A note on the geometry of three circles *

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Abstract

In the present note, we deduce some nice results concerning the geometry of three-circles from an easy incidence lemma in plane projective geometry. By particularizing to the case of the three excircles of a triangle, this lemma provides a unified geometric characterization of many interesting Kimberling centers.

In [5], the author applied Desargues' Theorem to prove the celebrated Three-Circle Theorem: *three circles* $\mathcal{C}_1$, $\mathcal{C}_2$ *and* $\mathcal{C}_3$ *are taken in pairs* $(\mathcal{C}_1, \mathcal{C}_2)$, $(\mathcal{C}_1, \mathcal{C}_3)$, *and* $(\mathcal{C}_2, \mathcal{C}_3)$; *then the external similarity points of the three pairs lie on a straight line* α. If one takes the poles of α with respect to the given circles and connect them with the radical center, then one gets the Gergonne's construction for obtaining a pair of solutions to the tangency problem of Apollonius [3]. From these observations transpire that Projective Geometry plays a fundamental role in the study of the geometry behind a configuration of three circles.

In the present note, we shall be able to deduce some nice results concerning the geometry of three-circles from an easy incidence lemma in plane projective

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geometry. By particularizing to the case of the three excircles of a triangle, this lemma, which we have not found in the literature, provides a unified geometric characterization of many interesting Kimberling centers [4]. For geometrical background we recommend [1, 2].

I. Consider three distinct points Q_1, Q_2, Q_3 on a line α. For each $i = 1, 2, 3$ take three distinct lines r_i, s_i, t_i through Q_i. Let R_{ij} be the intersection point of r_i and r_j, with $i < j$. Similarly, define the points S_{ij} and T_{ij}, as illustrated in Figure 1. Let D_{ij} be the intersection point of the diagonals of the quadrilateral defined by the two pairs of lines r_i, s_i and r_j, s_j, with $i < j$.

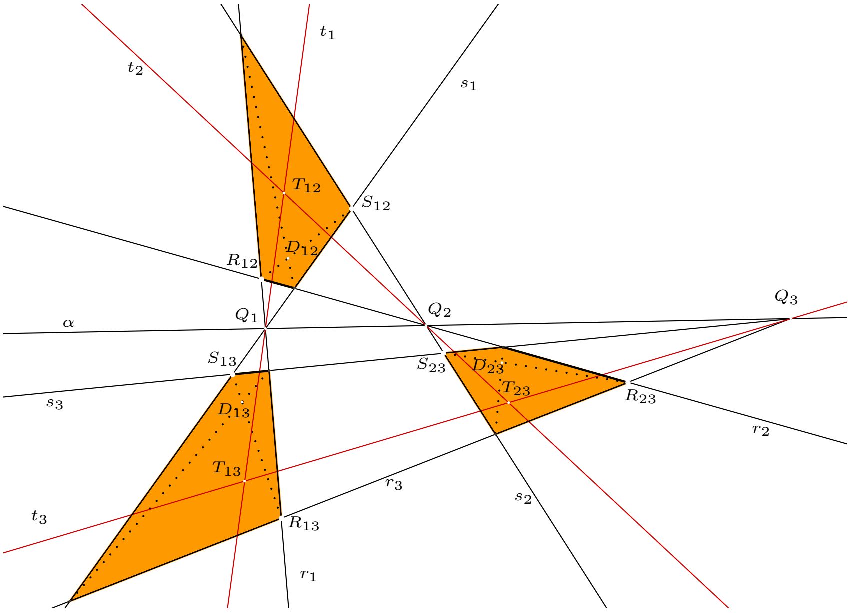

Figure 1.

By the converse of Desargues's Theorem, we have

i) $\triangle S_{12}S_{13}S_{23}$ and $\triangle R_{12}R_{13}R_{23}$ are perspective from a point U;

ii) $\triangle S_{12}S_{13}S_{23}$ and $\triangle T_{12}T_{13}T_{23}$ are perspective from a point V;

ii) $\triangle R_{12}R_{13}R_{23}$ and $\triangle T_{12}T_{13}T_{23}$ are perspective from a point W.

We have not found any reference in the literature to the following:

Lemma 1. *$\triangle T_{12}T_{13}T_{23}$ and $\triangle D_{12}D_{13}D_{23}$ are perspective from a point X. The four points U, V, W, X are collinear.*

Proof. Let Ω be the (unique) projective collineation that transforms the complete quadrilateral $r_1s_1r_2s_2$ in the complete quadrilateral $r_1s_1r_3s_3$. Since Ω preserves the relation of incidence, we have $\Omega(Q_1) = Q_1$ and $\Omega(Q_2) = Q_3$, hence Ω leaves invariant three distinct lines – r_1, s_1 and α – through Q_1. Then, by the Fundamental Theorem of Projective Geometry, Ω leaves invariant every line on the pencil through Q_1. Since $\Omega(D_{12}) = D_{13}$, this implies that $Q_1 \in D_{12}D_{13}$. Similarly, we can prove that $Q_2 \in D_{12}D_{23}$ and $Q_3 \in D_{23}D_{13}$. In particular, $\triangle T_{12}T_{13}T_{23}$ and $\triangle D_{12}D_{13}D_{23}$ are perspective from α, which means, by the converse of Desargues's Theorem, that $\triangle T_{12}T_{13}T_{23}$ and $\triangle D_{12}D_{13}D_{23}$ are perspective from a point X.

Now, to prove that U, V, W, X are collinear we proceed as follows. Let Ω_V be the perspective collineation transforming the complete quandrangle $S_{12}S_{13}S_{23}V$ into the complete quadrangle $T_{12}T_{13}T_{23}V$; Ω_W the perspective collineation transforming $R_{12}R_{13}R_{23}W$ into $T_{12}T_{13}T_{23}W$; Ω_U the perspective collineation transforming $S_{12}S_{13}S_{23}U$ into $R_{12}R_{13}R_{23}U$. In general, the composition of two perspective collienations is not a perspective collineation. However, since the perspective collineations Ω_U, Ω_W and Ω_V share the same axis α, the composition $\Omega_W \circ \Omega_U$ is also a perspective collineation and $\Omega_V = \Omega_W \circ \Omega_U$. Hence $\Omega_V(U) = \Omega_W \circ \Omega_U(U) = \Omega_W(U)$ belongs simultaneously to VU and WU. Thus, the three points U, V, W are collinear. Now, observe that $\triangle S_{12}S_{13}S_{23}$ and $\triangle D_{12}D_{13}D_{23}$ are also perspective from U. Hence, by applying the same argument, we can prove that U, V, X are also collinear, and we are done. ☐

II. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be three circles with non-collinear centers at A_1, A_2 and A_3, respectively, and radii a_1, a_2 and a_3, respectively. We assume that none of these circles lies completely inside another. The line A_1A_2 and the common external tangents to $\mathcal{C}_1$ and $\mathcal{C}_2$, say r_1 and s_1, intersect at a point Q_3 – the external similarity point of $\mathcal{C}_1$ and $\mathcal{C}_2$. Similarly, we construct the points Q_1 and Q_2. By the Three-Circle Theorem, Q_1, Q_2 and Q_3 are collinear. Denote this line by α and define S_{ij}, R_{ij} and D_{ij} $(i < j)$ as in the previous section. In this case, observe that:

i) by Brianchon Theorem, D_{ij} is the pole of $\mathcal{C}_k$ $(k \neq i, j)$ with respect to α;

ii) $\triangle S_{12}S_{13}S_{23}$ and $\triangle R_{12}R_{13}R_{23}$ are perspective from a point U;

iii) the center of perpectivity of $\triangle S_{12}S_{13}S_{23}$ and $\triangle A_1A_2A_3$ is the incenter I_S of $\triangle S_{12}S_{13}S_{23}$;

iv) the center of perpectivity of $\triangle R_{12}R_{13}R_{23}$ and $\triangle A_1A_2A_3$ is the incenter I_R of $\triangle R_{12}R_{13}R_{23}$.

Theorem 1. *The incenters I_S and I_R of $\triangle S_{12}S_{13}S_{23}$ and $\triangle R_{12}R_{13}R_{23}$ are collinear with the point U. The three lines A_3D_{12}, A_2D_{13} and A_1D_{23} are parallel to the line β defined by I_S and I_R (Figure 2), that is, the triangle of the poles and the triangle of the centers are perspective from the point X of intersection of β with the line at the infinity.*

Proof. Since the homothety with center at Q_1 and ratio $\frac{a_3}{a_2}$ transforms the line A_2D_{12} into the line A_3D_{13} and any homothety transforms each line into a parallel line, we conclude that A_2D_{12} and A_3D_{13} are parallel. The remaining is a straightforward consequence of Lemma 1. $\square$

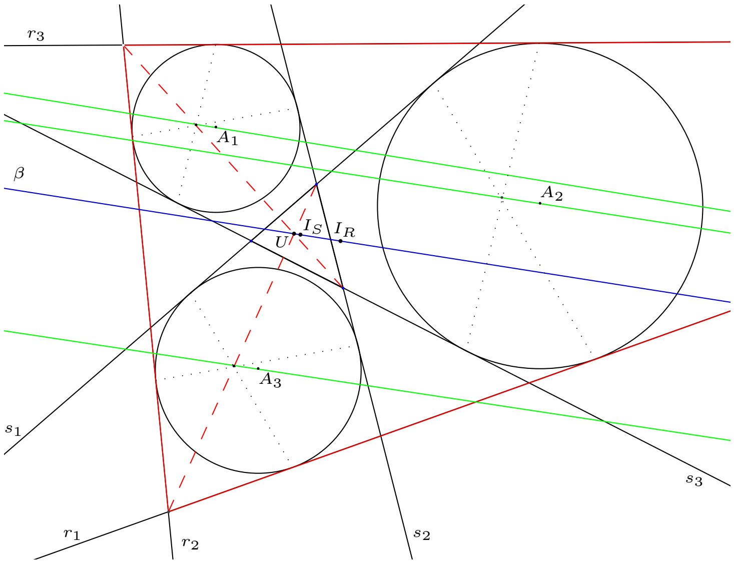

Figure 2.

Remark 1. Similarly, given three pairs of circles $(\mathcal{C}_1, \mathcal{C}_2)$, $(\mathcal{C}_1, \mathcal{C}_3)$, and $(\mathcal{C}_2, \mathcal{C}_3)$, it is well known [3] that the external similarity point of one pair and the two internal similarity points of the other two pairs lie upon a straight line. Hence, Theorem 1 also holds when we consider one pair of common external tangents and two pairs of common internal tangents.

Remark 2. Lines α and β are perpendicular. This is an immediate consequence of the following: given a circle $\mathcal{C}$ with center at P and a line α, the line joining P and the pole D of α with respect to $\mathcal{C}$ is perpendicular to α, as can be easily deduced taking account the following figure:

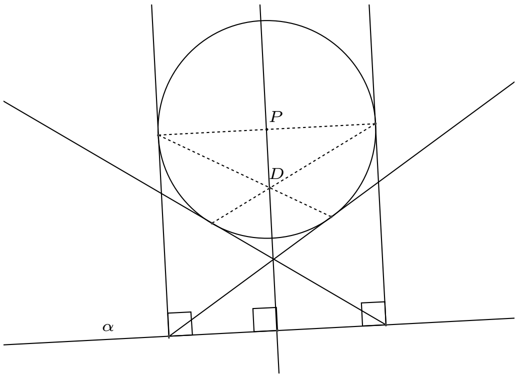

Figure 3.

Remark 3. When $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are the excircles of the triangle $\triangle S_{12}S_{13}S_{23}$, the triangle $\triangle R_{12}R_{13}R_{23}$ is the extangent triangle, U is the perspector of $\triangle S_{12}S_{13}S_{23}$ and $\triangle R_{12}R_{13}R_{23}$. Hence, in this case U coincides with the Kimberling center [4] $X(65)$ of $\triangle S_{12}S_{13}S_{23}$. At same time, with respect to $\triangle S_{12}S_{13}S_{23}$, I_R and I_S coincide with the Kimberling center $X(40)$ (Bevan point) and $X(1)$ (incenter), respectively. Hence, β will be the Kimberling line $(1,3)$.

III. We assume now that $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are three nonintersecting mutually external circles. The common internal tangents to $\mathcal{C}_2$ and $\mathcal{C}_3$ intersect at a point B_1. Similarly we define the points B_2 and B_3. Since $\frac{|A_jQ_i|}{|Q_iA_k|} = \frac{|A_jB_i|}{|B_iA_k|} = \frac{a_j}{a_k}$, we have:

i) by Ceva's Theorem, the cevians A_1B_1, A_2B_2 and A_3B_3 concur in a point Y (Figure 4);

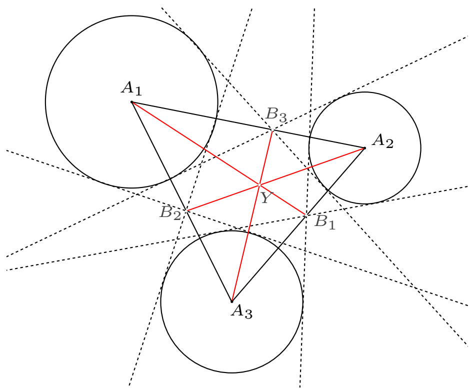

Figure 4.

ii) Q_i is the harmonic conjugate of B_i with respect to A_j and A_k, with $i \neq j, k$; hence, the line α is the trilinear polar of Y (Figure 5), otherwise said, $\triangle B_1B_2B_3$ and $\triangle A_1A_2A_3$ are perspective from the line α.

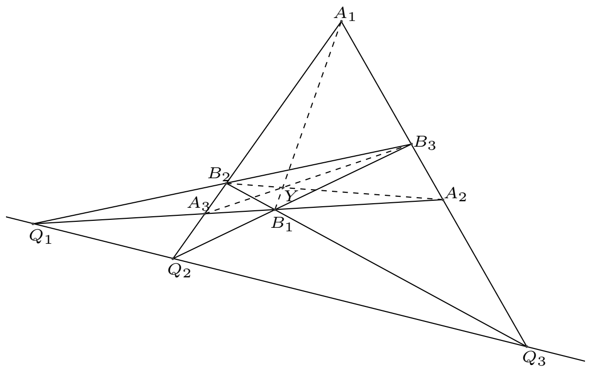

Figure 5.

Choose r_i and s_i as in II. The following is a direct a consequence of Lemma 1:

Theorem 2. a) *$\triangle R_{12}R_{13}R_{23}$ and $\triangle B_1B_2B_3$ are perspective from a point R and R is collinear with I_R and Y;* b) *$\triangle S_{12}S_{13}S_{23}$ and $\triangle B_1B_2B_3$ are perspective from a point S and S is collinear with I_S and Y;* c) *$\triangle D_{12}D_{13}D_{23}$ and $\triangle B_1B_2B_3$ are perspective from a point D;* d) *R, S, D and U are collinear.*

Remark 4. When $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are the excircles of the triangle $\triangle S_{12}S_{13}S_{23}$ we have: Y coincides with the incenter of $\triangle S_{12}S_{13}S_{23}$; $R = D = U = X(65)$; S is not defined; and $I_RY = I_SY = (1, 3)$.

IV. Consider again three nonintersecting mutually external circles. $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, with centers at A_1, A_2, and A_3, respectively. Draw a new circle $\mathcal{C}$ tangent to the three given circles, with center at O. Suppose that $\mathcal{C}_1$, $\mathcal{C}_2$ and C_3 are all external or all internal to $\mathcal{C}$. Let U_{12} be the point of tangency of $\mathcal{C}$ with $\mathcal{C}_3$. Similarly, define the points U_{13} and U_{23}. The line $U_{23}U_{13}$, passes through Q_3, the intersection point of the common external tangents (Figure 6). In fact, this is a simple application of Menelaus Theorem.

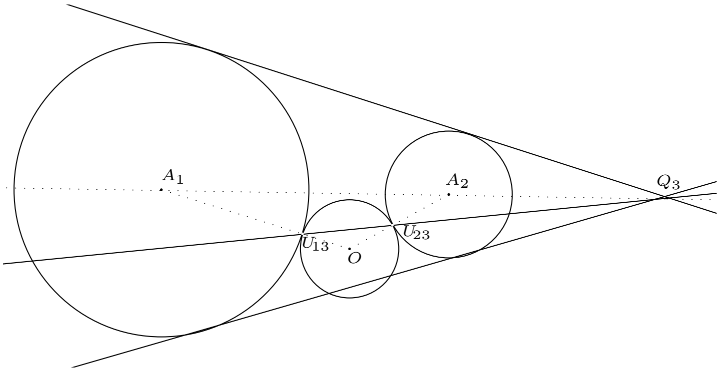

Figure 6.

Choose r_i and s_i as in II. Again, by Lemma 1 we have:

Theorem 3. a) *Triangles* $\triangle U_{12}U_{13}U_{23}$ *and* $\triangle R_{12}R_{13}R_{23}$ *are perspective from a point* L_R*;* b) *Triangles* $\triangle U_{12}U_{13}U_{23}$ *and* $\triangle S_{12}S_{13}S_{23}$ *are perspective from a point* L_S*;* c) *Triangles* $\triangle U_{12}U_{13}U_{23}$ *and* $\triangle D_{12}D_{13}D_{23}$ *are perspective from a point* L_D; d) L_R*,* L_S*,* L_D *and* U *are collinear (Figure 7).*

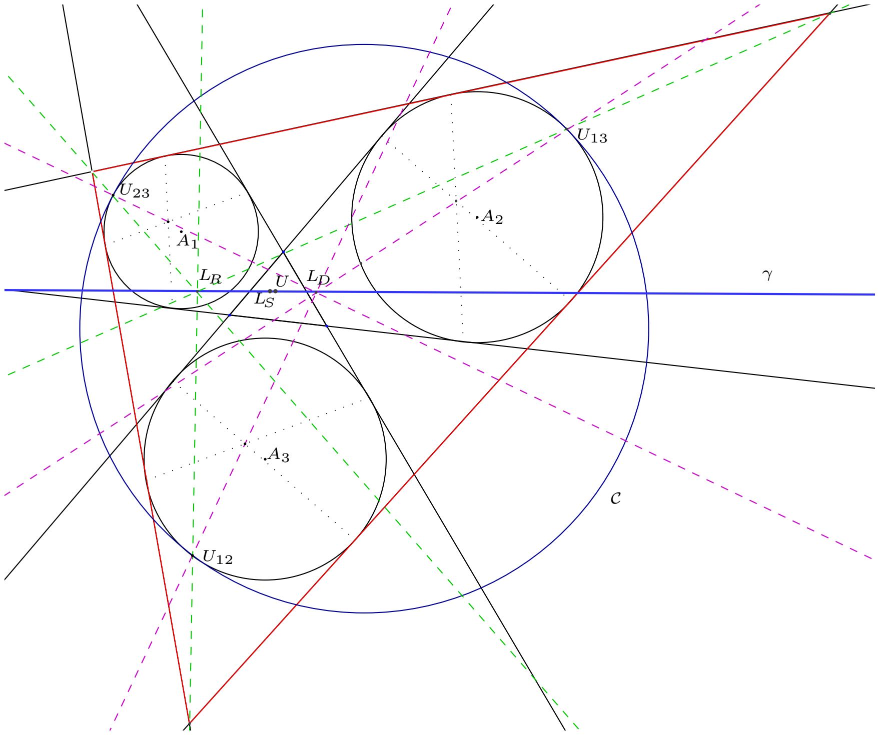

Figure 7.

Remark 5. L_D coincides with the radical center of the three given circles; the key observation in Gergonne's solution to the tangency problem of Apollonius is precisely that $\triangle U_{12}U_{13}U_{23}$ and $\triangle D_{12}D_{13}D_{23}$ are perspective from L_D (see [3]).

Remark 6. When $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are the excircles of the triangle $\triangle S_{12}S_{13}S_{23}$ and are external to $\mathcal{C}$, we have: $\mathcal{C}$ is the nine-point circle of $\triangle S_{12}S_{13}S_{23}$; the point L_S is the Kimberling center $X(12)$ of $\triangle S_{12}S_{13}S_{23}$; L_D is the Spieker point $X(10)$. The point L_R is the internal similitude center of the nine-point circle $\mathcal{N}$ and the incircle $\mathcal{I}$ of the extangents triangle. In fact: the external similitude center of $\mathcal{I}$ and $\mathcal{C}_1$ is R_{23} and the internal similitude center of $\mathcal{C}_1$ and $\mathcal{N}$ is U_{23}, hence the internal similitude center P of $\mathcal{I}$ and $\mathcal{N}$ belongs to the line $R_{23}U_{23}$; similarly, P belongs to the lines $R_{12}U_{12}$ and $R_{13}U_{13}$; then $P = L_R$. We have not found any reference to this point in the Kimberling list.

Remark 7. When $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are the excircles of the triangle $\triangle S_{12}S_{13}S_{23}$ and are internal to $\mathcal{C}$, we have: $\mathcal{C}$ is the Apollonius circle of $\triangle S_{12}S_{13}S_{23}$; the point L_S is the Apollonius center $X(181)$ of $\triangle S_{12}S_{13}S_{23}$; L_D is the Spieker point $X(10)$. We can argue as in Remark 6 in order to conclude that the point L_R, in this case, is the external similitude center of the Appolonius circle and the incircle of the extangent triangle. Again, we have not found any reference to this point in the Kimberling list.

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