t,s)\theta_{ij}(\hat{x}_j(t+s)) ds \right|. \end{aligned}$$

From (H1), (3.30) and the continuity of the functions h_{ij}, f_{ij}, and θ_{ij}, we have that $x_j'(t)$ is a bounded function, thus $x_j(t)$ is uniformly continuous for all $j = 1, \dots, n$, hence one can conclude that $x_j(t)$ is uniformly continuous. Moreover, from (H3), (H5), (3.29), and (3.30), we obtain that

$$\lim_{t\to+\infty} \delta_i(t) = 0. \tag{3.32}$$

Defining

$$u_i = \limsup_{t\to+\infty} |x_i(t) - \hat{x}_i(t)|. \tag{3.33}$$

From (3.30), we have

$$u = \max_i\{u_i\} \in [0, +\infty).$$

The result is proved if we show that $u = 0$.
Choose an $i \in \{1, \dots, n\}$ such that $u_i = u$. From our hypotheses,

$$\lim_{t\to+\infty} |x_i(t) - \hat{x}_i(t)|$$

exists. By contradiction, assume that $u > 0$. From (H6), there are $T > 0$ and $\rho < 0$

such that

$$-\alpha_i a_{ii}(t) + \sum_{\substack{j=1\\j\neq i}}^{n} \eta_{ij} a_{ij}(t) + \sum_{j=1}^{n} \beta_{ij} b_{ij}(t) + \sum_{j=1}^{n} \gamma_{ij} c_{ij}(t) < \rho, \quad t \geq T. \tag{3.34}$$

From (H1), we can define

$$L = \max_{i=1,\ldots,n} \left\{ \sup_{t\geq 0} \left(\alpha_i a_{ii}(t) + \sum_{\substack{j=1\\j\neq i}}^{n} \eta_{ij} a_{ij}(t) + \sum_{j=1}^{n} \beta_{ij} b_{ij}(t) + 2\sum_{j=1}^{n} \gamma_{ij} c_{ij}(t) \right) \right\}. \tag{3.35}$$

For

$$\varepsilon \in \left(0, \min\left\{u, \frac{-\rho u}{2(L+1)}\right\}\right), \tag{3.36}$$

as $t - \hat{\tau}_{ij}(t) \to +\infty$, then there is $T_1 = T_1(\varepsilon) > T$ such that, for $t \geq T_1$,

$$\begin{aligned} &|x_j(t - \hat{\tau}_{ij}(t)) - \hat{x}_j(t - \hat{\tau}_{ij}(t))| \leq u + \varepsilon, \; j = 1, \ldots, n, \\ &|x_i(t) - \hat{x}_i(t)| > u - \varepsilon. \end{aligned}$$

From (H5), there exists $T_2 > T_1$ such that

$$\int_{-\infty}^{-T_2} D^*_{ij}(s) ds < \frac{\varepsilon}{z},$$

where

$$z = \max_j \sup_{t\in\mathbb{R}} |x_j(t) - \hat{x}_j(t)| \in (0, +\infty), \; j = 1, \ldots, n.$$

Therefore, by (3.31), for $t \geq 2T_2$, we have

$$\begin{aligned} V'_i(t) \;\; \leq \;\; & -\alpha_i a_{ii}(t)|x_i(t) - \hat{x}_i(t)| + \sum_{\substack{j=1\\j\neq i}}^{n} \eta_{ij} a_{ij}(t) |\hat{x}_j(t) - x_j(t)| \\ & + \sum_{j=1}^{n} \beta_{ij} b_{ij}(t) |\hat{x}_j(t - \hat{\tau}_{ij}(t)) - x_j(t - \hat{\tau}_{ij}(t))| + \delta_i(t) \\ & + \sum_{j=1}^{n} \gamma_{ij} c_{ij}(t) \int_{-\infty}^{-T_2} D^*_{ij}(s) \big|\hat{x}_j(t+s) - x_j(t+s)\big| ds \\ & + \sum_{j=1}^{n} \gamma_{ij} c_{ij}(t) \int_{-T_2}^{0} D^*_{ij}(s) \big|\hat{x}_j(t+s) - x_j(t+s)\big| ds \end{aligned}$$

$$
\begin{aligned}
V_i'(t) \;\leq\; & -\alpha_i a_{ii}(t)(u-\varepsilon) + \sum_{\substack{j=1\\ j\neq i}}^{n} \eta_{ij} a_{ij}(t)(u+\varepsilon) + \sum_{j=1}^{n} \beta_{ij} b_{ij}(t)(u+\varepsilon) + \delta_i(t) \\
& + \sum_{j=1}^{n} \gamma_{ij} c_{ij}(t)\varepsilon + \sum_{j=1}^{n} \gamma_{ij} c_{ij}(t) \int_{-T_2}^{0} D_{ij}^*(s)(u+\varepsilon)ds \\
\leq\; & \Big(-\alpha_i a_{ii}(t) + \sum_{\substack{j=1\\ j\neq i}}^{n} \eta_{ij} a_{ij}(t) + \sum_{j=1}^{n} \beta_{ij} b_{ij}(t) + \sum_{j=1}^{n} \gamma_{ij} c_{ij}(t) \Big) u \\
& + \Big(\alpha_i a_{ii}(t) + \sum_{\substack{j=1\\ j\neq i}}^{n} \eta_{ij} a_{ij}(t) + \sum_{j=1}^{n} \beta_{ij} b_{ij}(t) + 2\sum_{j=1}^{n} \gamma_{ij} c_{ij}(t) \Big) \varepsilon + \delta_i(t).
\end{aligned}
$$

From (3.34) and (3.35), we have

$$
V_i'(t) \;\leq\; \rho u + L\varepsilon + \delta_i(t).
$$

From (3.32), taking $T^* = T^*(\varepsilon) \geq 2T_2$ such that $\delta_i(t) < \varepsilon$, for $t \geq T^*$ and $i = 1, \ldots, n$, we obtain, using (3.36),

$$
\begin{aligned}
V_i'(t) \;&\leq\; \rho u + (L+1)\varepsilon \\
&<\; \rho u - \frac{\rho u}{2} = \frac{\rho u}{2}.
\end{aligned}
$$

Thus, for $t \geq T^*$, we have $V_i'(t) \leq \frac{\rho u}{2}$. By integrating on $[T^*, t]$, we obtain

$$
\begin{aligned}
& V_i(t) - V_i(T^*) \leq \frac{\rho}{2} u(t - T^*) \\
& V_i(t) - \frac{\rho}{2} u(t - T^*) \leq V_i(T^*)
\end{aligned}
$$

which is a contradiction because $u > 0$, $\rho < 0$, and $V_i(t) \geq 0$ for all $t \geq 0$. Thus $u = 0$, and the proof is concluded. □

Obviously, the system (1.10) can be considered as an asymptotic system of itself. Thus, we have the following results

Corollary 3.2.4. *Assume (H1)-(H6) and that system* (1.10) *is permanent. Then, for all $x(t)$ and $\hat{x}(t)$ solutions of system* (1.10)*, with bounded initial conditions* (3.1)*, such that*

$$
\lim_{t\to+\infty} |x(t) - \hat{x}(t)| \;\; \textit{exists},
$$

we have

$$
\lim_{t\to+\infty} |x(t) - \hat{x}(t)| = 0.
$$

3.3 Numerical example

Example 3.1. Let $0 < 2B < A < 9B$ and, consider the system

$$\begin{aligned} x_1'(t) &= x_1(t)\Bigg[A - 3x_1(t) - \frac{1}{t+1}\,\mathrm{e}^{\sin(x_2(t-|\sin t|))} \\ &\qquad -(\cos t + 1 + \mathrm{e}^{-t})\int_{-\infty}^{0} \mathrm{e}^{s}\,x_2(t+s)\,ds\Bigg] \\ x_2'(t) &= x_2(t)\Bigg[B - \frac{1}{2}x_2(t) - \frac{1}{t+2}\,\mathrm{e}^{\cos(x_1(t-t/2))} \\ &\qquad -\frac{2}{3}(t+1)\sin\left(\frac{1}{t+1}\right)\int_{-\infty}^{0} \mathrm{e}^{2s}\,x_1(t+s)\,ds\Bigg]. \end{aligned} \tag{3.37}$$

This system is a particular case of (1.10) with $f_{11}(u) = f_{22}(u) = 0$, $f_{12}(u) = \mathrm{e}^{\sin u}$, $f_{21}(u) = \mathrm{e}^{\cos u}$, $h_{11}(u) = h_{22}(u) = u$, $h_{12}(u) = h_{21}(u) = 0$, $\theta_{12}(u) = \theta_{21}(u) = u$, $\theta_{11}(u) = \theta_{22}(u) = 0$, $r_1(t) = A$, $r_2(t) = B$, $a_{11}(t) = 3$, $a_{22}(t) = 1/2$, $b_{12}(t) = 1/(t+1)$, $b_{21}(t) = 1/(t+2)$, $c_{12}(t) = \cos t + 1 + \mathrm{e}^{-t}$, $c_{21}(t) = \frac{1}{3}(t+1)\sin(1/(1+t))$, $\tau_{12}(t) = |\sin t|$, $\tau_{21}(t) = t/2$, $D_{12}(t,s) = \mathrm{e}^{s}$, and $D_{21}(t,s) = 2\,\mathrm{e}^{2s}$. In Figures 3.1 and 3.2, three solutions $(x_1(t), x_2(t))$ of the system (3.37) are plotted for the case $A = 7$ and $B = 2$, with initial conditions $\varphi(s) = (\mathrm{e}^{s} + 2, \cos(s) + 1)$, $\varphi(s) = (2, \sin(s) + 1)$, and $\varphi(s) = (e^{s}, 1)$ for $s \leq 0$, at $t_0 = 0$.

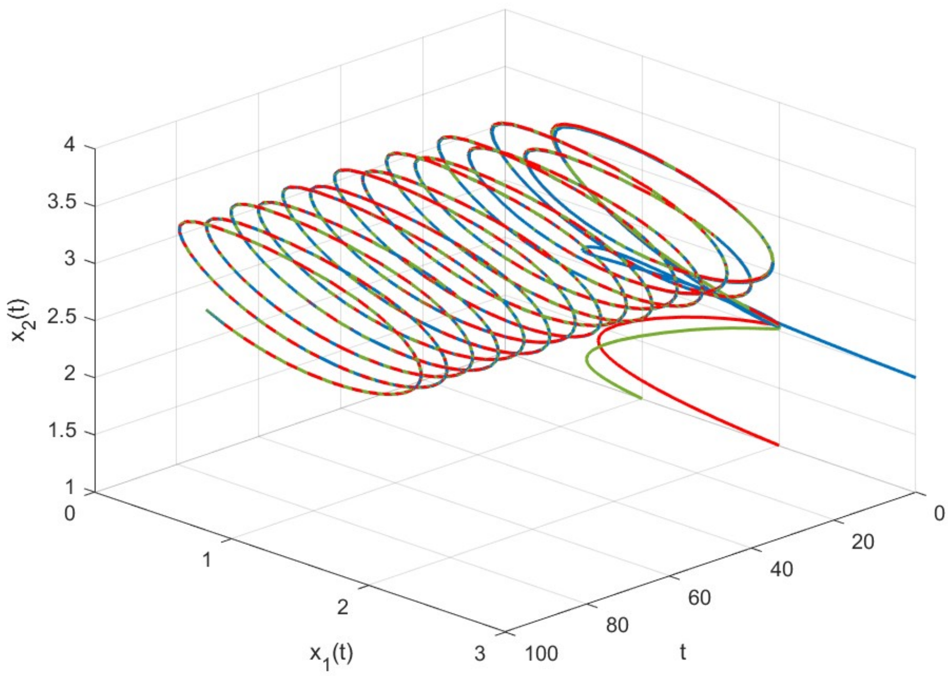

Figure 3.1: Numerical simulation of three solutions of system (3.37) with $A = 7$ and $B = 2$, $(x_1(t), x_2(t))$, with initial conditions $\varphi(s) = (\mathrm{e}^{s} + 2, \cos(s) + 1)$, $\varphi(s) = (2, \sin(s) + 1)$, $\varphi(s) = (e^{s}, 1)$ for $s \leq 0$, at $t_0 = 0$.

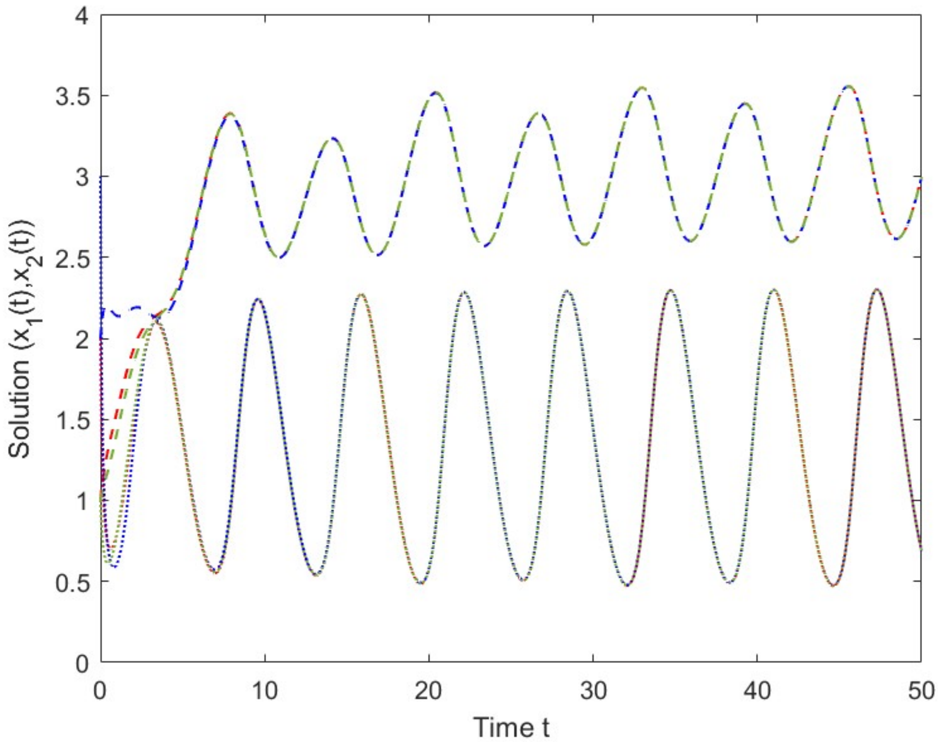

Figure 3.2: Numerical simulation of three solutions of system (3.37) with $A = 7$ and $B = 2$, $(x_1(t), x_2(t))$, with initial conditions $\varphi(s) = (\mathrm{e}^s + 2, \cos(s) + 1)$, $\varphi(s) = (2, \sin(s) + 1)$, $\varphi(s) = (e^s, 1)$ for $s \leq 0$, at $t_0 = 0$.

We will now check that this system satisfies the conditions (H1) to (H7). Condition (H1) is immediate, taking into account the parameter functions that are considered in the system. Condition (H2) is also immediate and holds with $\alpha_1 = \alpha_2 = 1$, since h_{11} and h_{22} coincide with the identity function, thus the quotient on the left-hand side of the inequality is identically 1. Condition (H3) holds with $\beta_{12} = \beta_{21} = \mathrm{e}$ (we can obtain a better estimate for β_{12}, β_{21} but this is enough for our purposes), and $\gamma_{12} = \gamma_{21} = 1$. Since $\tau_{12}(t)$ is bounded, and $t - \tau_{21}(t) = t/2$, we conclude that (H4) holds. To justify that (H5) holds, we simply note that, since $D_{12}(t, s)$ and $D_{21}(t, s)$ are independent of t and the integrals on the interval $(-\infty, 0]$ with respect to s of each of these functions is 1, so we can take $D^*_{12}(s) = D_{12}(t, s) = \mathrm{e}^s$ and $D^*_{21}(s) = D_{21}(t, s) = 2\,\mathrm{e}^{2s}$. Since

$$\begin{aligned}\limsup_{t\to+\infty}\Big[-\alpha_1 a_{11}(t) + \beta_{12} b_{12}(t) + \gamma_{12} c_{12}(t)\Big] &= \limsup_{t\to+\infty}\Big[-3 + \frac{\mathrm{e}}{t+1} + \cos t + 1 + \mathrm{e}^{-t}\Big] \\ &= -1 < 0,\end{aligned}$$

and

$$\begin{aligned}\limsup_{t\to+\infty}\Big[-\alpha_2 a_{22}(t) + \beta_{21} b_{21}(t) + \gamma_{21} c_{21}(t)\Big] &= \limsup_{t\to+\infty}\Big[-\frac{1}{2} + \frac{\mathrm{e}}{t+2} + \frac{1}{3}\frac{\sin\frac{1}{1+t}}{\frac{1}{1+t}}\Big] \\ &= -\frac{1}{6} < 0,\end{aligned}$$

thus, we conclude that (H6) holds.

To justify that (H7) holds, we note that, in this case, $g_1(t,u) = A - 3u$ and $g_2(t,u) = B - u/2$, and we can take $(x_1^*(t), x_2^*(t)) = (A/3, 2B)$, for $t \in \mathbb{R}$. Thus,

$$\widetilde{f}_{12}(t, (x_2^*)_t + \varepsilon_1) = \sup\{\mathrm{e}^{\sin x} : 0 \leq x \leq A/3 + \varepsilon_1\} \leq \mathrm{e},$$

and

$$\widetilde{f}_{21}(t, (x_1^*)_t + \varepsilon_1) = \sup\{\mathrm{e}^{\cos x} : 0 \leq x \leq 2B + \varepsilon_1\} \leq \mathrm{e}\,.$$

Therefore, taking $\omega_1 = 2\pi$ and since $A > 2B$,

$$\begin{aligned}
& \liminf_{t\to+\infty} \int_t^{t+2\pi} \left[r_1(s) - b_{12}(s)\widetilde{f}_{12}(s, (x_2^*)_s + \varepsilon_1) - \gamma_{12}c_{12}(s) \int_{-s}^0 D_{12}^*(r)x_2^*(s+r)dr \right] ds \\
\geq\ & \liminf_{t\to+\infty} \int_t^{t+2\pi} \left[A - \frac{\mathrm{e}}{s+1} - (\cos s + 1 + \mathrm{e}^{-s}) \int_{-s}^0 2B\,\mathrm{e}^r\, dr \right] ds \\
\geq\ & \liminf_{t\to+\infty} \int_t^{t+2\pi} \left[A - \frac{\mathrm{e}}{s+1} - 2B(\cos s + 1 + \mathrm{e}^{-s})(1 - \mathrm{e}^{-s}) \right] ds \\
=\ & \liminf_{t\to+\infty} \Big[(A-2B)s - \mathrm{e}\ln(s+1) - 2B\sin s + B\,\mathrm{e}^{-s}(\sin s - \cos s - \mathrm{e}^{-s}) \Big]_t^{t+2\pi} \\
=\ & \lim_{t\to+\infty} \Big[2\pi(A-2B) + \mathrm{e}\ln\frac{t+1}{t+1+2\pi} \\
& \quad -B\,(\sin t - \cos t)\,\mathrm{e}^{-t}\left(1 + \mathrm{e}^{-\pi}\right) - B\,\mathrm{e}^{-2t}(\mathrm{e}^{-2\pi} - 1) \Big] \\
=\ & 2\pi(A-2B) > 0
\end{aligned}$$

and, for any $\omega_2 > 0$, we obtain, noting that $A < 9B$ and $|t\sin(1/t)| < 1$, for $t > 0$,

$$\begin{aligned}
& \liminf_{t\to+\infty} \int_t^{t+\omega_2} \left[r_2(s) - b_{21}(s)\widetilde{f}_{21}(s, (x_1^*)_s + \varepsilon_1) - \gamma_{21}c_{21}(s) \int_{-s}^0 D_{21}^*(r)x_1^*(s+r)dr \right] ds \\
\geq\ & \liminf_{t\to+\infty} \int_t^{t+\omega_2} \left[B - \frac{\mathrm{e}}{s+2} - \frac{1}{9}(s+1)\sin\left(\frac{1}{s+1}\right) \int_{-s}^0 2A\,\mathrm{e}^{2r}\, dr \right] ds \\
\geq\ & \liminf_{t\to+\infty} \int_t^{t+\omega_2} \left[B - \frac{\mathrm{e}}{s+2} - \frac{A}{9}(1 - \mathrm{e}^{-2s}) \right] ds \\
=\ & \liminf_{t\to+\infty} \left[Bs - \mathrm{e}\ln(s+2) - \frac{A}{9}\left(s + \frac{\mathrm{e}^{-2s}}{2} \right) \right]_t^{t+\omega_2} \\
=\ & \lim_{t\to+\infty} \left[\omega_2 B + \mathrm{e}\ln\frac{t+2}{t+\omega_2} - \frac{A}{9}\left(\omega_2 - (1 - \mathrm{e}^{-2\omega_2})\frac{\mathrm{e}^{-2t}}{2} \right) \right] \\
=\ & \omega_2(B - A/9) > 0.
\end{aligned}$$

Thus (H7) holds.

Finally, note that $h_{11}(0) = h_{22}(0) = 0$ and that, for any $\omega_0 > 0$,

$$\liminf_{t\to+\infty} \int_t^{t+\omega_0} a_{11}(s)ds = \liminf_{t\to+\infty} \int_t^{t+\omega_0} 3ds = 3\omega_0 > 0,$$

and

$$\liminf_{t\to+\infty} \int_t^{t+\omega_0} a_{22}(s)ds = \liminf_{t\to+\infty} \int_t^{t+\omega_0} \frac{1}{2}ds = \frac{\omega_0}{2} > 0.$$

The conditions above show that we can use Theorem 3.2.1 to conclude that system (3.37) is permanent. Since the condition (H6) hold, using Corollary 3.2.4, we can conclude that whenever

$$\lim_{t\to+\infty} |x(t) - \hat{x}(t)| \text{ exists,}$$

for any solutions, $x(t) = (x_1(t), x_2(t))$ and $\tilde{x}(t) = (\tilde{x}_1(t), \tilde{x}_2(t))$, of system (3.37) with bounded initial conditions (3.1), we have

$$\lim_{t\to+\infty} |x(t) - \hat{x}(t)| = 0.$$

Moreover, we note that an asymptotic system of (3.37) is

$$\begin{aligned} x_1'(t) &= x_1(t)\left[7 - 3x_1(t) - (\cos t + 1)\int_{-\infty}^0 \mathrm{e}^s\, x_2(t+s)\, ds\right] \\ x_2'(t) &= x_2(t)\left[2 - \frac{1}{2}x_2(t) - \frac{1}{3}\int_{-\infty}^0 2\,\mathrm{e}^{2s}\, x_1(t+s)\, ds\right], \end{aligned} \tag{3.38}$$

which is a $2\pi-$periodic system that can be written in the following form:

$$\begin{aligned} x_1'(t) &= x_1(t)\left[7 - 3x_1(t) - (\cos t + 1)\int_{-\infty}^t \mathrm{e}^{(\rho-t)}\, x_2(\rho)\, ds\right] \\ x_2'(t) &= x_2(t)\left[2 - \frac{1}{2}x_2(t) - \frac{1}{3}\int_{-\infty}^t 2\,\mathrm{e}^{2(\rho-t)}\, x_1(\rho)\, ds\right]. \end{aligned} \tag{3.39}$$

It is easy to verify that system (3.39) satisfies all the hypotheses of [4, Theorem 3.2]. Consequently, system (3.38) has a $2\pi-$periodic solution, denoted by $\hat{x}^*(t)$. From similar computations to those done for system (3.37), we obtain that system (3.38) is permanent. By Theorem 3.2.3, we conclude that for any solution of system (3.37), $x(t)$, with initial conditions in BC_+^2 at $t_0 = 0$, if

$$\lim_{t\to+\infty} |x(t) - \hat{x}^*(t)| \text{ exists,}$$

then

$$\lim_{t\to+\infty} |x(t) - \hat{x}^*(t)| = 0.$$

This result implies that every solution of system (3.37) with positive bounded initial conditions – provided the aforementioned limit exists – converges to the periodic solution $\hat{x}^*(t)$ of system (3.38). Based on our numerical simulations, we conjecture that Theorem 3.2.3 holds without the strong assumption regarding the existence of the limit.

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