Classification of Laguerre-Hahn orthogonal polynomials of class one

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We study orthogonal polynomials related to Stieltjes functions satisfying Riccati type differential equations with polynomial coefficients, $AS' = BS^2 + CS + D$, with $\max\{\deg(A), \deg(B)\} \leq 3$, $\deg(C) \leq 2$. We derive recurrences for the three-term recurrence relation coefficients of the orthogonal polynomials, including connections with some forms of discrete Painlevé equations.

1 Motivation

Orthogonal polynomials $\{P_n(x) = x^n + \text{lower degree terms}\}_{n\geq 0}$ on the real line may be fully characterized by the orthogonality relation

$$\int_I P_n(x)P_m(x)d\mu(x) = h_n\delta_{m,n}\,, \quad n, m \geq 0\,, \;\; h_n \neq 0\,, \tag{1}$$

or, equivalently, by a three-term recurrence relation [27]

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 1, 2, \dots\,, \tag{2}$$

with $P_0(x) = 1,\ P_1(x) = x - \beta_0$ and $\gamma_n \neq 0,\ n \geq 1$. Here, $\delta_{m,n}$ is the Kronecker delta, and μ is a positive Borel measure supported on a finite or infinite interval of the real line, I, with finite moments

$$u_n = \int_I x^n d\mu(x)\,, \quad n \geq 0\,.$$

The numbers β_n, γ_n, commonly called the recurrence relation coefficients, can be expressed as

$$\beta_n = \frac{1}{h_n}\int_I xP_n^2(x)d\mu(x)\,, \quad \gamma_{n+1} = \frac{1}{h_n}\int_I P_{n+1}^2(x)d\mu(x)\,, \;\; n \geq 0\,. \tag{3}$$

Note that $\gamma_n > 0$, taking into account (3).

A very important topic of research, often encountered in the literature of orthogonal polynomials and special functions, concerns the so-called direct problem [29]: to deduce information on β_n, γ_n, given a measure μ. The list of references is quite vast, many connections with problems from Mathematical Physics have been studied (see, for instance, the introduction and the references list of [21, 29]). In the account of (3), if explicit representations of the polynomials P_n are given, one can apply computation techniques that, under some conditions,

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give the coefficients β_n, γ_n (see [8, 16]). This is the case of the classical orthogonal polynomials - the Hermite, Laguerre and Jacobi polynomials. These families of polynomials are orthogonal with respect to measures defined by an absolutely continuous part with respect to the Lebesgue measure, w, satisfying $Aw' = Cw$, with A, C polynomials such that $\deg(A) \leq 2, \deg(C) = 1$. Increasing the degrees on A, C, or doing perturbations on the orthogonality measure produces new families of orthogonal polynomials, with increasing complexity of the equations involving the recurrence relation coefficients. In general, one can deduce recurrences in the form $F_j(\beta_n, \beta_{n-1}, \ldots, \beta_0, \gamma_n, \gamma_{n-1}, \ldots, \gamma_1, n, \ldots) = 0,\ j = 1, \ldots, N$, where F_j are non-linear functions in the β's and γ's. In the literature of orthogonal polynomials, such systems of recurrences are known as Laguerre-Freud equations [4, 15, 19, 20]. Some of these recurrences were identified as forms of discrete Painlevé equations. The first identification, with a discrete Painlevé I - dP_I, took place in [13]. Since then, many other cases have been studied (see, for instance, [9, 29], and the recent monograph [28]).

In the present paper we focus on extensions of the above mentioned direct problem: we take the Laguerre-Hahn orthogonal polynomials [11, 18, 24, 30], that is, the sequences of orthogonal polynomials whose Stieltjes function,

$$S(x) = \int_I \frac{d\mu(y)}{x-y}\,,$$

satisfies a Riccati type differential equation with polynomial coefficients,

$$AS' = BS^2 + CS + D\,, \quad A \neq 0\,, \tag{4}$$

and we seek formulae for β_n and γ_n, given A, B, C, D in (4). We shall consider the so-called class one, that is, $1 = \max\{\deg(C) - 1, d - 2\}$, $d = \max\{\deg(A), \deg(B)\}$, with A, B, C, D co-prime [2, 24].

On a more general framework, the Stieltjes function is defined in terms of a formal series [24, 30]

$$S(x) = \sum_{n=0}^{+\infty} \frac{u_n}{x^{n+1}}\,,$$

with $(u_n)_{n\geq 0}$ the sequence of moments of a linear functional u, $u_n = \langle u, x^n \rangle,\ n \geq 0$, where we take $u_0 = 1$, without loss of generality. There holds the equivalence between [11, 24]: (i) the Riccati equation (4), where D is a polynomial defined in terms of A, B, C; (ii) the distributional equation for the corresponding linear functional u,

$$\mathcal{D}(Au) = \psi u + B(x^{-1}u^2)\,, \quad \psi = A' + C\,. \tag{5}$$

Here, the following operations in the algebraic dual space of the polynomials hold: the left product of u by a polynomial, defined as $\langle g\,u, p \rangle = \langle u, g\,p \rangle,\ p \in P$; the derivative $\mathcal{D}u$, defined as $\langle \mathcal{D}u, p \rangle = -\langle u, p' \rangle,\ p \in P$; the functional $x^{-1}u$, defined as $\langle x^{-1}u, p \rangle = \langle u, \theta_0 p \rangle,\ \ (\theta_0 p)(x) = \frac{p(x)-p(0)}{x}$; the product of two linear functionals, u and v, defined as $\langle u\,v, p \rangle = \langle u, v\,p \rangle,\ p \in P$, with the right product given by $v\,p = \sum_{k=0}^{n} \left(\sum_{j=k}^{n} p_j v_{j-k} \right) x^k$, being that $p(x) = \sum_{j=0}^{n} p_j x^j$.

If $B \equiv 0$ in the previous equations, then we have the semi-classical case [24]. Furthermore, when u is represented in terms of a weight function w, that is,

$$u_n = \int_I x^n\, w(x) dx\,,\ n \geq 0,$$

then $\mathcal{D}(Au) = \psi u$ is equivalent to $Aw' = Cw$, $C = \psi - A'$, under the boundary conditions

$$x^n A(x) w(x)\big|_{a,b} = 0\,, \quad n \geq 0\,,$$

where a, b (eventually a or b infinite) are linked with the roots of A. In such a case, w is the weight function on the support $I = [a, b]$. The class of a semi-classical linear functional u is the (unique) nonnegative integer s defined by $s = \min_{(\phi,\psi)\in\mathcal{E}} \max\{\deg(\phi) - 2,\ \deg(\psi) - 1\}$, where $\mathcal{E}$ is the set of all pairs of polynomials (ϕ, ψ) with $\deg(\psi) \geq 1$ satisfying the distributional equation $\mathcal{D}(Au) = \psi u$. The semi-classical class $s = 1$ has been revisited many times in the literature, some references on classification include [3, 4, 22, 23].

In the present paper we will use the differential systems from [5, 11] combined with the recurrence relation - the compatibility conditions, to deduce non-linear difference equations for the recurrence relation coefficients of Laguerre-Hahn polynomials. Some of such difference equations give a recursive way to compute the β_n and γ_n,

$$\beta_n = F(\beta_{n-1}, \ldots, \beta_0, \hat{f}_n)\,, \quad \gamma_{n+1} = G(\gamma_n, \ldots, \gamma_1, \beta_n, \hat{g}_n)\,, \quad n \geq 1\,,$$

with $\hat{f}_n, \hat{g}_n$ explicit expressions given in terms of A, B, C, D. In the symmetric case, that is, $\beta_n = 0,\ n \geq 0$, we recover the closed formulae for γ_n given in [2]. We also show connections between Laguerre-Hahn orthogonal polynomials and discrete Painlevé equations: in case $\deg(A) = 2$, difference equations which are similar to a form of dP_V are deduced; the case $\deg(A) \leq 1$ was analyzed in [12].

The reminder of the paper is organized as follows: in Section 2 we show difference-differential equations for Laguerre-Hahn orthogonal polynomials to be used throughout the paper, and we discuss some remarks on the canonical forms of Laguerre-Hahn functional equations; in Section 3 we deduce difference equations for the recurrence relation coefficients; in Section 4 we show connections between the orthogonal polynomials and discrete Painlevé equations; in Section 5 we show examples that illustrate the results obtained in sections 3 and 4. We state final remarks and conclusions in Section 6.

2 Preliminary results

We shall take sequences of monic orthogonal polynomials, $P_n(x) = x^n+$ lower degree terms, $n \geq 0$, satisfying (2), and we denote them by SMOP. Let us also consider the sequence of associated polynomials of the first kind of $\{P_n\}_{n\geq 0}$, denoted by $\{P_n^{(1)}\}_{n\geq 0}$, satisfying the three-term recurrence relation

$$P_n^{(1)}(x) = (x-\beta_n)P_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x)\,, \quad n = 1, 2, ... \tag{6}$$

with $P_{-1}^{(1)}(x) = 0,\ P_0^{(1)}(x) = 1$. We combine the recurrence relations (2) and (6) in the matrix form,

$$Y_n = \mathcal{A}_n Y_{n-1}\,, \ \ Y_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}\,, \ \ \mathcal{A}_n = \begin{bmatrix} x-\beta_n & -\gamma_n \\ 1 & 0 \end{bmatrix}\,, \quad n \geq 1\,, \tag{7}$$

with initial conditions $Y_0 = \begin{bmatrix} x-\beta_0 & 1 \\ 1 & 0 \end{bmatrix}$.

The SMOP related to (4), $AS' = BS^2 + CS + D$, satisfies the matrix Sylvester equation [5]

$$AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}\,, \quad n \geq 0\,, \tag{8}$$

where

$$\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & l_{n-1} + (x-\beta_n)\Theta_{n-1}/\gamma_n \end{bmatrix}\,, \ \ \mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix}$$

with l_n, Θ_n polynomials of uniformly bounded degrees. We will take the convention $\Theta_{-1}/\gamma_0 = D$. The initial conditions for l_n, Θ_n are

$$l_{-1} = C/2\,, \ \ l_0 = -C/2 - (x-\beta_0)D\,, \ \ \Theta_0 = A + (x-\beta_0)(C/2 - l_0) + B\,. \tag{9}$$

Combining the recurrence relation (7) with the differential system (8) yields the Lax pair

$$\begin{cases} Y_n = \mathcal{A}_n Y_{n-1}\,, \\ AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}\,. \end{cases} \tag{10}$$

As a consequence, we get the compatibility conditions for the matrices $\mathcal{A}_n$,

$$A\mathcal{A}_n' = \mathcal{B}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{B}_{n-1}\,, \quad n \geq 1\,. \tag{11}$$

Equation (11) yields two non-trivial equations, respectively, from positions $(1,1)$ and $(1,2)$:

$$A=(x-\beta_n)(l_n-l_{n-1})+\Theta_n-\gamma_n\frac{\Theta_{n-2}}{\gamma_{n-1}}\,, \tag{12}$$

$$l_n+(x-\beta_n)\frac{\Theta_{n-1}}{\gamma_n}=l_{n-2}+(x-\beta_{n-1})\frac{\Theta_{n-2}}{\gamma_{n-1}}\,. \tag{13}$$

After some basic computations, (12) implies (15) and (13) implies (14) (see [5, Corollary 1])

$$\operatorname{tr}\mathcal{B}_n=0\,,\quad n\geq 0\,, \tag{14}$$

$$\det\mathcal{B}_n=\det\mathcal{B}_0+A\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k}\,,\quad n\geq 1\,, \tag{15}$$

with $\det\mathcal{B}_0=D(A+B)-(C/2)^2$.

Equations (14) and (15) read, respectively, as

$$l_n(x)+l_{n-1}(x)+(x-\beta_n)\frac{\Theta_{n-1}(x)}{\gamma_n}=0\,,\ \ n\geq 0\,, \tag{16}$$

$$-l_n^2(x)+\Theta_n(x)\frac{\Theta_{n-1}(x)}{\gamma_n}=\det\mathcal{B}_0+A\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k}\,,\quad n\geq 1\,. \tag{17}$$

Under suitable degrees of the polynomials A,B,C,D, the equations (12)–(15), together with the initial conditions from (9), will be used to get recurrences for β_n,γ_n. Recall that we will consider the class one, that is,

$$1=\max\{\deg(C)-1,d-2\}\,,\ \ d=\max\{\deg(A),\deg(B)\}\,,$$

where the polynomials A,B,C,D are co-prime [2, Prop. 2.5]. It turns out that D is a polynomial of degree one, defined in terms of A,B,C through equations (21)–(22) below.

The next lemma gives us fundamental quantities to be used in the sequel. Throughout the paper we will use the following convention: if $i>j$, then $\sum_i^j\cdot=0$.

Lemma 2.1 [12] *Let S be a Stieltjes function satisfying $AS'=BS^2+CS+D$. Let $\{P_n\}_{n\geq 0}$ be the SMOP associated with S, satisfying the recurrence relation (2), $P_{n+1}(x)=(x-\beta_n)P_n(x)-\gamma_nP_{n-1}(x)\,,\quad n=0,1,2,\ldots$.*

Set

$$A(x)=a_3x^3+a_2x^2+a_1x+a_0\,,\ \ B(x)=b_3x^3+b_2x^2+b_1x+b_0\,, \tag{18}$$

$$C(x)=c_2x^2+c_1x+c_0\,,\ \ D(x)=d_1x+d_0\,, \tag{19}$$

$$l_n(x)=\ell_{n,2}x^2+\ell_{n,1}x+\ell_{n,0}\,,\ \ \Theta_n(x)=\theta_{n,1}x+\theta_{n,0}\,, \tag{20}$$

where $|a_3|+|b_3|+|c_2|\neq 0$. With the functions defined above, we have, for all $n\geq 1$,

$$d_1=-a_3-b_3-c_2\,, \tag{21}$$

$$d_0=-(2a_3+2b_3+c_2)\beta_0-a_2-b_2-c_1\,, \tag{22}$$

$$\ell_{n,2}=(n+1)a_3+\eta/2\,, \tag{23}$$

$$\ell_{n,1}=a_3S_n+(n+1)a_2+\lambda/2\,, \tag{24}$$

$$\ell_{n,0}=-2a_3(\nu_n+\beta_0\eta_n)+(n+1)a_1+\mu+a_3S_n^2+a_2S_n-\theta_{n,1}\,, \tag{25}$$

$$\frac{\theta_{n,1}}{\gamma_{n+1}}=-((2n+3)a_3+\eta)\,, \tag{26}$$

$$\frac{\theta_{n,0}}{\gamma_{n+1}}=-\left\{2a_3(S_n+(n+2)\beta_{n+1})+(2n+3)a_2+\eta\beta_{n+1}+\lambda\right\}\,, \tag{27}$$

where

$$\eta=2b_3+c_2\,,\ \ \lambda=2b_3\beta_0+2b_2+c_1\,,\ \ \mu=b_3(\gamma_1+\beta_0^2)+b_2\beta_0+b_1+c_0/2\,, \tag{28}$$

and

$$\eta_n = \sum_{k=1}^{n} \beta_k \,, \quad S_n = \beta_0 + \eta_n \,, \quad \nu_n = \sum_{1\le i<j\le n} \beta_i\beta_j - \sum_{k=1}^{n} \gamma_k \,.$$

Also, we have

$$\frac{\theta_{0,1}}{\gamma_1} = -3a_3 - \eta \,, \quad \frac{\theta_{0,0}}{\gamma_1} = -\frac{(a_0 + b_0 - \beta_0(c_0 - \beta_0 d_0))(3a_3 + \eta)}{a_1 + b_1 + c_0 + (2a_2 + 2b_2 + c_1)\beta_0 + (3a_3 + 3b_3 + c_2)\beta_0^2} \,, \tag{29}$$

$$\ell_{0,2} = a_3 + \frac{\eta}{2} \,, \quad \ell_{0,1} = -\frac{c_1}{2} - d_0 - \beta_0(a_3 + b_3 + c_2) \,, \quad \ell_{0,0} = -\frac{c_0}{2} + \beta_0 d_0 \,. \tag{30}$$

Additionally, there holds the condition

$$a_1 + b_1 + c_0 + (2a_2 + 2b_2 + c_1)\beta_0 + (3a_3 + 3b_3 + c_2)\beta_0^2 + (3a_3 + 2b_3 + c_2)\gamma_1 = 0 \,. \tag{31}$$

Remark 2.2 If the polynomials A, B, C, D are not co-prime, then some of the quantities given in the previous lemma may simplify. For instance, if A, B, C, D have one common root, say α, then, by dividing the Riccati equation $AS' = BS^2 + CS + D$ by $x - \alpha$, we obtain the equation $\tilde{A}S' = \tilde{B}S^2 + \tilde{C}S + \tilde{D}$, with lower degree polynomials $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$. In this case, the class would decrease and the corresponding quantities $d_1, l_{n,2}$, and $\theta_{n,1}$ would be zero.

In the sequel we shall assume $|a_3| + |b_3| + |c_2| \neq 0$. Furthermore, whenever $a_3 = 0$, we will take $\eta \neq 0$.

2.1 Remarks on the canonical forms of Laguerre-Hahn functional equations

The coefficients of the polynomials A, ψ, B involved in the distributional equation (5), $\mathcal{D}(Au) = \psi u + B(x^{-1}u^2)$, are linearly related. Indeed, in the account of the notation from (18) together with $\psi(x) = \psi_2 x^2 + \psi_1 x + \psi_0$ (cf. (5)), condition $\langle \mathcal{D}(Au), x^m\rangle = \langle \psi u, x^m\rangle + \langle B(x^{-1}u^2), x^m\rangle \,, \quad m \geq 0$, gives us the following system of equations involving the moments $u_k,\ k = 0, \ldots, m+2$:

$$\begin{cases} 0 = \psi_2 u_2 + \psi_1 u_1 + \psi_0 u_0 + \sum_{k=0}^{2} \left(\sum_{j=k}^{2} \check{b}_j u_{j-k}\right) u_k \,, & \text{if } m = 0 \,, \\ -m(a_3 u_{m+2} + a_2 u_{m+1} + a_1 u_m + a_0 u_{m-1}) & \\ \qquad = \psi_2 u_{m+2} + \psi_1 u_{m+1} + \psi_0 u_m + \sum_{k=0}^{m+2} \left(\sum_{j=k}^{m+2} \check{b}_j u_{j-k}\right) u_k \,, & \text{if } m \geq 1 \,, \end{cases} \tag{32}$$

where $\check{b}_j,\ j = 0, \ldots, m+2$, are the coefficients of the polynomial $\check{B} = \theta_0(x^m B(x))$, that is,

$$\check{B}(x) = \begin{cases} x^{m-1}B(x) \,, & \text{if } m \geq 1 \,, \\ b_3 x^2 + b_2 x + b_1 \,, & \text{if } m = 0 \,. \end{cases}$$

On the other hand, note that a displacement (which amounts to a linear change of variable in the Riccati equation) does not change neither the Laguerre-Hahn character nor the class of a Laguerre-Hahn linear functional (more details can be found in [26, Sec. 4.2]). Thus, in order to get the so-called canonical forms of functional equations, we can consider the polynomial A given by the following cases:

$$\begin{aligned} &\deg(A) = 3 : \quad A(x) = (x^2 - 1)(x - c) \,,\ c \neq \pm 1 \,, \quad A(x) = x^2(x - 1) \,, \quad A(x) = x^3 \,; \\ &\deg(A) = 2 : \quad A(x) = x^2 - 1 \,, \quad A(x) = x^2 \,; \\ &\deg(A) = 1 : \quad A(x) = x \,; \\ &\deg(A) = 0 : \quad A(x) = 1 \,. \end{aligned}$$

Hence, given the polynomial A, the system of equations (32) with $m = 0, \ldots, 6$, yields the coefficients of ψ and B, defined in terms of the a_j's as well as in terms of $u_0 (= 1), u_1, \ldots, u_8$. Taking into account that the first eight moments are functions of $\beta_0, \ldots, \beta_3, \gamma_1, \ldots, \gamma_4$, then a finite number of (the first) recurrence coefficients will be acting as arbitrary parameters in the coefficients of ψ, B.

Let us remark that an equivalent system of equations to get ψ, B (given A), can be obtained through the use of the initial conditions, as follows: by taking the linear and independent terms in (9); the independent terms in (12) with $n=1$ as well as $n=2$; the linear and independent terms in (13) with $n=1$; the independent term in (13) with $n=2$. In such a case, we obtain the system of equations that we now write in the matrix form as $\mathcal{M}\mathcal{V}=\mathcal{A}$, with $\mathcal{V}=\begin{bmatrix}\psi_2 & \psi_1 & \psi_0 & b_3 & b_2 & b_1 & b_0\end{bmatrix}^T$, with T denoting the transpose, and

$$\mathcal{M}=\begin{bmatrix}\gamma_1+\beta_0^2 & \beta_0 & 1 & 2\gamma_1+3\beta_0^2 & 2\beta_0 & 1 & 0\\ -\beta_0^3 & -\beta_0^2 & -\beta_0 & -2\beta_0^3 & -\beta_0^2 & 0 & 1\\ \mathcal{M}_{3,1} & -\beta_0\beta_1-\gamma_2+\gamma_1 & -\beta_1 & \mathcal{M}_{3,4} & -2\beta_0\beta_1-2\gamma_2+\gamma_1 & -\beta_1 & 0\\ \beta_1 & 1 & 0 & 2\beta_0+2\beta_1 & 2 & 0 & 0\\ \gamma_2-\beta_0^2 & -\beta_0 & 0 & \gamma_1-\beta_0^2+2\gamma_2 & 0 & 1 & 0\\ \mathcal{M}_{6,1} & -\gamma_3 & 0 & \mathcal{M}_{6,4} & -2\gamma_3 & 0 & 0\\ \beta_0^2+\beta_2^2+\gamma_3 & \beta_0+\beta_2 & 1 & \mathcal{M}_{7,4} & 2\beta_0+2\beta_2 & 1 & 0\end{bmatrix},$$

$$\mathcal{A}=\begin{bmatrix}0\\ -a_3\beta_0^3-a_2\beta_0^2-a_1\beta_0-a_0+\theta_{0,0}\\ \mathcal{A}_{3,1}\\ -a_3(2\beta_0+\beta_1)-a_2-\theta_{0,0}/\gamma_1\\ -a_3(2\beta_0^2+\beta_1^2+2\gamma_1+2\gamma_2)-a_2(2\beta_0+\beta_1)-2a_1+\beta_1\frac{\theta_{0,0}}{\gamma_1}\\ \mathcal{A}_{6,1}\\ -a_3(2\beta_2(\beta_0+\beta_1)+\beta_1^2+4\beta_2^2+2\gamma_1+2\gamma_2+4\gamma_3)-a_2(\beta_1+4\beta_2)-2a_1-\beta_1\frac{\theta_{0,0}}{\gamma_1}\end{bmatrix},$$

where

$$\begin{aligned}
\mathcal{M}_{3,1} &= -\beta_1\gamma_2-\beta_0\beta_1-\beta_2\gamma_2+\gamma_1\,,\\
\mathcal{M}_{3,4} &= -\beta_1(\gamma_1+\beta_0^2)-2\beta_1\gamma_2-2\beta_0\beta_1-2\beta_2\gamma_2-2\beta_0\gamma_2+2\gamma_1\,,\\
\mathcal{M}_{6,1} &= \beta_2(\gamma_2-\gamma_3)-\beta_3\gamma_3\,,\\
\mathcal{M}_{6,4} &= 2\beta_2(\gamma_2-\gamma_3)-2\beta_0\gamma_3-2\beta_3\gamma_3\,,\\
\mathcal{M}_{7,4} &= -\beta_0^2+2\beta_0\beta_2+2\beta_2^2+\gamma_1+2\gamma_3\,,\\
\mathcal{A}_{3,1} &= a_3(2\beta_1\gamma_1+3\beta_0^2\beta_1+\beta_1^3+4\beta_1\gamma_2+2\beta_0\gamma_2+3\beta_2\gamma_2-2\beta_0\gamma_1-3\beta_0\beta_1+3\gamma_1)\\
&\qquad +a_2(\beta_1^2+3\gamma_2+\gamma_1)+a_1\beta_1+a_0\,,\\
\mathcal{A}_{6,1} &= a_3\left[2\beta_2(3\gamma_3-\gamma_2)+2\beta_0\gamma_3+2\beta_1\gamma_3+5\beta_3\gamma_3+\beta_2(\gamma_2-2\beta_2(\beta_0+\beta_1))+(S_2^2-S_1^2)\right]\\
&\qquad +a_2(5\gamma_3+\beta_2^2)+a_1\beta_2+a_0+\gamma_2\frac{\theta_{0,0}}{\gamma_1}\,.
\end{aligned}$$

Solving for $\mathcal{V}$ we get the coefficients of ψ, B, as required. When $B\equiv 0$ we get the semi-classical case, thus, we get the canonical cases from [3]. In the Laguerre-Hahn symmetric case we get the canonical cases from [2] (note that we have to adjust the notation, by taking a minus sign in the polynomials B and ψ).

Some of the functional equations obtained in the Laguerre-Hahn class one correspond to the following sequences of orthogonal polynomials: the associated and anti-associated polynomials related to functionals $\tilde{u}$ such as $\tilde{u}=R(x)u+\sum_{k=1}^{n_1}\lambda_k\delta(x-\xi_k)$, where R is some function of the type $R(x)=\prod_{k=1}^{n_0}|x-\alpha_k|^{\mu_k}$, with α_k,μ_k suitable real numbers, being the α_k's located outside the support of u, $\delta(x-\xi_k)$ is the Dirac Delta functional at ξ_k, for some suitable real numbers ξ_k, and u is the linear functional corresponding to the Hermite, Laguerre, Jacobi, and Bessel polynomials (see [3] and [2, pg. 317]); the associated and anti-associated polynomials related to some limiting cases of $q-$Racah polynomials when $q\to -1$ (see [3, pg. 264] and the references therein); the generalized co-dilated with parameter $k=1$ (see [10, Eqs. (5), (21)]) of the previous sequences.

3 Main results

The goal of this section is to obtain recurrences for the recurrence relation coefficients β_n,γ_n, given the polynomials A,B,C,D in the Riccati equation (4). We shall deduce formulae that compute the β_n and γ_n recursively.

We start by using equation (17) to get difference equations involving β_n, γ_n. The coefficients of x^2 and x of (17) yield, respectively, for all $n \geq 1$,

$$-2\ell_{n,0}\ell_{n,2} - \ell_{n,1}^2 + \theta_{n,1}\frac{\theta_{n-1,1}}{\gamma_n} = \xi_{0,2} + a_2\sum_{k=1}^{n}\frac{\theta_{k-1,0}}{\gamma_k} + a_1\sum_{k=1}^{n}\frac{\theta_{k-1,1}}{\gamma_k} , \tag{33}$$

$$-2\ell_{n,1}\ell_{n,0} + \theta_{n,0}\frac{\theta_{n-1,1}}{\gamma_n} + \theta_{n,1}\frac{\theta_{n-1,0}}{\gamma_n} = \xi_{0,1} + a_1\sum_{k=1}^{n}\frac{\theta_{k-1,0}}{\gamma_k} + a_0\sum_{k=1}^{n}\frac{\theta_{k-1,1}}{\gamma_k} . \tag{34}$$

The quantities $\xi_{0,j}$ denote the coefficient of x^j in $\det \mathcal{B}_0,\ j = 0, \ldots, 4$.

It is important to emphasize that equations (33) and (34) involve sums and products of the β_k's and γ_k's. A much simpler equation, expressing γ_{n+1} in terms of the β_k's only, is obtained in the following theorem.

Theorem 3.1 *The coefficients γ_n satisfy, for all $n \geq 1$,*

$$w_n\gamma_{n+1} = \vartheta_n , \tag{35}$$

with

$$w_n = \ell_{n,1}\frac{\theta_{n,1}}{\gamma_{n+1}}\frac{\theta_{n-1,1}}{\gamma_n} - \ell_{n,2}\left(\frac{\theta_{n,0}}{\gamma_{n+1}}\frac{\theta_{n-1,1}}{\gamma_n} + \frac{\theta_{n,1}}{\gamma_{n+1}}\frac{\theta_{n-1,0}}{\gamma_n}\right) ,$$

$$\vartheta_n = \ell_{n,1}\xi_{0,2} - \ell_{n,2}\xi_{0,1} + \ell_{n,1}^3 + (a_2\ell_{n,1} - a_1\ell_{n,2})\sum_{k=1}^{n}\frac{\theta_{k-1,0}}{\gamma_k} + (a_1\ell_{n,1} - a_0\ell_{n,2})\sum_{k=1}^{n}\frac{\theta_{k-1,1}}{\gamma_k} . \tag{36}$$

Thus, if $w_n \neq 0$, the coefficients γ_n are defined by

$$\gamma_{n+1} = \frac{\vartheta_n}{w_n} , \quad n \geq 1 , \tag{37}$$

$$\gamma_1 = -\frac{a_1 + b_1 + c_0 + (2a_2 + 2b_2 + c_1)\beta_0 + (3a_3 + 3b_3 + c_2)\beta_0^2}{3a_3 + \eta} . \tag{38}$$

P r o o f. The elimination of $\ell_{n,0}$ between (33) and (34) yields (35). The equation for γ_1 comes from (31). □

Remark 3.2 The quantities w_n, ϑ_n do not depend on the γ_k's.

Now we use equations (12), (13) and (16).

Lemma 3.3 *The coefficients γ_n satisfy the following difference equations:*
for all $n \geq 1$,

$$r_n\gamma_{n+1} = s_n\gamma_n + t_n , \tag{39}$$

with

$$r_n = \frac{\Theta_n}{\gamma_{n+1}}(\beta_n) , \quad s_n = \frac{\Theta_{n-2}}{\gamma_{n-1}}(\beta_n) , \quad t_n = A(\beta_n) ;$$

for all $n \geq 3$,

$$((2n+3)a_3 + \eta)\gamma_{n+1} + 2a_3\gamma_n - ((2n-3)a_3 + \eta)\gamma_{n-1} = -2a_1 + v_n , \tag{40}$$

with

$$v_n = -2a_3\beta_n^2 - (2a_2 + a_3(-\beta_{n-1} + \beta_n))\beta_n - (\beta_{n-1} + \beta_n)(a_2 + a_3(\beta_{n-1} + \beta_n)) + (\beta_n - \beta_{n-1})\frac{\Theta_{n-2}}{\gamma_{n-1}}(\beta_n) ;$$

for all $n \geq 2$,

$$
\begin{aligned}
&-((2n+1)a_3+\eta)\beta_n^2 - (2a_3S_{n-1}+a_3\beta_n+(2n+1)a_2+\lambda)\beta_n \\
&+2a_3(2\nu_{n-1}+\eta_{n-1}(\beta_n+2\beta_0)+\beta_0\beta_n)-(2n+1)a_1-2\mu \\
&\qquad -a_3(S_n^2+S_{n-1}^2)-a_2(S_n+S_{n-1}) = ((2n+3)a_3+\eta)(\gamma_n+\gamma_{n+1}). \quad (41)
\end{aligned}
$$

P r o o f. Equation (12) gives us (39) when evaluated at β_n.
Equation (13) gives us (40) when evaluated at β_n or at β_{n-1}, where we used l_k and Θ_k for $k \geq 1$.
Equation (16) gives us (41) when evaluated at β_n, where we used l_k and Θ_k for $k \geq 1$. □

Remark 3.4 Also, (13) gives us the following equations: taking $n=2$ and evaluating at β_2 and taking $n=1$ and evaluating at β_1 we get, respectively,

$$l_2(\beta_2) = l_0(\beta_2) + (\beta_2-\beta_1)\frac{\Theta_0}{\gamma_1}(\beta_2)\,, \quad (42)$$

$$l_1(\beta_1) = \frac{C}{2}(\beta_1) + (\beta_1-\beta_0)D(\beta_1)\,. \quad (43)$$

Furthermore, taking $n=1$ in (16) and evaluating at β_1, we get

$$l_1(\beta_1) + l_0(\beta_1) = 0\,. \quad (44)$$

Let us emphasize that r_n, s_n, t_n, v_n do not depend on the γ_k's.

Using Lemma 3.3 we now deduce a formula that allows to compute recursively the coefficients β_n, given β_0.

Theorem 3.5 *Under the previous notations, the coefficients β_n are defined recursively through*

$$\beta_{n+1} = \frac{\{((2n+3)a_3+\eta)\,\beta_n + 2a_3S_n + (2n+3)a_2 + \lambda\}\,\vartheta_n + y_n\varpi_n}{-(2(n+2)a_3+\eta)\vartheta_n + z_n\varpi_n}\,, \quad n \geq 2\,, \quad (45)$$

assuming $-(2(n+2)a_3+\eta)\vartheta_n + z_n\varpi_n \neq 0$.
There hold the initial conditions:

$$\beta_2 = \frac{((5a_3+\eta)\,\beta_1 + 2a_3(\beta_0+\beta_1) + 5a_2 + \lambda)\,\vartheta_1 + y_1(\gamma_1D(\beta_1)+A(\beta_1))}{-(6a_3+\eta)\vartheta_1 + z_1(\gamma_1D(\beta_1)+A(\beta_1))}\,, \quad (46)$$

$$\beta_1 = \frac{2a_3\beta_0 + 3a_2 + \theta_{0,0}/\gamma_1 + \lambda}{-4a_3-\eta}\,, \quad (47)$$

with γ_1 given in (38). For all $n \geq 1$,

$$
\begin{aligned}
y_n &= \ell_{n,1}((2n+1)a_3+\eta)((2n+3)a_3+\eta) \\
&+ \ell_{n,2}\left\{-(2a_3S_n+(2n+3)a_2+\lambda)((2n+1)a_3+\eta) + \frac{\theta_{n-1,0}}{\gamma_n}((2n+3)a_3+\eta)\right\}, \quad (48)\\
z_n &= ((n+1)a_3+\eta/2)\,(2(n+2)a_3+\eta)\,((2n+1)a_3+\eta)\,, \quad (49)
\end{aligned}
$$

and, for all $n \geq 2$,

$$\varpi_n = s_n\frac{\vartheta_{n-1}}{y_{n-1}-z_{n-1}\beta_n} + A(\beta_n)\,.$$

P r o o f. Let us write (37) as

$$\gamma_{n+1} = \frac{\vartheta_n}{y_n - z_n\beta_{n+1}}\,, \quad n \geq 1\,, \quad (50)$$

where ϑ_n is the function of $\beta_0, \ldots, \beta_n$ defined previously, y_n is the function of $\beta_0, \ldots, \beta_n$ given by (48), and z_n is given by (49).

Using (50) into equation (39) we have, for all $n \geq 1$,

$$r_n \frac{\vartheta_n}{y_n - z_n\beta_{n+1}} = \varpi_n \,, \tag{51}$$

where we write $\varpi_n = s_n \dfrac{\vartheta_{n-1}}{y_{n-1} - z_{n-1}\beta_n} + A(\beta_n)$. Taking into account that, for all $n \geq 1$,

$$r_n = -((2n+3)a_3 + \eta)\beta_n - 2a_3 S_n - (2n+3)a_2 - \lambda - (2(n+2)a_3 + \eta)\beta_{n+1} \,,$$

from (51) we get

$$\{-((2n+3)a_3 + \eta)\beta_n - 2a_3 S_n - (2n+3)a_2 - \lambda \\ -(2(n+2)a_3 + \eta)\beta_{n+1}\}\, \vartheta_n = \varpi_n (y_n - z_n\beta_{n+1}) \,. \tag{52}$$

Note that ϖ_n and y_n are functions of $\beta_0, \ldots, \beta_n$. Solving (52) for β_{n+1} yields (45).

To obtain β_2 take $n = 1$ in (39), that is, $\dfrac{\Theta_1}{\gamma_2}(\beta_1)\gamma_2 = \gamma_1 D(\beta_1) + A(\beta_1)$, and use (50) with $n = 1$, that is, $\gamma_2 = \dfrac{\vartheta_1}{y_1 - z_1\beta_2}$. Solving the resulting equation for β_2 we get (46). The coefficient β_1 may be obtained as follows: take $n = 1$ in (16) and substitute it into the equation that results from position $(1, 2)$ of (8) for $n = 1$. The coefficient of x^2 yields (47). □

3.1 Simplifications when $\deg(A) \leq 2$.

Some simplifications occur under lower degrees of A. Indeed, if $\deg(A) \leq 2$, then $\ell_{n,0}$ involves linear terms in β_k's and γ_k's (cf. (25)). Therefore, equations (33), (34) and (41) substantially simplify. We note that equation (33) produces no new identities, due to cancellations.

Equations (34) and (41) will now be used to obtain much simpler formulae for γ_n, as well as for β_n.

Theorem 3.6 *Under the previous notations, the coefficients γ_n are defined in terms of the β_k's. There hold the formulae, for all $n \geq 1$:*

$$\gamma_{n+1} = \frac{f_n/\eta}{2(n+1)a_2 + \lambda + \eta(\beta_{n+1} + \beta_n)} \,, \tag{53}$$

$$\gamma_1 = -\frac{a_1 + b_1 + c_0 + (2a_2 + 2b_2 + c_1)\beta_0 + (3b_3 + c_2)\beta_0^2}{\eta} \,, \tag{54}$$

assuming $2(n+1)a_2 + \lambda + \eta(\beta_{n+1} + \beta_n) \neq 0$. *Here,*

$$f_n = f_n(\beta_0, \ldots, \beta_n) = (2(n+1)a_2 + \lambda)((n+1)a_1 + \mu + a_2 S_n) + \xi_{0,1} \\ + a_1 \left(\frac{\theta_{0,0}}{\gamma_1} - (n-1)(n+3)a_2 - \eta \sum_{k=2}^{n} \beta_k - (n-1)\lambda \right) - n\eta a_0 \,. \tag{55}$$

P r o o f. Equation (34) yields (53). Let us detail.

Taking $a_3 = 0$ in (24)-(27) we have, for all $n \geq 1$,

$$\ell_{n,1} = (n+1)a_2 + \lambda/2 \,, \quad \ell_{n,0} = (n+1)a_1 + \mu + a_2 S_n + \eta\gamma_{n+1} \,,$$
$$\frac{\theta_{n,1}}{\gamma_{n+1}} = -\eta \,, \quad \frac{\theta_{n,0}}{\gamma_{n+1}} = -((2n+3)a_2 + \eta\beta_{n+1} + \lambda) \,,$$

together with the corresponding initial conditions (29). Also, we have

$$\sum_{k=1}^{n} \frac{\theta_{k-1,1}}{\gamma_k} = -n\eta \,, \quad \sum_{k=1}^{n} \frac{\theta_{k-1,0}}{\gamma_k} = \frac{\theta_{0,0}}{\gamma_1} - (n-1)(n+3)a_2 - \eta \sum_{k=2}^{n} \beta_k - (n-1)\lambda \,.$$

Using this data into (34) and solving for γ_{n+1}, we get (53). The equation for γ_1 follows by taking $a_3 = 0$ in (38). □

The equations in the following lemma are obtained by taking $a_3 = 0$ in (39), (40), and (41), respectively, and using the data from (23)-(27) for $n \geq 1$. Note that s_n in (39) reads differently, according to $n = 1$, $n = 2$, or $n \geq 3$. Therefore, we state the following results.

Lemma 3.7 *The following equations hold:*
for all $n \geq 3$,

$$\gamma_{n+1}\left(\eta(\beta_{n+1}+\beta_n)+(2n+3)a_2+\lambda\right) = \gamma_n\left(\eta(\beta_n+\beta_{n-1})+(2n-1)a_2+\lambda\right) - A(\beta_n)\,; \tag{56}$$

for all $n \geq 3$,

$$3a_2\beta_n + 2a_1 + a_2\beta_{n-1} + (\beta_n - \beta_{n-1})\left(\eta(\beta_n+\beta_{n-1})+(2n-1)a_2+\lambda\right) = \eta(\gamma_{n-1}-\gamma_{n+1})\,; \tag{57}$$

for all $n \geq 2$,

$$-\eta\beta_n^2 - (2(n+1)a_2+\lambda)\beta_n - (2n+1)a_1 - 2\mu - 2a_2S_{n-1} = \eta(\gamma_n+\gamma_{n+1})\,. \tag{58}$$

Using Theorem 3.6 and Lemma 3.7 we now deduce formulae that compute recursively the coefficients β_n and γ_n, given β_0.

Theorem 3.8 *Under the previous notations, the coefficients* β_n *and* γ_n *can be computed recursively through the following equations:*

$$\beta_{n+1} = -\beta_n + \frac{\kappa_n - (2(n+1)a_2+\lambda)}{\eta}\,, \quad n \geq 2\,, \tag{59}$$

$$\gamma_{n+1} = \frac{-2a_2\sum_{k=3}^{n}\gamma_k - \sum_{k=3}^{n}A(\beta_k) + \gamma_3(\eta(\beta_3+\beta_2)+7a_2+\lambda)}{(2n+3)a_2+\lambda+\eta(\beta_{n+1}+\beta_n)}\,, \quad n \geq 3\,, \tag{60}$$

assuming $(2n+3)a_2+\lambda+\eta(\beta_{n+1}+\beta_n) \neq 0$. *There hold the initial conditions (depending on* β_0*):*

$$\beta_2 = \frac{(\eta\beta_1+5a_2+\lambda)\,\vartheta_1 + y_1(\gamma_1D(\beta_1)+A(\beta_1))}{-\eta\vartheta_1 + z_1(\gamma_1D(\beta_1)+A(\beta_1))}\,, \tag{61}$$

$$\beta_1 = \frac{3a_2+\theta_{0,0}/\gamma_1+\lambda}{-\eta}\,, \tag{62}$$

and

$$\gamma_3 = -\gamma_2 - \frac{\left(\eta\beta_2^2+(6a_2+\lambda)\beta_2+5a_1+2\mu+2a_2(\beta_0+\beta_1)\right)}{\eta}\,,$$

$$\gamma_2 = \frac{(\eta/2)\beta_1^2+(2a_2+\lambda/2)\beta_1+2a_1+\mu+a_2(\beta_0+\beta_1)-(C/2)(\beta_1)-(\beta_1-\beta_0)D(\beta_1)}{\eta}\,,$$

$$\gamma_1 = -\frac{a_1+b_1+c_0+(2a_2+2b_2+c_1)\beta_0+(3b_3+c_2)\beta_0^2}{\eta}\,.$$

The quantities κ_n *are given by*

$$\kappa_n = \frac{(2na_2+\lambda+\eta(\beta_n+\beta_{n-1}))f_n}{(2na_2+\lambda+\eta(\beta_n+\beta_{n-1}))g_n - f_{n-1}}\,, \tag{63}$$

where $g_n = -\eta\beta_n^2 - (2(n+1)a_2+\lambda)\beta_n - (2n+1)a_1 - 2\mu - 2a_2S_{n-1}$, *and* f_n *is given by (55).*

Proof. To deduce (59) we take equation (58), written as

$$\eta(\gamma_n+\gamma_{n+1}) = g_n\,, \quad n \geq 2\,,$$

and we use (53). Noting that g_n is a function of $\beta_0,\ldots,\beta_n$, solving for β_{n+1} yields (59).

To deduce (60) we start with equation (56) written as

$$\gamma_{n+1}\left(\eta(\beta_{n+1}+\beta_n)+(2n+3)a_2+\lambda\right)=\gamma_n\left(\eta(\beta_n+\beta_{n-1})+(2n+1)a_2+\lambda\right) \\ -2a_2\gamma_n-A(\beta_n)\,,\quad n\geq 3\,.$$

Using the notation $T_n=\gamma_n\left(\eta(\beta_n+\beta_{n-1})+(2n+1)a_2+\lambda\right)$, we have

$$T_{n+1}=T_n-2a_2\gamma_n-A(\beta_n)\,,\quad n\geq 3\,. \tag{64}$$

Iterating, we get

$$T_{n+1}=T_3-2a_2\sum_{k=3}^{n}\gamma_k-\sum_{k=3}^{n}A(\beta_k)\,,\quad n\geq 3\,,$$

and (60) follows.

The coefficients β_2 and β_1 follow from (46) and (47) , respectively, taking $a_3=0$.

The coefficient γ_3 is obtained by taking $n=2$ in (58); γ_2 is obtained from (43) using $a_3=0$; γ_1 is given in (54). □

Remark 3.9 The equations from Theorems 3.6 and 3.8 and Lemma 3.7 apply for $\deg(A)=1$ or $\deg(A)=0$ by taking, respectively, $a_2=0$ or $a_2=a_1=0$.

Remark 3.10 In the symmetric case of class one we have $\beta_n=0,\ n\geq 0$, and $AS'=BS^2+CS+D$ with A,B odd and C even (see [2, Prop. 3.1]). This case was analysed in [2]. Our results agree with those, noting the following adjustment in the notation: to take a minus sign in the polynomials B and ψ.

4 Connections with discrete Painlevé equations

Some of the difference equations satisfied by the recurrence relation coefficients of Laguerre-Hahn orthogonal polynomials obtained previously can be put into the form of discrete Painlevé equations. When $\deg(A)\leq 1$, the identification with discrete Painlevé equations was done in [12].

We now look at the case $\deg(A)=2$ when A has two different zeroes, say, $A(x)=a_2(x-\alpha_1)(x-\alpha_2)$. We will deduce a similar form of a dP_V (equations (67)-(68) below). Through a suitable linear change of variable, the zeroes of A can be re-situated, thus, without loss of generality, in what follows we take $A(x)=a_2(x-\alpha_1)(x-\alpha_2)$ with $\alpha_2=-\alpha_1,\alpha_1\neq 0$.

Theorem 4.1 *Let the previous notations hold. Let $A(x)=a_2(x-\alpha_1)(x+\alpha_1)$, with $\alpha_1\neq 0$. The quantities*

$$F_n=\mu+a_2S_n+\eta\gamma_{n+1}+((n+1)a_2+\lambda/2)\alpha_1\,, \tag{65}$$

$$G_n=\frac{\eta(-\alpha_1+\beta_n)+(2n+1)a_2+\lambda}{\eta(\alpha_1+\beta_n)+(2n+1)a_2+\lambda}\,, \tag{66}$$

satisfy the system of equations, for all $n\geq 1$,

$$\frac{(F_n-a_{n,1})(F_n-b_{n,1})}{(F_n-a)(F_n-b)}=G_{n+1}G_n\,, \tag{67}$$

$$F_n+F_{n-1}=-\alpha_1^2\eta+\frac{-2\alpha_1(2\alpha_1\eta+\lambda+(2n+1)a_2)}{G_n-1}+\frac{-4\alpha_1^2\eta}{(G_n-1)^2}\,, \tag{68}$$

with

$$a_{n,1}=2\left((n+1)a_2+\frac{\lambda}{2}\right)\alpha_1-\frac{\eta}{2}\alpha_1^2+\sqrt{\zeta_2}\,, \tag{69}$$

$$b_{n,1}=2\left((n+1)a_2+\frac{\lambda}{2}\right)\alpha_1-\frac{\eta}{2}\alpha_1^2-\sqrt{\zeta_2}\,, \tag{70}$$

$$a=-\frac{\eta}{2}\alpha_1^2+\sqrt{\zeta_1}\,,\quad b=-\frac{\eta}{2}\alpha_1^2-\sqrt{\zeta_1}\,, \tag{71}$$

where $\zeta_1 = -\det \mathcal{B}_0(\alpha_1)$, $\zeta_2 = -\det \mathcal{B}_0(-\alpha_1)$.

The initial conditions hold:

$$F_0 = \mu + a_2\beta_0 + \eta\gamma_1 + (a_2 + \lambda/2)\alpha_1\,, \quad G_1 = \frac{\eta(-\alpha_1+\beta_1) + 3a_2 + \lambda}{\eta(\alpha_1+\beta_1)+3a_2+\lambda}\,. \tag{72}$$

P r o o f. For the sake of simplicity, let us write $\alpha_2 = -\alpha_1$. Evaluating (17) at $\alpha_j,\ j = 1, 2$, yields

$$-l_n^2(\alpha_1) + \Theta_n(\alpha_1)\frac{\Theta_{n-1}}{\gamma_n}(\alpha_1) = -\zeta_1\,,$$
$$-l_n^2(\alpha_2) + \Theta_n(\alpha_2)\frac{\Theta_{n-1}}{\gamma_n}(\alpha_2) = -\zeta_2\,,$$

where $\zeta_j = -\det \mathcal{B}_0(\alpha_j),\ j = 1, 2$.

The ratio of the two equations above is equivalent to

$$\frac{l_n^2(\alpha_2)-\zeta_2}{l_n^2(\alpha_1)-\zeta_1} = \frac{\Theta_n(\alpha_2)/\gamma_{n+1}\ \Theta_{n-1}(\alpha_2)/\gamma_n}{\Theta_n(\alpha_1)/\gamma_{n+1}\ \Theta_{n-1}(\alpha_1)/\gamma_n}\,. \tag{73}$$

Factorizing, we get

$$\frac{(l_n(\alpha_2)-\sqrt{\zeta_2})(l_n(\alpha_2)+\sqrt{\zeta_2})}{(l_n(\alpha_1)-\sqrt{\zeta_1})((l_n(\alpha_1)+\sqrt{\zeta_1}))} = \frac{\Theta_n(\alpha_2)/\gamma_{n+1}\ \Theta_{n-1}(\alpha_2)/\gamma_n}{\Theta_n(\alpha_1)/\gamma_{n+1}\ \Theta_{n-1}(\alpha_1)/\gamma_n}\,. \tag{74}$$

Taking into account that $\alpha_2 = -\alpha_1$, let us define

$$F_n = \ell_{n,0} + ((n+1)a_2 + \lambda/2)\alpha_1\,, \quad G_n = \frac{\Theta_{n-1}(-\alpha_1)/\gamma_n}{\Theta_{n-1}(\alpha_1)/\gamma_n}\,. \tag{75}$$

Thus, we re-write (74) as (67) with the quantities $a_{n,1}, b_{n,1}, a, b$ given in (69)–(71).

Furthermore, we take (16) evaluated at β_n, thus obtaining

$$\ell_{n,0} + \ell_{n-1,0} = -\eta\beta_n^2 - ((2n+1)a_2 + \lambda)\beta_n\,.$$

Adding $((2n+1)a_2 + \lambda)\alpha_1$ to both sides of the above equation we obtain, in the notation from (75),

$$F_n + F_{n-1} = -\eta\beta_n^2 + ((2n+1)a_2 + \lambda)(\alpha_1 - \beta_n)\,, \tag{76}$$

which can be written as (68). □

Remark 4.2 Let us emphasize that the previous result does not hold when $\alpha_2 = \alpha_1$, because a key step in the previous proof is the ratio (73). Obviously, if $\alpha_2 = \alpha_1$, i.e., A has a double root, then such a ratio is one, and (73) gives us no further information.

5 Examples

5.1 Example 1.

We start by taking the SMOP $\{\tilde{P}_n\}_{n\geq 0}$ with respect to the modified Jacobi weight [7]

$$w(x) = e^{-t/x}x^{\alpha}(1-x)^{\beta}\,, \quad x \in [0,1]\,,\ \alpha > 0\,,\ \beta > 0\,,\ t \geq 0\,. \tag{77}$$

Let $(\tilde{\beta}_n)_{n\geq 0}, (\tilde{\gamma}_n)_{n\geq 1}$ denote the recurrence relation coefficients of $\{\tilde{P}_n\}_{n\geq 0}$.

The function w satisfies $\tilde{A}w' = \tilde{C}w$ with

$$\tilde{A}(x) = x^2 - x^3\,, \quad \tilde{C}(x) = (-\alpha-\beta)x^2 + (\alpha - t)x + t\,. \tag{78}$$

Thus, the Stieltjes function of w, henceforth denoted by $\tilde{S}$, satisfies $\tilde{A}\tilde{S}' = \tilde{C}\tilde{S} + \tilde{D}$, $\tilde{D}(x) = \tilde{d}_1 x + \tilde{d}_0$, with

$$\tilde{d}_1 = 1 + \alpha + \beta\,, \quad \tilde{d}_0 = (2 + \alpha + \beta)\tilde{\beta}_0 - 1 - \alpha + t\,. \tag{79}$$

In the account of (3),

$$\tilde{\beta}_0 = \int_0^1 e^{-t/x} x^{\alpha+1} (1-x)^\beta dx \Big/ \int_0^1 e^{-t/x} x^{\alpha} (1-x)^\beta dx\,. \tag{80}$$

Also, from (38), we have

$$\tilde{\gamma}_1 = \frac{t + (2 + \alpha - t)\tilde{\beta}_0 - (3 + \alpha + \beta)\tilde{\beta}_0^2}{3 + \alpha + \beta}\,. \tag{81}$$

Let us now take the sequence of associated polynomials $\{\tilde{P}_n^{(1)}\}_{n\geq 0}$, which, for the sake of simplification of notation, we henceforth denote by $\{P_n\}_{n\geq 0}$, and let us denote by S its Stieltjes function. The function S satisfies

$$\tilde{\gamma}_1 S(x) = -\frac{1}{\tilde{S}(x)} + x - \tilde{\beta}_0\,, \tag{82}$$

thus, S satisfies the Riccati equation

$$AS' = BS^2 + CS + D\,, \tag{83}$$

with

$$A = \tilde{A}\,, \;\; B = \tilde{\gamma}_1 \tilde{D}\,, \;\; C = -\tilde{C} - 2(x - \tilde{\beta}_0)\tilde{D}\,, \;\; D = \frac{1}{\tilde{\gamma}_1}\left(\tilde{A} + (x - \tilde{\beta}_0)\tilde{C} + (x - \tilde{\beta}_0)^2 \tilde{D}\right)\,. \tag{84}$$

After elementary computations, we get

$$A(x) = x^2 - x^3\,, \quad B(x) = \tilde{\gamma}_1(1 + \alpha + \beta)x + \tilde{\gamma}_1((2 + \alpha + \beta)\tilde{\beta}_0 - 1 - \alpha + t)\,, \tag{85}$$

$$C(x) = -(\alpha + \beta + 2)x^2 + (\alpha - t + 2 - 2\tilde{\beta}_0)x - t + 2\tilde{\beta}_0\tilde{d}_0\,, \tag{86}$$

$$D(x) = \frac{t - (\alpha - t)\tilde{\beta}_0 - 2\tilde{\beta}_0\tilde{d}_0 + \tilde{\beta}_0^2\tilde{d}_1}{\tilde{\gamma}_1}x + \frac{-\tilde{\beta}_0 t + \tilde{\beta}_0^2\tilde{d}_0}{\tilde{\gamma}_1}\,. \tag{87}$$

In the account of (83) and (85)–(87), $\{P_n\}_{n\geq 0}$ is a sequence of Laguerre-Hahn orthogonal polynomials of class one. The recurrence relation coefficients of $\{P_n\}_{n\geq 0}$, which we denote by $(\beta_n)_{n\geq 0}, (\gamma_n)_{n\geq 1}$, are defined through the equations given in theorems 3.1 and 3.5. Thus, from (37) and (45), we have, respectively,

$$\gamma_{n+1} = \frac{\vartheta_n}{w_n}\,, \quad n \geq 1\,, \tag{88}$$

$$\beta_{n+1} = \frac{\{(-(2n+3) + \eta)\,\beta_n - 2S_n + 2n + 3 + \lambda\}\,\vartheta_n + y_n\varpi_n}{-(-2(n+2) + \eta)\vartheta_n + z_n\varpi_n}\,, \quad n \geq 2\,, \tag{89}$$

with

$$\vartheta_n = \ell_{n,1}\left(\xi_{0,2} + \ell_{n,1}^2 + \sum_{k=1}^{n} \frac{\theta_{k-1,0}}{\gamma_k}\right) - \ell_{n,2}\xi_{0,1}\,, \tag{90}$$

$$w_n = \ell_{n,1}\frac{\theta_{n,1}}{\gamma_{n+1}}\frac{\theta_{n-1,1}}{\gamma_n} - \ell_{n,2}\left(\frac{\theta_{n,0}}{\gamma_{n+1}}\frac{\theta_{n-1,1}}{\gamma_n} + \frac{\theta_{n,1}}{\gamma_{n+1}}\frac{\theta_{n-1,0}}{\gamma_n}\right)\,, \tag{91}$$

$$\begin{aligned} y_n &= \ell_{n,1}(2n + 1 - \eta)(2n + 3 - \eta) \\ &\quad + \ell_{n,2}\left\{(-2S_n + 2n + 3 + \lambda)(2n + 1 - \eta) + \frac{\theta_{n-1,0}}{\gamma_n}(-2n - 3 + \eta)\right\}\,, \end{aligned} \tag{92}$$

$$z_n = (-n - 1 + \eta/2)(-2n - 4 + \eta)(-2n - 1 + \eta)\,, \tag{93}$$

$$\varpi_n = \left(\frac{\theta_{n-2,1}}{\gamma_{n-1}}\beta_n + \frac{\theta_{n-2,0}}{\gamma_{n-1}}\right)\frac{\vartheta_{n-1}}{y_{n-1} - z_{n-1}\beta_n} + \beta_n^2 - \beta_n^3\,, \tag{94}$$

where

$$\ell_{n,2} = -n-1+\eta/2\,, \quad \ell_{n,1} = -S_n+n+1+\lambda/2\,, \quad n \geq 1\,, \tag{95}$$

$$\frac{\theta_{n,1}}{\gamma_{n+1}} = 2n+3-\eta\,, \quad n \geq 0\,, \tag{96}$$

$$\frac{\theta_{k,0}}{\gamma_{k+1}} = -\left\{-2(S_k+(k+2)\beta_{k+1})+2k+3+\eta\beta_{k+1}+\lambda\right\}\,, \quad k=1,\ldots,n-1\,. \tag{97}$$

Recalling that $\xi_{0,j}$ denotes the coefficient of x^j in $D(A+B)-(C/2)^2$, as well as equations (28) and (29), we obtain, in the account of (85)–(87),

$$\xi_{0,2} = \left(t-\tilde{\beta}_0(\alpha-t)-2\tilde{\beta}_0\tilde{d}_0+\tilde{\beta}_0^2\tilde{d}_1\right)\tilde{d}_1+\frac{-\tilde{\beta}_0 t+\tilde{\beta}_0^2\tilde{d}_0}{\tilde{\gamma}_1} \tag{98}$$

$$-\frac{((\alpha-t+2-2\tilde{\beta}_0)^2-2(-t+2\tilde{\beta}_0\tilde{d}_0))(\alpha+\beta+2)}{4}\,, \tag{99}$$

$$\xi_{0,1} = \left(t-\tilde{\beta}_0(\alpha-t)-2\tilde{\beta}_0\tilde{d}_0+\tilde{\beta}_0^2\tilde{d}_1\right)\tilde{d}_0+(-\tilde{\beta}_0 t+\tilde{\beta}_0^2\tilde{d}_0)\tilde{d}_1$$

$$-\frac{(-t+2\tilde{\beta}_0\tilde{d}_0)(\alpha-t+2-2\tilde{\beta}_0)}{2}\,, \tag{100}$$

$$\frac{\theta_{0,0}}{\gamma_1} = -\frac{\tilde{\gamma}_1\tilde{d}_0-\beta_0\left[-t+2\tilde{\beta}_0\tilde{d}_0-\beta_0(-\tilde{\beta}_0 t+\tilde{\beta}_0^2\tilde{d}_0)/\tilde{\gamma}_1\right](-3+\eta)}{\tilde{\gamma}_1\tilde{d}_1-t+2\tilde{\beta}_0\tilde{d}_0+(4+\alpha-t-2\tilde{\beta}_0)\beta_0-(5+\alpha+\beta)\beta_0^2}\,, \tag{101}$$

$$\eta = -\alpha-\beta-2\,, \quad \lambda = \alpha-t+2-2\tilde{\beta}_0\,. \tag{102}$$

The coefficients $\gamma_1, \beta_2, \beta_1$ are obtained from (38), (46) and (47), respectively, using the data (85)–(87), (101), and (102).

5.2 Example 2.

We start by taking the SMOP $\{\tilde{P}_n\}_{n\geq 0}$ with respect to the modified Laguerre weight [6]

$$w(x) = x^\alpha e^{-x} e^{-s/x}\,, \quad x \in [0,+\infty[\,, \;\; \alpha > 0\,, \;\; s > 0\,. \tag{103}$$

Let $(\tilde{\beta}_n)_{n\geq 0}, (\tilde{\gamma}_n)_{n\geq 1}$ denote the recurrence relation coefficients of $\{\tilde{P}_n\}_{n\geq 0}$.

The function w satisfies $\tilde{A}w' = \tilde{C}w$ with

$$\tilde{A}(x) = x^2\,, \quad \tilde{C}(x) = -x^2+\alpha x+s\,. \tag{104}$$

Thus, the Stieltjes function of w, henceforth denoted by $\tilde{S}$, satisfies $\tilde{A}\tilde{S}' = \tilde{C}\tilde{S}+\tilde{D}$, $\tilde{D}(x) = \tilde{d}_1 x+\tilde{d}_0$, with

$$\tilde{d}_1 = 1\,, \quad \tilde{d}_0 = \tilde{\beta}_0-1-\alpha\,. \tag{105}$$

In the account of (3),

$$\tilde{\beta}_0 = \int_0^{+\infty} x^{\alpha+1}e^{-x}e^{-s/x}dx \Big/ \int_0^{+\infty} x^\alpha e^{-x}e^{-s/x}dx\,. \tag{106}$$

From (54), we obtain

$$\tilde{\gamma}_1 = s+(2+\alpha)\tilde{\beta}_0-\tilde{\beta}_0^2\,. \tag{107}$$

Let us now take the sequence of associated polynomials $\{\tilde{P}_n^{(1)}\}_{n\geq 0}$, which, for the sake of simplification of notation, we henceforth denote by $\{P_n\}_{n\geq 0}$, and let us denote by S its Stieltjes function. The function S satisfies the Riccati equation of type (83), with coefficients

$$A(x) = x^2\,, \quad B(x) = \tilde{\gamma}_1 x + \tilde{\gamma}_1(\tilde{\beta}_0 - 1 - \alpha)\,, \tag{108}$$

$$C(x) = -x^2 + (\alpha+2)x - s + 2\tilde{\beta}_0\tilde{d}_0\,, \tag{109}$$

$$D(x) = \frac{s + \tilde{\beta}_0(\alpha + 2 - \tilde{\beta}_0)}{\tilde{\gamma}_1}x + \frac{\tilde{\beta}_0(\tilde{\beta}_0\tilde{d}_0 - s)}{\tilde{\gamma}_1}\,. \tag{110}$$

In the account of (83) and (108)–(110), $\{P_n\}_{n\geq 0}$ is a sequence of Laguerre-Hahn orthogonal polynomials of class one. The recurrence relation coefficients of $\{P_n\}_{n\geq 0}$, which we denote by $(\beta_n)_{n\geq 0}, (\gamma_n)_{n\geq 1}$, are defined through the equations given in theorems 3.6 and 3.8. Thus, from (53) and (59), we have, respectively,

$$\gamma_{n+1} = \frac{f_n/\eta}{2n + 2 + \lambda + \eta(\beta_{n+1} + \beta_n)}\,, \quad n \geq 1\,, \tag{111}$$

$$\beta_{n+1} = -\beta_n + \frac{\kappa_n - (2n+2+\lambda)}{\eta}\,, \quad n \geq 2\,, \tag{112}$$

with

$$f_n = (2n+2+\lambda)(\mu + S_n) + \xi_{0,1}\,, \quad \kappa_n = \frac{(2n + \lambda + \eta(\beta_n + \beta_{n-1}))f_n}{(2n + \lambda + \eta(\beta_n + \beta_{n-1}))g_n - f_{n-1}}\,, \tag{113}$$

$$g_n = -\eta\beta_n^2 - (2n+2+\lambda)\beta_n - 2\mu - 2S_{n-1}\,. \tag{114}$$

Recalling that $\xi_{0,j}$ denotes the coefficient of x^j in $D(A+B) - (C/2)^2$, and recalling equation (28), we obtain, the account of (108)–(110),

$$\xi_{0,1} = \left(s + \tilde{\beta}_0(\alpha + 2 - \tilde{\beta}_0)\right)\tilde{d}_0 - s\tilde{\beta}_0 + \tilde{\beta}_0^2\tilde{d}_0 - (\alpha+2)(-s + 2\tilde{\beta}_0\tilde{d}_0)/2\,, \tag{115}$$

$$\eta = -1\,, \quad \lambda = \alpha + 2\,, \quad \mu = \tilde{\gamma}_1 + (-s + 2\tilde{\beta}_0\tilde{d}_0)/2\,. \tag{116}$$

The coefficients $\gamma_1, \beta_2, \beta_1$ are obtained from (54), (61) and (62), respectively, using the data (108)–(110) and (116).

5.3 Example 3.

We start by taking the SMOP $\{\tilde{P}_n\}_{n\geq 0}$ with respect to the modified Laguerre weight [14]

$$w(x) = (1 - \zeta H(x - t/2))|x - t/2|^{\omega}(x + t/2)^{\alpha}e^{-x-t/2}\,, \quad x \in]0, +\infty[\,, \;\; \zeta < 1\,, \;\; \alpha > 0\,, \tag{117}$$

where H denotes the Heaviside function, $H(y) = 1$ for $y > 0$, $H(y) = 0$ otherwise. Let $(\tilde{\beta}_n)_{n\geq 0}, (\tilde{\gamma}_n)_{n\geq 1}$ denote the recurrence relation coefficients of $\{\tilde{P}_n\}_{n\geq 0}$.

The function w satisfies $\tilde{A}w' = \tilde{C}w$ (here, $=$ is to be interpreted as equals almost everywhere), with

$$\tilde{A}(x) = x^2 - (t/2)^2\,, \quad \tilde{C}(x) = -x^2 + (\omega + \alpha)x + (\omega - \alpha)t/2 + (t/2)^2\,.$$

Thus, the Stieltjes function of w, henceforth denoted by $\tilde{S}$, satisfies $\tilde{A}S' = \tilde{C}\tilde{S} + \tilde{D}$, $\tilde{D}(x) = \tilde{d}_1 x + \tilde{d}_0$, with

$$\tilde{d}_1 = 1\,, \quad \tilde{d}_0 = \tilde{\beta}_0 - 1 - \omega - \alpha\,. \tag{118}$$

In the account of (3),

$$\tilde{\beta}_0 = \int_0^{+\infty} xw(x)dx \Big/ \int_0^{+\infty} w(x)dx\,. \tag{119}$$

Also, from (54) we obtain

$$\tilde{\gamma}_1 = (\omega - \alpha)t/2 + t^2/4 + (2 + \omega + \alpha)\tilde{\beta}_0 - \tilde{\beta}_0^2\,. \tag{120}$$

Let us now take the sequence of associated polynomials $\{\tilde{P}_n^{(1)}\}_{n\geq 0}$, which, for the sake of simplification of notation, we henceforth denote by $\{P_n\}_{n\geq 0}$, and let us denote by S its Stieltjes function. The function S satisfies the Riccati equation of type (83), with coefficients

$$A(x) = x^2 - (t/2)^2\,, \quad B(x) = \tilde{\gamma}_1 x + \tilde{\gamma}_1 \tilde{d}_0\,, \tag{121}$$

$$C(x) = -x^2 + (\omega+\alpha+2)x - (\omega-\alpha)\frac{t}{2} - \frac{t^2}{4} + 2\tilde{\beta}_0\tilde{d}_0\,, \tag{122}$$

$$D(x) = \frac{1}{\tilde{\gamma}_1}\left\{((\omega-\alpha)\frac{t}{2} + \frac{t^2}{4} - \tilde{\beta}_0(\tilde{\beta}_0 - \omega - \alpha - 2))x - \frac{t^2}{4} - \tilde{\beta}_0((\omega-\alpha)\frac{t}{2} + \frac{t^2}{4} - \tilde{\beta}_0\tilde{d}_0)\right\}. \tag{123}$$

In the account of (83) and (121)–(123), $\{P_n\}_{n\geq 0}$ is a sequence of Laguerre-Hahn orthogonal polynomials of class one. The recurrence relation coefficients of $\{P_n\}_{n\geq 0}$, which we denote by $(\beta_n)_{n\geq 0}, (\gamma_n)_{n\geq 1}$, are defined through the equations given in Theorem 4.1. Thus, the quantities

$$F_n = \mu + S_n + \eta\gamma_{n+1} + (n+1+\lambda/2)t/2\,, \quad G_n = \frac{\eta(-t/2+\beta_n) + 2n+1+\lambda}{\eta(t/2+\beta_n)+2n+1+\lambda}\,, \tag{124}$$

satisfy the system of equations (67)–(68), for all $n \geq 1$,

$$\frac{(F_n - a_{n,1})(F_n - b_{n,1})}{(F_n - a)(F_n - b)} = G_{n+1}G_n\,, \tag{125}$$

$$F_n + F_{n-1} = -\frac{\eta t^2}{4} + \frac{-t(\eta t + \lambda + 2n + 1)}{G_n - 1} + \frac{-\eta t^2}{(G_n - 1)^2}\,, \tag{126}$$

with

$$a_{n,1} = 2\left(n+1+\frac{\lambda}{2}\right)\frac{t}{2} - \frac{\eta t^2}{8} + \sqrt{\zeta_2}\,, \quad b_{n,1} = 2\left(n+1+\frac{\lambda}{2}\right)\frac{t}{2} - \frac{\eta t^2}{8} - \sqrt{\zeta_2}\,, \tag{127}$$

$$a = -\frac{\eta t^2}{8} + \sqrt{\zeta_1}\,, \quad b = -\frac{\eta t^2}{8} - \sqrt{\zeta_1}\,. \tag{128}$$

Recalling (28) and $\zeta_1 = -(D(A+B) - (C/2)^2)(t/2),\ \zeta_2 = -(D(A+B) - (C/2)^2)(-t/2)$, we have, in the account of (121)–(123),

$$\eta = -1\,, \quad \lambda = \omega + \alpha + 2\,, \quad \mu = \tilde{\gamma}_1 - \frac{1}{2}\left((\omega-\alpha)t/2 + t^2/4 - 2\tilde{\beta}_0\tilde{d}_0\right)\,,$$

$$\zeta_1 = -(DB)(t/2) + \frac{C^2(t/2)}{4}\,, \quad \zeta_2 = -(DB)(-t/2) + \frac{C^2(-t/2)}{4}\,.$$

Remark 5.1 In the previous examples we exhibited sequences of Laguerre-Hahn orthogonal polynomials of class one, built from semi-classical polynomials. In general, if we start with a semi-classical SMOP of class one, then the sequence of associated polynomials is Laguerre-Hahn of class one, as the Stieltjes functions related to the associated polynomials satisfy Riccati differential equations (83) with coefficients given as in (84).

Following the notations in examples 1 – 3, and recalling the shift relation for the recurrence relation coefficients, $\tilde{\beta}_n = \beta_{n-1},\ \ n \geq 1,\ \tilde{\gamma}_n = \gamma_{n-1},\ \ n \geq 2$, then the difference equations obtained in the previous examples yield difference equations for the coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ related to the semi-classical weights.

Remark 5.2 When (77) reduces to $w(x) = x^\alpha(1-x)^\beta$, and (103) reduces to $w(x) = x^\alpha e^{-x}$, integral representations for the linear functionals corresponding to the Stieltjes function related to the associated polynomials are known. See [25, Eq. (3.31)] and [17, Eq. (8)] for the first case, and [1] for the second case.

In general, the integral representation of Laguerre-Hahn linear functionals is an open problem.

6 Final remarks and conclusions

In this paper we computed difference equations for the recurrence coefficients of Laguerre-Hahn orthogonal polynomials, obtained trough the so-called compatibility conditions that result from combining the matrix Sylvester equation with the recurrence relation for the orthogonal polynomials (cf. Eq. (10)). Some of the difference equations give a recursive way to compute the recurrence relation coefficients. We also showed forms of discrete Painlevé equations, labelled in terms of $\deg(A)$ in the Riccati equation (4): the case $\deg(A) \leq 1$ was studied in [12]; in case $\deg(A) = 2$ with A having two different roots we deduced a similar form of a discrete Painlevé V; the case $\deg(A) = 3$ remains an open problem, for further consideration.

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